band to match any load, and this letter is concerned with the screw spacing to meet this criterion.

The frequency ranges used for most waveguide sizes cause λ_g to vary by about 2, so that the screw spacing should be less than $\lambda_g/2$ at the highest frequency for the tuner to be useful at any frequency in the band. This can be seen from Fig. 1, which illustrates at (*a*) a tuner with screws 1, 2 and 3 spaced at $k\lambda_g$ and shows at (*b*) the corresponding Smith[3] chart. On this, *A* represents the load presented to the plane of screw 3, *B* to screw 2 and *D* to screw 1. Adjustment of screw 1 adds sufficient susceptance to rotate the load point *D* round the unit conductance circle (I) to the centre of the chart. This requires that screw 2 has been adjusted to add sufficient susceptance to translate the load point *B* around the constant conductance circle on which it lies to *C*, a point on circle II,

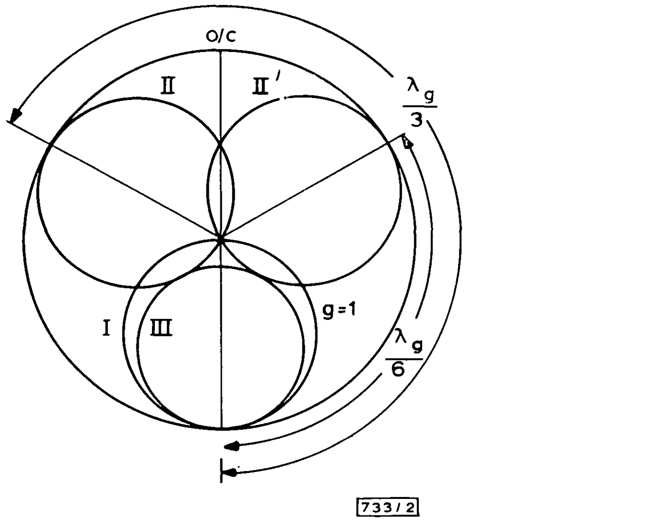

Fig. 2

this being the transformation of I through the $k\lambda_g$ section between screws 1 and 2. If *B* lies within the constant-conductance circle tangential to II (circle III), *B* cannot be translated onto circle II without adjustment of screw 3. This serves to add sufficient susceptance to load point *A* so that *B* (which is *A* transformed by the $k\lambda_g$ section between screws 2 and 3) lies outside circle III. As the frequency increases, the radius of circle III decreases, but, if $k < \frac{1}{2}$ throughout the 2:1 wavelength range, the radius of circle III > 0 and tuning is possible. However, if $k = \frac{1}{2}$ at any frequency, circles I and II coincide and tuning is impossible.

The design can be optimised, in the sense that the action of the screws can be made the same at the band edges, if the screw spacing be $\lambda_g/6$ at the lowest frequency of the band, or, more generally, if it be $k\lambda_g$, where $k = \frac{1}{2}(1+r)^{-1}$, *r* being the ratio of λ_g at the lowest frequency to that at the highest. The Smith chart illustrating this is shown at Fig. 2, in which circle I is the unit-conductance circle and circles II and II′ represent, respectively, its transformation by the $\lambda_g/6$ section between screws 1 and 2 at the lowest frequency and $\lambda_g/3$ at the highest. The radius of circle III is then the same at both band edges ($g \simeq 1{\cdot}35$) and its variation with frequency is minimised.

Although the foregoing is elementary, the author is unaware of any statement of it, and some suppliers of tuners do not incorporate these features in their designs.

E. J. GRIFFIN

8th November 1976

Royal Signals & Radar Establishment
St Andrews Road, Great Malvern, Worcs. WR14 3PS, England

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METHOD OF CONSTRUCTING DE-BRUIJN SEQUENCES

Indexing terms: Binary sequences, Shift registers

It is shown how a de-Bruijn sequence of length 2^n, which is considered to have 'good' random properties, can be generated by starting from a p.n. sequence of length $2^{n-1}-1$. The number of different sequences which can be generated from the same p.n. sequence is 2^n-4.

