DE-2013

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Ch 3.1: Second Order Linear Homogeneous Equations with Constant Coefficients

- A **second order ordinary differential equation** has the general form

$$y'' = f(t, y, y')$$

where *f* is some given function.

- This equation is said to be **linear** if *f* is linear on *y* and *y'*:

$$y'' = g(t) - p(t)y' - q(t)y \qquad P(t)y'' + Q(t)y' + R(t)y = G(t)$$

Give emphasis on superposition of linear parts.

- If $G(t) = 0$ for all *t*, then the equation is called **homogeneous**. Otherwise the equation is **nonhomogeneous**.
- IVP $\quad ay'' + by' + cy = 0 \qquad y(t_0) = y_0,\ y'(t_0) = y_0'$

Example: Infinitely Many Solutions

- Linear DE, two solutions, and combinations of these two in any numbers, and

  $y'' - y = 0 \quad y_1(t) = e^t, \quad y_2(t) = e^{-t}$

  $y_3(t) = 3e^t, \quad y_4(t) = 5e^{-t}, \quad y_5(t) = 3e^t + 5e^{-t}$

- Characteristic Eq. three possible cases.

  $y(t) = c_1 e^t + c_2 e^{-t}$

$$y'' - y = 0, \quad y(0) = 3, \; y'(0) = 1 \quad \left.\begin{array}{l} y(0) = c_1 + c_2 = 3 \\ y'(0) = c_1 - c_2 = 1 \end{array}\right\} \Rightarrow c_1 = 2, \; c_2 = 1$$

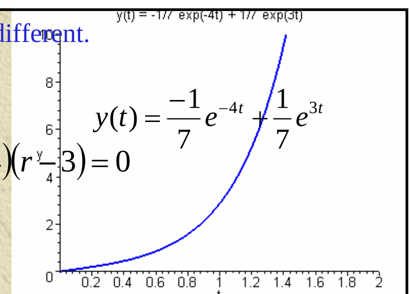
$$ay'' + by' + cy = 0, \quad ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0$$

$$ar^2 + br + c = 0, \qquad r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

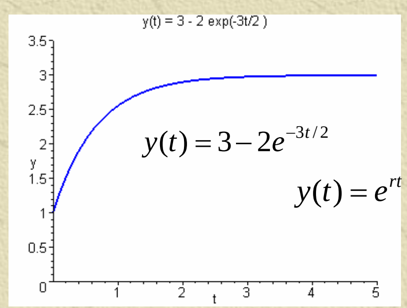
Check these examples for the case where roots are different.

$$y'' + y' - 12y = 0, \;\; y(0) = 0, \;\; y'(0) = 1$$

$$y(t) = e^{rt} \;\Rightarrow\; r^2 + r - 12 = 0 \;\Leftrightarrow\; (r+4)(r-3) = 0$$



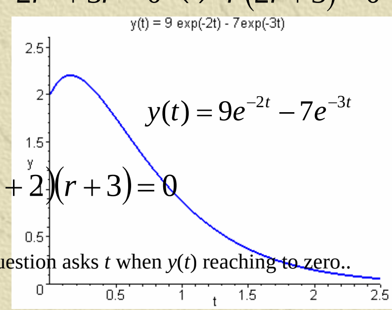
$$y(t) = \frac{-1}{7} e^{-4t} + \frac{1}{7} e^{3t}$$



$$y(t) = 3 - 2e^{-3t/2}$$

$$2y'' + 3y' = 0, \;\; y(0) = 1, \;\; y'(0) = 3$$

$$y(t) = e^{rt} \;\Rightarrow\; 2r^2 + 3r = 0 \;\Leftrightarrow\; r(2r+3) = 0$$



$$y(t) = 9e^{-2t} - 7e^{-3t}$$

$$y'' + 5y' + 6y = 0, \;\; y(0) = 2, \;\; y'(0) = 3$$

$$y(t) = e^{rt} \;\Rightarrow\; r^2 + 5r + 6 = 0 \;\Leftrightarrow\; (r+2)(r+3) = 0$$

$$y(t) = -18e^{-2t} + 21e^{-3t} \overset{set}{=} 0 \text{ ==> } 6e^{-2t} = 7e^{-3t}$$

this question asks t when $y(t)$ reaching to zero..

$$e^t = 7/6 \text{ ==> } t \approx .1542; \; y \approx 2.204$$

Ch 3.2: Fundamental Solutions of Linear Homogeneous Equations

- p, q are continuous funtns on an interval $I = (\alpha, \beta)$, can be [∞].
- For a function y twice differentiable on I, differential operator L by

$$L[y](t) = y''(t) + p(t)\,y'(t) + q(t)\,y(t)$$

- ie in pieces, $p(t) = t^2,\ q(t) = e^{2t},\ y(t) = \sin(t),\ I = (0, 2\pi)$

$$L[y](t) = -\sin(t) + t^2\cos(t) + 2e^{2t}\sin(t)$$

- $L[y](t) = 0$, Linear homogeneous equation, along with TWO initial conditions, and if so, are they unique.

$$y(t_0) = y_0,\ y'(t_0) = y_1$$

Theorem 3.2.1 Work out the example below yourself.

- Consider the initial value problem

$$y'' + p(t)\,y' + q(t)\,y = g(t)$$
$$y(t_0) = y_0,\ y'(t_0) = y'_0$$

- where p, q, and g are continuous on **an open interval I that contains t_0**. Then there exists a unique solution $y = \phi(t)$ on I.
- Note: While this theorem says that a solution to the ivp above exists, it is often not possible to write down a useful expression for the solution which is a major difference between first and second order linear equations.
- Determine the longest interval on which the given initial value problem is certain to have a unique twice differentiable solution. Do not attempt to find the solution.

$$(t+1)y'' - (\cos t)y' + 3y = 1,\ y(0) = 1,\ y'(0) = 0$$

- First put differential equation into standard form:

$$y'' - \frac{\cos t}{t+1}y' + \frac{3}{t+1}y = \frac{1}{t+1},\ y(0) = 1,\ y'(0) = 0$$

- The longest interval containing the point $t = 0$ on which the coefficient functions are continuous is (-1, ∞)!!!!!.
- It follows from Theorem 3.2.1 that the longest interval on which this initial value problem is certain to have a twice differentiable solution is also (-1, ∞).

Theorem 3.2.2 (Principle of Superposition)

- If $y_1=f$ and $y_2=g$ are solutions to the 2nd order LDE, then the linear combination $c_1y_1 + c_2y_2$ is also a solution, for all constants c_1 and c_2. To prove, substitute $c_1y_1 + y_2c_2$ and check…!!!! Please remember the notes given before (Proving linearity if always an exam question, in terms of additivity and scalability).
- Wronskian determinant: Can all solutions can be written this way, or do some solutions have a different form altogether?
- Suppose y_1 and y_2 are solutions to $L[y] = y'' + p(t)y'=g(t)$, with initial values of $y(t_0)=y_0$ and $y'(t_0)= y_0'$,
- from Thr 3.2.2, $y = c_1y_1 + c_2y_2$ is a solution to this equation such that $y = c_1y_1 + c_2y_2$ satisfies the given initial conditions ****

$$\begin{aligned} c_1y_1(t_0)+c_2y_2(t_0) &= y_0 \\ c_1y_1'(t_0)+c_2y_2'(t_0) &= y_0' \end{aligned}$$

$$c_1 = \frac{y_0y_2'(t_0)-y_0'y_2(t_0)}{y_1(t_0)y_2'(t_0)-y_1'(t_0)y_2(t_0)}$$

$$c_2 = \frac{-y_0y_1'(t_0)+y_0'y_1(t_0)}{y_1(t_0)y_2'(t_0)-y_1'(t_0)y_2(t_0)}$$

The Wronskian Determinant

- Arbitrary coefficients, in terms of determinants **Wronskian determinant**,
- If solution exists then, the determinant W cannot be zero.

$$c_1 = \frac{y_0y_2'(t_0)-y_0'y_2(t_0)}{y_1(t_0)y_2'(t_0)-y_1'(t_0)y_2(t_0)}$$

$$c_2 = \frac{-y_0y_1'(t_0)+y_0'y_1(t_0)}{y_1(t_0)y_2'(t_0)-y_1'(t_0)y_2(t_0)}$$

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}, \quad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{\begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}}$$

$$c_1 = \frac{\begin{vmatrix} y_0 & y_2(t_0) \\ y_0' & y_2'(t_0) \end{vmatrix}}{W}, \quad c_2 = \frac{\begin{vmatrix} y_1(t_0) & y_0 \\ y_1'(t_0) & y_0' \end{vmatrix}}{W}$$

$$W = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0)-y_1'(t_0)y_2(t_0)$$

$$W(y_1, y_2)(t_0)$$

Theorem 3.2.3

- The opposite of theorems given before, suppose y_1 and y_2 are solutions to the equation at a given t_0, if the Wronskian

  W= $y_1 y'_2 + y'_1 y_2 \neq 0$, then there is a choice of constants c_1, c_2 for any of which $y = c_1 y_1 + c_2 y_2$ is a solution to the DE for given initial conditions $y(t_0)=y_0$ and $y'(t_0)= y_0'$.
- Recall the following initial value problem and its solution:

$$y'' - y = 0,\ y(0) = 3,\ \ y'(0) = 1 \ \ \Rightarrow \ \ y(t) = 2e^{t} + e^{-t}$$