Introduction: A class of binary sequences which are considered to have 'good' random properties are the de-Bruijn sequences.[1, 2] Such a sequence is of length 2^n with all possible strings of *n* successive bits distinct. Throughout this letter, the first bit of a finite sequence is considered to be successor of the last bit, owing to the cyclic properties of the sequence.

A known general method of constructing de-Bruijn sequences is by joining several short sequences.[1, 3-5] This method is usually implemented by a complicated design. This letter shows how a class of de-Bruijn sequences can be constructed by joining two sequences, using a modified version of a linear shift register which generates a p.n. sequence.

Theory:
Postulate 1: Let *A* be a p.n. sequence of length $2^{n-1}-1$. Let *S* be the set of all $2^{n-1}-1$ possible strings of *n* successive bits taken from *A*. Let $x = (x_1, x_2, \ldots, x_n)$ be a string of *n* bits and let $\bar{x}$ be its complement. If $x \in S$, then $\bar{x} \notin S$.

Proof: A basic property of a linear feedback shift register which generates a p.n. sequence is that an even number of stages are connected to the feedback loop. If a string $\hat{x}$ consists of the first $n-1$ bits of x, x_n is the sum of an even number of bits taken from $\hat{x}$, and therefore the bit which follows $\bar{\hat{x}}$ in *A* is also x_n. This means that the string $\bar{x}$ does not exist in *A*, and therefore $\bar{x} \notin S$.

Let $\bar{S}$ denote the set whose elements are the complements of the strings in *S*. If $\bar{A}$ is the complement of *A*, then $\bar{S}$ contains all strings of length *n* in $\bar{A}$. To illustrate postulate 1, as well as the following postulates, the strings of *S* and $\bar{S}$ are listed for $A = 1110010$:

$$S = \{1110, 1100, 1001, 0010, 0101, 1011, 0111\}$$

$$\bar{S} = \{0001, 0011, 0110, 1101, 1010, 0100, 1000\}$$

Notation: Let *a* and *b* be strings of binary bits. The string *ab* is obtained by attaching *b* to the end of *a*.

Postulate 2: If $x = x_1 b x_n \in S$ and $y = y_1 b y_n \in \bar{S}$, the common string *b* being any binary string of length $n-2$, then either $x_1 = y_1$ or $x_n = y_n$.

Proof:
(*a*) *b is 'all 0':* A basic property of the p.n. sequence *A* is that there exists in it one and only one string of 0s whose length is $n-2$. The length of *A* was defined to be $2^{n-1}-1$. The only $x \in S$ whose form is $x_1 b x_n$ is therefore $1b1$.

Since there is one and only one string of $n-1$ successive 1s in *A*, there are two and only two strings in *S* of the form $x_1 \bar{b} x_n$. ($\bar{b}$ being 'all 1'). These are $1\bar{b}0$ and $0\bar{b}1$. It follows that there are two and only two strings in $\bar{S}$ of the form $y_1 b y_n$. These are $0b1$ and $1b0$. Since it has been shown that $x = 1b1$, the postulate is proved.

(*b*) *b is 'all 1':* Since the strings in $\bar{S}$ are the complements of the strings in *S*, it follows that the only $y = y_1 b y_n \in \bar{S}$ is $0b0$, which is the complement of *x* mentioned in (*a*). It has also been shown that the only strings in *S* of the form $x_1 b x_n$ are $1b0$ and $0b1$, and the postulate is proved.

(*c*) *b is neither 'all 0' nor 'all 1':* In view of postulate 1, the set $S \cup \bar{S}$ contains 2^n-2 distinct strings. In view of what was said above, the missing two strings, out of the 2^n possible ones, are of the form $z_1 b z_n$, where *b* is either 'all 0' or 'all 1'. For *b* being any other string of length $n-2$, the set $S \cup \bar{S}$ contains all of the following four possible sequences: $0b0$, $0b1$, $1b0$ and $1b1$. Since no string of length $n-1$ in *A* repeats itself and the same applies to $\bar{A}$, it follows that the only way of distributing these four sequences between *S* and $\bar{S}$ is that

$0b0$ and $1b1$ belong to one set, where $1b0$ and $0b1$ belong to the other; otherwise either $0b$, $b0$, $1b$ or $b1$ occur twice. This means that if $x_1 bx_n \in S$ and $y_1 by_n \in \bar{S}$ then either $x_1 = y_1$ or $x_n = y_n$.