- The two functions that are are solutions: $y_1 = e^{t}, y_2 = e^{-t}$
- The Wronskian of y_1 and y_2 is

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 = -e^{t}e^{-t} - e^{t}e^{-t} = -2e^{0} = -2$$

- Since $W \neq 0$ for all t, linear combinations of y_1 and y_2 can be used to construct solutions of the IVP for any initial value t_0.
- $y = c_1 y_1 + c_2 y_2$

Theorem 3.2.4 (Fundamental Solutions)

- Suppose y_1 and y_2 are solutions to the equation L[y]=0,
- If there is a point t_0 such that $W(y_1,y_2)(t_0) \neq 0$, then the family of solutions $y = c_1 y_1 + c_2 y_2$ with arbitrary coefficients c_1, c_2 includes every solution to the differential equation.
- The expression $y = c_1 y_1 + c_2 y_2$ is called the **general solution** of the differential equation above, and in this case y_1 and y_2 are said to form a **fundamental set of solutions** to the differential equation.
- In previous example… $y_1 = e^{r_1 t}, y_2 = e^{r_2 t}, \ \ r_1 \neq r_2$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{r_1 t} & e^{r_2 t} \\ r_1 e^{r_1 t} & r_2 e^{r_2 t} \end{vmatrix} = (r_2 - r_1)e^{(r_1 + r_2)t} \neq 0 \ \text{ for all } t.$$

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Example: $2t^2 y'' + 3t\,y' - y = 0,\ t > 0$

- Show that the functions below are fundamental solutions:
  $y_1 = t^{1/2}, y_2 = t^{-1}$
- To show this, first substitute y_1 into the equation, similar for y_2:

$$2t^2\left(\frac{-t^{-3/2}}{4}\right) + 3t\left(\frac{t^{-1/2}}{2}\right) - t^{1/2} = \left(-\frac{1}{2} + \frac{3}{2} - 1\right)t^{1/2} = 0 \qquad 2t^2\left(2t^{-3}\right) + 3t\left(-t^{-2}\right) - t^{-1} = (4 - 3 - 1)t^{-1} = 0$$

- To show both solutions form fundamental set of solutions.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} t^{1/2} & t^{-1} \\ \frac{1}{2}t^{-1/2} & -t^{-2} \end{vmatrix} = -t^{-3/2} - \frac{1}{2}t^{-3/2} = -\frac{3}{2}t^{-3/2} = -\frac{3}{2\sqrt{t^3}}$$

- Since $W \neq 0$ for $t > 0$, y_1, y_2 form a fundamental set of solutions for the differential equation

$$2t^2 y'' + 3t\,y' - y = 0,\ t > 0$$

Summary

- To find a general solution of the differential equation

$$y'' + p(t)\,y' + q(t)\,y = 0,\ \alpha < t < \beta$$

  we first find two solutions y_1 and y_2.
- Then make sure there is a point t_0 in the interval such that $W(y_1, y_2)(t_0) \neq 0$.
- It follows that y_1 and y_2 form a fundamental set of solutions to the equation, with general solution $y = c_1 y_1 + c_2 y_2$.
- If initial conditions are prescribed at a point t_0 in the interval where $W \neq 0$, then c_1 and c_2 can be chosen to satisfy those conditions.
- Exact and adjoint and self adjoint functions?
  Page 126, questions 26 and32

Ch 3.3: Linear Independence and the Wronskian

- Two functions *f* and *g* are **linearly dependent** if they are multiples of each other. $c_1 f(t) + c_2 g(t) = 0$
  If the only solution to this equation is $c_1 = c_2 = 0$, then *f* and *g* are **linearly independent**. For example, $f(x) = \sin 2x$ and $g(x) = \sin x \cos x$, and their linear combination $c_1 \sin 2x + c_2 \sin x \cos x = 0$
  This is satisfied if we choose $c_1 = 1$, $c_2 = -2$, and hence *f* and *g* are linearly dependent.
- Note that if $a = b = 0$, then the only solution to this system of equations is $c_1 = c_2 = 0$, provided $D \neq 0$.

$$\begin{aligned} c_1 x_1 + c_2 x_2 &= a \\ c_1 y_1 + c_2 y_2 &= b \end{aligned} \qquad c_1 = \frac{a y_2 - b x_2}{x_1 y_2 - y_1 x_2} = \frac{a y_2 - b x_2}{D},$$

$$c_2 = \frac{-a y_1 + b x_1}{x_1 y_2 - y_1 x_2} = \frac{-a y_1 + b x_1}{D}, \quad \text{where } D = \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$$

Example 1: Linear Independence

- Show that the following two functions are linearly independent on any interval:
$$f(t) = e^t, \quad g(t) = e^{-t}$$
- Suppose for all *t* in an arbitrary interval (α, β).
$$c_1 f(t) + c_2 g(t) = 0$$
- We want to show the equation holds only for $c_1 = c_2 = 0$ for all *t* in (α, β), where $t_0 \neq t_1$, except $t_0 = t_1$. Then

$$D = \begin{vmatrix} e^{t_0} & e^{-t_0} \\ e^{t_1} & e^{-t_1} \end{vmatrix} = e^{t_0} e^{-t_1} - e^{-t_0} e^{t_1} = e^{t_0 - t_1} - e^{t_1 - t_0}$$

$$\begin{aligned} c_1 e^{t_0} + c_2 e^{-t_0} &= 0 \\ c_1 e^{t_1} + c_2 e^{-t_1} &= 0 \end{aligned} \qquad D = 0 \;\Leftrightarrow\; e^{t_0 - t_1} = e^{t_1 - t_0} \;\Leftrightarrow\; t_0 = t_1$$

- $D \neq 0$, and therefore *f* and *g* are linearly independent.

Theorem 3.3.1 from Wronskian p o view

- If f and g are a) **differentiable** functions on an open interval I and b) **if $W(f, g)(t_0) \neq 0$** for some point **t_0 in I**, then f and g are **linearly independent on** I. Moreover, if f and g are linearly dependent on I, then $W(f, g)(t) = 0$ for all t in I.

- Proof (outline): Let c_1 and c_2 be scalars, and suppose $c_1 f(t)+c_2 g(t)=0$
- for all t in I. In particular, when $t = t_0$ we have

$$c_1 f(t_0) + c_2 g(t_0) = 0$$
$$c_1 f'(t_0) + c_2 g'(t_0) = 0$$

- Since $W(f, g)(t_0) \neq 0$, it follows that $c_1 f(t)+c_2 f(t)=0$ only at $c_1 = c_2 = 0$, and hence f and g are linearly independent.

Theorem 3.3.2 (Abel's Theorem)

- Suppose y_1 and y_2 are solutions to the equation

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0$$

where p and q are continuous on some open interval I. Then $W(y_1,y_2)(t)$ is given by

$$W(y_1, y_2)(t) = ce^{-\int p(t)\,dt}$$

where c is a constant that depends on y_1 and y_2 but not on t.

- Note that **$W(y_1,y_2)(t)$ is either zero for all t in I only if $c = 0$** or **else is never zero in I (if $c \neq 0$).**

Example 2: Wronskian and Abel's Theorem

- Recall the following equation and two of its solutions:

$$y'' - y = 0,\ \ y_1 = e^t, y_2 = e^{-t}$$

- The Wronskian of y_1 and y_2 is

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = -e^t e^{-t} - e^t e^{-t} = -2e^0 = -2 \neq 0 \text{ for all } t.$$

- Thus y_1 and y_2 are linearly independent on any interval I, by Theorem 3.3.1. Now compare W with Abel's Theorem:

$$W(y_1, y_2)(t) = ce^{-\int p(t)dt} = ce^{-\int 0 dt} = c$$

- Choosing $c = -2$, we get the same W as above.

Theorem 3.3.3

- Suppose y_1 and y_2 are solutions to equation below, whose coefficients p and q are continuous on some open interval I:

$$L[y] = y'' + p(t)\,y' + q(t)\,y = 0$$

Then y_1 and y_2 are linearly dependent on I iff $W(y_1, y_2)(t) = 0$ for all t in I. Also, y_1 and y_2 are linearly independent on I iff $W(y_1, y_2)(t) \neq 0$ for all t in I.

Summary

- Let y_1 and y_2 be solutions of

$$y'' + p(t)\,y' + q(t)\,y = 0$$

where p and q are continuous on an open interval I.

- Then the following statements are equivalent:
  - The functions y_1 and y_2 form a fundamental set of solutions on I.
  - The functions y_1 and y_2 are linearly independent on I.
  - $W(y_1,y_2)(t_0) \neq 0$ for some t_0 in I.
  - $W(y_1,y_2)(t) \neq 0$ for all t in I.

Linear Algebra Note

- Let V be the set

$$V = \{y : y'' + p(t)\,y' + q(t)\,y = 0,\ t \in (\alpha,\beta)\}$$

Then V is a vector space of dimension two, whose bases are given by any fundamental set of solutions y_1 and y_2.

- For example, the solution space V to the differential equation

$$y'' - y = 0$$

has bases

$$S_1 = \{e^t, e^{-t}\},\quad S_2 = \{\cosh t, \sinh t\}$$

with

$$V = \text{Span}\, S_1 = \text{Span}\, S_2$$

Unit Circle, Taylor and Mc Laurin series.