Implementation: If A is a p.n. sequence of length $2^{n-1}-1$ and $\bar{A}$ is its complement, all strings of length $n-1$ which appear in A appear in $\bar{A}$, except for the 'all-1' sequence which is replaced by an 'all 0'.

For any cyclic shift of A, except for that starting with $n-1$ successive 1s, it is possible to shift $\bar{A}$ cyclically until $\bar{A}'$ is obtained, such that both A and $\bar{A}'$ start with the same $n-1$ bits. If $\bar{A}'$ is now attached to the tail of A, a sequence of length 2^n-2 is obtained. This sequence is denoted by B.

Illustration 1 will clarify some of the following arguments:

Illustration 1: Let $k = 2^{n-1}-1$, $A = (a_1, a_2, \ldots, a_k)$ and $\bar{A}' = (a_1', a_2', \ldots, a_k')$, where $a_i = a_i'$, $i = 1, 2, \ldots, n-1$. Then $B = (a_1, a_2, \ldots, a_k, a_1', a_2', \ldots, a_k')$. It will be shown now that the 2^n-2 strings of length n in B are all the strings in $S \cup \bar{S}$ and are distinct, in view of postulate 1.

There are $n-1$ strings of length n in S, which are obtained from A by an 'end-around' process. These are the strings which start with any of the last $n-1$ bits of A. These strings are also found in B, since the second half of B, namely $\bar{A}'$, starts with the same $n-1$ bits as A, such that the 'end around' is replaced by continuing with the bits which start the second half of B. For example, the string $(a_k, a_1, a_2, \ldots, a_{n-1}) \in S$ is replaced in B by the string $(a_k, a_1', a_2', \ldots, a'_{n-1})$.

The $n-1$ strings of length n in $\bar{S}$, which are obtained from $\bar{A}'$ by 'end around', are also found in B. This is because B starts with the same $n-1$ bits as $\bar{A}'$, so that 'end around' for the complete B produces the same strings as those produced when 'end around' is performed only in $\bar{A}'$. For example, the string $(a_k', a_1', a_2', \ldots, a'_{n-1}) \in \bar{S}$ is replaced in B by the string $(a_k', a_1, a_2, \ldots, a_{n-1})$.

Of course, B contains all the remaining strings in S and $\bar{S}$. It follows that all the 2^n-2 distinct strings in $S \cup \bar{S}$ are found in B. If a 1 and a 0 are now added to the longest strings of 1s and 0s, respectively, a de-Bruijn sequence of length 2^n is obtained.

The following postulate shows how many different sequences B can be produced by A (where two cyclic shifts of the same B are not considered different).

Postulate 3: Each cyclic shift of A produces a different sequence B.

Proof: Assume that there exists a sequence B' which is obtained by shifting B cyclically for l places, and assume that the first and second halves of B' are a cyclic shift of A and $\bar{A}$, respectively, where both halves start with the same $n-1$ bits. If $n \leqslant l \leqslant 2^n-2-n$, the result is that $\bar{A}'$ and A have n or more successive bits in common (n or more bits of $\bar{A}'$ are now in the first half of B'), which is impossible. If $0 < l < n$ or $2^n-2-n < l < 2^n-2$, then either a_n and a_n', or a_k' and a_k, are in corresponding places within the $n-1$ bits which start each half of B' (refer to illustration 1). This means that $a_n = a_n'$ or $a_k' = a_k$. On the other hand, since A and $\bar{A}'$ (in B) have their first $n-1$ bits in common, it follows that $a_n \neq a_n'$ and $a_k' \neq a_k$, otherwise A and $\bar{A}'$ have a string of length n in common. It is concluded that B' cannot exist and all sequences B generated by any cyclic shift of A are different.