- z=x+iy, [x, y]=rcos(Θ), r(sin(Θ));
- (r, Θ)={sqrt(x^2+y^2), arctang(x/y)}
- z=x+iy=|z|cos(Θ)+isin(Θ)=|z| $e^{i\Theta}$
- f(a)+f'(a)(x-a)/1!+f''(a)$(x-a)^2$/2!+f'''(a)$(x-a)^3$/2!+….
- {1/(1-x)}=1+x+x^2+x^3+..
- Wikipedia..

Ch 3.4: Complex Roots of Characteristic Equation

- Recall our discussion of the equation
  $$ay'' + by' + cy = 0$$
  where *a*, *b* and *c* are constants.
- Assuming an exponential soln leads to characteristic equation:
  $$y(t) = e^{rt} \quad \Rightarrow \quad ar^2 + br + c = 0$$
- Quadratic formula (or factoring) yields two solutions, r_1 & r_2:
  $$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
- If $b^2 - 4ac < 0$, then complex roots: $r_1 = \lambda + i\mu$, $r_2 = \lambda - i\mu$
- Thus
  $$y_1(t) = e^{(\lambda + i\mu)t}, \quad y_2(t) = e^{(\lambda - i\mu)t}$$

Euler's Formula; Complex Valued Solutions

- Substituting *it* into Taylor series for e^t, we obtain **Euler's formula**:

$$e^{it} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i\sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^{2n-1}}{(2n-1)!} = \cos t + i \sin t$$

- Generalizing Euler's formula, we obtain

$$e^{i\mu t} = \cos \mu t + i \sin \mu t$$

- Then

$$e^{(\lambda + i\mu)t} = e^{\lambda t} e^{i\mu t} = e^{\lambda t}\left[\cos \mu t + i \sin \mu t\right] = e^{\lambda t} \cos \mu t + i e^{\lambda t} \sin \mu t$$

- Therefore

$$y_1(t) = e^{(\lambda + i\mu)t} = e^{\lambda t}(\cos \mu t + i \sin \mu t)$$
$$y_2(t) = e^{(\lambda - i\mu)t} = e^{\lambda t}(\cos \mu t - i \sin \mu t)$$

Real Valued Solutions, The Wronskian

To achieve this, recall that linear combinations of solutions are themselves solutions. Ignoring constants

$$y_1(t) + y_2(t) = 2e^{\lambda t} \cos \mu t \qquad y_3(t) = e^{\lambda t} \cos \mu t,$$
$$y_1(t) - y_2(t) = 2ie^{\lambda t} \sin \mu t \qquad y_4(t) = e^{\lambda t} \sin \mu t$$

Checking the Wronskian

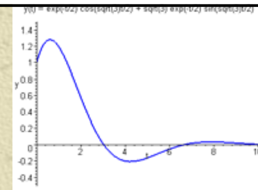
$$W = \begin{vmatrix} e^{\lambda t} \cos \mu t & e^{\lambda t} \sin \mu t \\ e^{\lambda t}(\lambda \cos \mu t - \mu \sin \mu t) & e^{\lambda t}(\lambda \sin \mu t + \mu \cos \mu t) \end{vmatrix}$$
$$= \mu e^{2\lambda t} \neq 0$$

Thus y_3 and y_4 form a fundamental solution set, and the general solution

- All Exam Quest:
- Example 1: $y'' + y' + y = 0$

$$y(t) = e^{rt} \Rightarrow r^2 + r + 1 = 0$$

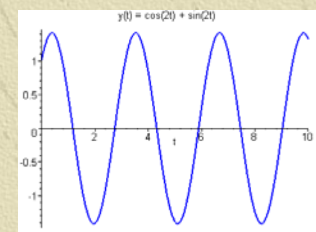
$$r = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}\,i}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$



- Therefore and thus $\lambda = -1/2,\ \mu = \sqrt{3}/2$
- the general solution is $y(t) = e^{-t/2}\left(c_1 \cos\left(\sqrt{3}t/2\right) + c_2 \sin\left(\sqrt{3}t/2\right)\right)$
- Example 2: $y'' + 4y = 0$

$$y(t) = e^{rt} \Rightarrow r^2 + 4 = 0 \Leftrightarrow r = \pm 2i \qquad \lambda = 0,\ \mu = 2$$

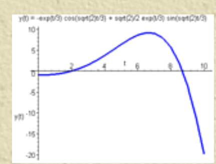
$$y(t) = c_1 \cos(2t) + c_2 \sin(2t)$$

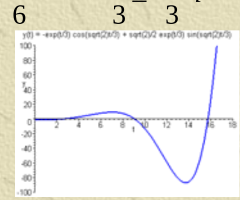


- Example 3: $y'' - 2y' + y = 0$

$$y(t) = e^{rt} \Rightarrow 3r^2 - 2r + 1 = 0 \Leftrightarrow r = \frac{2 \pm \sqrt{4-12}}{6} = \frac{1}{3} \pm \frac{\sqrt{2}}{3} i$$

$$y(t) = e^{t/3}\left(c_1 \cos\left(\sqrt{2}t/3\right) + c_2 \sin\left(\sqrt{2}t/3\right)\right)$$

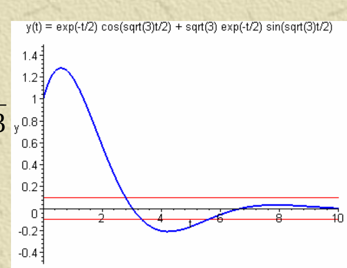




- Cnt.ed fr Example 1: For the initial values $y(0)=1$, $y'(0)=1$, find (a) the solution $u(t)$ and (b) the smallest time T for which $|u(t)| \le 0.1$

$$u(t) = c_1 e^{-t/2} \cos\left(\sqrt{3}t/2\right) + c_2 e^{-t/2} \sin\left(\sqrt{3}t/2\right)$$

$$\left.\begin{aligned} c_1 \qquad\qquad &= 1 \\ -\frac{1}{2}c_1 + \frac{\sqrt{3}}{2}c_2 &= 1 \end{aligned}\right\} \Rightarrow c_1 = 1,\ c_2 = \frac{3}{\sqrt{3}} = \sqrt{3}$$



- Find the smallest time T for which $|u(t)| \le 0.1$
- graphing calculator or computer algebra system, we find that $T \cong 2.79$.

Ch 3.5: Repeated Roots; Reduction of Order

- Recall our 2nd order linear homogeneous ODE

$$ay'' + by' + cy = 0$$

- where a, b and c are constants.
- Assuming an exponential soln leads to characteristic equation:

$$y(t) = e^{rt} \quad \Rightarrow \quad ar^2 + br + c = 0$$

- Quadratic formula (or factoring) yields two solutions, r_1 & r_2:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- When $b^2 - 4ac = 0$, $r_1 = r_2 = -b/2a$, since method only gives one solution:

$$y_1(t) = ce^{-bt/2a}$$

Exam Question
Second Solution: Multiplying Factor $v(t)$

- We know that

$$y_1(t) \text{ a solution} \Rightarrow y_2(t) = cy_1(t) \text{ a solution}$$

- Since y_1 and y_2 are linearly dependent, we generalize this approach and multiply by a function v, and determine conditions for which y_2 is a solution:

- **Then ****** $\quad y_1(t) = e^{-bt/2a}$ a solution $\Rightarrow$ try $y_2(t) = v(t)e^{-bt/2a}$

$$y_2(t) = v(t)e^{-bt/2a}$$

$$y_2'(t) = v'(t)e^{-bt/2a} - \frac{b}{2a}v(t)e^{-bt/2a}$$

$$y_2''(t) = v''(t)e^{-bt/2a} - \frac{b}{2a}v'(t)e^{-bt/2a} - \frac{b}{2a}v'(t)e^{-bt/2a} + \frac{b^2}{4a^2}v(t)e^{-bt/2a}$$

Finding Multiplying Factor $v(t)$

- Substituting derivatives into ODE, seek a formula for v: $b^2-4ac=0$

$$e^{-bt/2a}\left\{a\left[v''(t)-\frac{b}{a}v'(t)+\frac{b^2}{4a^2}v(t)\right]+b\left[v'(t)-\frac{b}{2a}v(t)\right]+cv(t)\right\}=0$$

$$av''(t)-bv'(t)+\frac{b^2}{4a}v(t)+bv'(t)-\frac{b^2}{2a}v(t)+cv(t)=0$$

$$av''(t)+\left(\frac{b^2}{4a}-\frac{b^2}{2a}+c\right)v(t)=0$$

$$av''(t)+\left(\frac{b^2}{4a}-\frac{2b^2}{4a}+\frac{4ac}{4a}\right)v(t)=0 \quad\Leftrightarrow\quad av''(t)+\left(\frac{-b^2}{4a}+\frac{4ac}{4a}\right)v(t)=0$$

$$av''(t)-\left(\frac{b^2-4ac}{4a}\right)v(t)=0$$

$$v''(t)=0 \quad\Rightarrow v'(t)=k \quad\Rightarrow\quad v(t)=k_3t+k_4$$

- The General Solution

$$\begin{aligned}y(t)&=k_1e^{-bt/2a}+k_2v(t)e^{-bt/2a}\\&=k_1e^{-bt/2a}+\left(k_3t+k_4\right)e^{-bt/2a}\\&=c_1e^{-bt/2a}+c_2te^{-bt/2a}\end{aligned}$$