There are $2^{n-1}-2$ possible cyclic shifts of A which produce a sequence B. The only missing shift being the one which starts with $n-1$ successive 1s. The number of different sequences B, produced by A, is therefore $2^{n-1}-2$.

The following is a second method by which more de-Bruijn sequences are obtained:

Let B be obtained by attaching A and $\bar{A}$, where $\bar{A}'$ is obtained from the complement of A by a cyclic shift such that A and $\bar{A}'$ are identical only in their first $n-2$ bits, while their $(n-1)$th bit differs. Illustration 1 is still valid here. The only change is that $a_i = a_i'$, $i = 1, 2, \ldots, n-2$.

Of the $2(n-1)$ strings of length n which are in $S \cup \bar{S}$ and which are obtained from $\bar{A}$ or $\bar{A}'$ by an 'end-around' process, all but two can be found directly in B by using considerations identical to those mentioned before. The two strings in B which need a special consideration are those starting with a_k and a_k'. Let these strings be denoted by x' and y', respectively. If $x \in S$ starts with a_k, then $x \neq x'$, since they differ in their last bit (unlike the previous case where $x = x'$). If $y \in \bar{S}$ starts with a_k', then $y \neq y'$.

Postulate 4: $x' = y$ and $y' = x$.

Proof: $x = a_k ba_{n-1}$ and $y = a_k' ba'_{n-1}$, where b is a string of $n-2$ bits common to x and y and $a_{n-1} \neq a'_{n-1}$. It follows from postulate 2 that $a_k = a_k'$. Since $x' = a_k ba'_{n-1}$ and $y' = a_k' ba_{n-1}$, it follows that $x' = y$ and $y' = x$.

It follows that all 2^n-2 strings of length n in B are the strings of $S \cup \bar{S}$ and are therefore distinct.

The number of different de-Bruijn sequences obtained by the two described methods is $2(2^{n-1}-2) = 2^n-4$. In the first version, the sequence B is generated by a linear shift register which generates a p.n. sequence of length $2^{n-1}-1$. After the initial state repeats itself, the bits which are fed back are complemented. In the second version, the sequence B is generated by first complementing the last bit of the repeated initial state and then carrying out the above procedure.

B. ARAZI *2nd November 1976*

National Electrical Engineering Research Institute
PO Box 395
Pretoria 0001, South Africa

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DESIGN OF STABLE 2-DIMENSIONAL DISCRETE RECURSIVE FILTERS

Indexing term: Digital filters

The letter presents preliminary results on the design of stable 2-dimensional (2d) discrete recursive filters. The method is based on the properties of multivariable positive real functions and multivariable passive networks, and results in an approach wherein the stability of the filter is guaranteed.

There has been recent interest in the design of 2d recursive filters for processing 2d discrete signals. Denoting the transfer function of a 2d recursive filter as

$$H(z_1, z_2) = \frac{N(n_{ij}, z_1, z_2)}{D(d_{ij}, z_1, z_2)} \equiv \frac{\sum_{i,j} n_{ij} z_1{}^i z_2{}^j}{\sum_{i,j} d_{ij} z_1{}^i z_2{}^j} \quad . \quad . \quad (1)$$

where N and D are polynomials in z_1 ($=\exp(s_1 T_1)$) and z_2 ($=\exp(s_2 T_2)$), the design problem is to obtain the polynomial coefficients $\{n_{ij}\}$ and $\{d_{ij}\}$ such that

(*a*) H approximates a given response

(*b*) the filter is stable. That is

$$D(d_{ij}, z_1, z_2) \neq 0 \quad \text{for} \quad |z_1|, |z_2| \geqslant 1 \quad . \quad . \quad . \quad (2)$$

The testing of the condition in expr. 2 is by no means an easy task,[1] inhibiting the widespread application of recursive filters, in spite of their significant advantages over the nonrecursive filters.