$$y(t)=c_1e^{-bt/2a}+c_2te^{-bt/2a}$$

- Thus every solution is a linear combination of

$$y_1(t)=e^{-bt/2a},\; y_2(t)=te^{-bt/2a}$$

- Wrnskian

$$W(y_1,y_2)(t)=\begin{vmatrix}e^{-bt/2a} & te^{-bt/2a}\\ -\frac{b}{2a}e^{-bt/2a} & \left(1-\frac{bt}{2a}\right)e^{-bt/2a}\end{vmatrix}=e^{-bt/a}\left(1-\frac{bt}{2a}\right)+e^{-bt/a}\left(\frac{bt}{2a}\right)$$

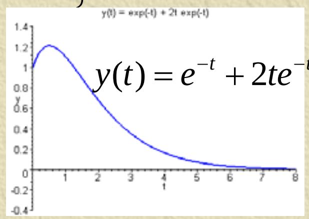
$$=e^{-bt/a}\neq 0 \quad \text{for all } t$$

- Thus y_1 and y_2 form a fundamental solution set for equation

Example 1: $y'' + 2y' + y = 0$, $y(0)=1$, $y'(0) = 1$,

$$y(t) = e^{rt} \Rightarrow r^2 + 2r + 1 = 0 \Leftrightarrow (r+1)^2 = 0 \Leftrightarrow r = -1$$

$$y(t) = c_1 e^{-t} + c_2 t e^{-t} \qquad \left.\begin{array}{rcl} c_1 & = & 1 \\ -c_1 + c_2 & = & 1 \end{array}\right\} \Rightarrow c_1 = 1,\ c_2 = 2$$



$$y(t) = e^{-t} + 2te^{-t}$$

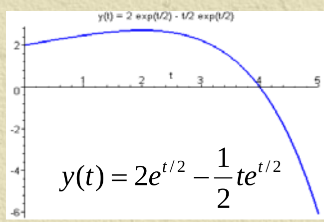
Example 2: $y'' - 2y' + 0.25y = 0$, $y(0)=2$, $y'(0) = 1/2$,

$$y(t) = e^{rt} \Rightarrow r^2 - r + 0.25 = 0$$

$$\Leftrightarrow (r - 1/2)^2 = 0 \Leftrightarrow r = 1/2$$

$$y(t) = c_1 e^{t/2} + c_2 t e^{t/2}$$

$$\left.\begin{array}{rcl} c_1 & = & 2 \\ \frac{1}{2}c_1 + c_2 & = & \frac{1}{2} \end{array}\right\} \Rightarrow c_1 = 2,\ c_2 = -\frac{1}{2}$$



$$y(t) = 2e^{t/2} - \frac{1}{2} t e^{t/2}$$

Reduction of Order also works for equations with nonconstant coefficients

$$y'' + p(t)y' + q(t)y = 0$$

- That is, given that y_1 is solution, and $y_2 = v(t)y_1$:

$$\begin{aligned} y_2(t) &= v(t)y_1(t) \\ y_2'(t) &= v'(t)y_1(t) + v(t)y_1'(t) \\ y_2''(t) &= v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t) \end{aligned}$$

$$y_1 v'' + (2y_1' + py_1)v' + (y_1'' + py_1' + qy_1)v = 0$$

- this last equation reduces to a first order equation in v': $y_1 v'' + (2y_1' + py_1)v' = 0$

Example of Reduction of Order Similar Exam questions

- Given the variable coefficient equation and solution y_1, use reduction of order method to find a second solution:

$$t^2 y'' + 3ty' + y = 0, \quad t > 0; \quad y_1(t) = t^{-1},$$

- Substituting these into ODE and collecting terms,

$$y_2(t) = v(t)\, t^{-1}$$
$$y_2'(t) = v'(t)\, t^{-1} - v(t)\, t^{-2}$$
$$y_2''(t) = v''(t)\, t^{-1} - 2v'(t)\, t^{-2} + 2v(t)\, t^{-3}$$

$$t^2\left(v''t^{-1} - 2v't^{-2} + 2vt^{-3}\right) + 3t\left(v't^{-1} - vt^{-2}\right) + vt^{-1} = 0$$
$$\Leftrightarrow v''t - 2v' + 2vt^{-1} + 3v' - 3vt^{-1} + vt^{-1} = 0$$
$$\Leftrightarrow tv'' + v' = 0$$
$$\Leftrightarrow tu' + u = 0, \text{ where } u(t) = v'(t)$$

$$t\frac{du}{dt} + u = 0 \quad \Leftrightarrow \ln|u| = -\ln|t| + C$$
$$\Leftrightarrow \quad |u| = |t|^{-1} e^C \quad \Leftrightarrow \quad u = ct^{-1}, \quad \text{since } t > 0.$$

$$v' = \frac{c}{t} \qquad v(t) = c\ln t + k$$

$$y_2(t) = (c\ln t + k)t^{-1} = ct^{-1}\ln t + k\, t^{-1} \qquad y_2(t) = t^{-1}\ln t.$$

$$y(t) = c_1 t^{-1} + c_2 t^{-1}\ln t$$

Ch 3.6: Nonhomogeneous Equations

- Recall the nonhomogeneous equation p, q, g are continuous functions on an open interval I.

$$y'' + p(t)y' + q(t)y = g(t)$$

- Theorem 3.6.1 (Exam question very potential)
- If Y_1, Y_2 are solutions of nonhomogeneous equation then Y_1 - Y_2 is a solution of the homogeneous equation
- If y_1, y_2 form a fundamental solution set of homogeneous equation, then there exists constants c_1, c_2 such that

$$L(y1) - L(y2) = L(y1 - y2) = g(t) - g(t) = 0 = c_1 y_1(t) + c_2 y_2(t)$$

- The general solution of nonhomogeneous equation, where Y is a specific solution to the nonhomogeneous equation.
$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

Method of Undetermined Coefficients: g(t)

- g(t) is exp $$y'' - 3y' - 4y = 3e^{2t}$$
- Since **exponentials replicate** through differentiation, a good guess for Y is: $$Y(t) = Ae^{2t} \Rightarrow Y'(t) = 2Ae^{2t},\ Y''(t) = 4Ae^{2t}$$
- g(t)=sine $$y'' - 3y' - 4y = 2\sin t$$

$$Y(t) = A\sin t \Rightarrow Y'(t) = A\cos t,\ Y''(t) = -A\sin t$$

$$-A\sin t - 3A\cos t - 4A\sin t = 2\sin t$$

$$\Leftrightarrow\ (2+5A)\sin t + 3A\cos t = 0$$

$$\Leftrightarrow\ c_1 \sin t + c_2 \cos t = 0$$

- Since sin(x) and cos(x) are linearly independent (they are not multiples of each other), we must have $c_1 = c_2 = 0$, **and hence 2 + 5*A* = 3*A* = 0, which is impossible** Our next attempt at finding a Y is

$$Y(t) = A\sin t + B\cos t$$

$$\Rightarrow Y'(t) = A\cos t - B\sin t,\ Y''(t) = -A\sin t - B\cos t$$

- Polynomial $g(t)$: $$y'' - 3y' - 4y = 4t^2 - 1$$

$$Y(t) = At^2 + Bt + C \Rightarrow Y'(t) = 2At + B,\ Y''(t) = 2A$$

$$Y(t) = -t^2 + \frac{3}{2}t - \frac{11}{8}$$

- Product $g(t)$

$$y'' - 3y' - 4y = -8e^t \cos 2t$$

$$Y(t) = Ae^t \cos 2t + Be^t \sin 2t$$

$$Y'(t) = Ae^t \cos 2t - 2Ae^t \sin 2t + Be^t \sin 2t + 2Be^t \cos 2t$$

$$= (A + 2B)e^t \cos 2t + (-2A + B)e^t \sin 2t$$

$$Y''(t) = (A+2B)e^t \cos 2t - 2(A+2B)e^t \sin 2t + (-2A+B)e^t \sin 2t$$

$$+ 2(-2A+B)e^t \cos 2t$$

$$= (-3A + 4B)e^t \cos 2t + (-4A - 3B)e^t \sin 2t$$

Sum $g(t)$ is sum of functions

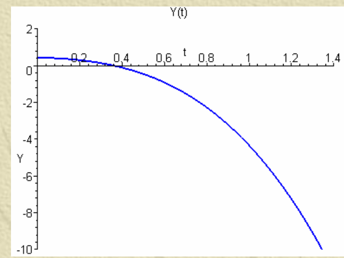
$$g(t) = g_1(t) + g_2(t)$$

If Y_1, Y_2 are solutions of

$$y'' + p(t)y' + q(t)y = g_1(t)$$
$$y'' + p(t)y' + q(t)y = g_2(t)$$

respectively, then $Y_1 + Y_2$ is a solution of the nonhomogeneous equation above.

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t \cos 2t$$
$$y'' - 3y' - 4y = 3e^{2t}$$
$$y'' - 3y' - 4y = 2\sin t$$
$$y'' - 3y' - 4y = -8e^t \cos 2t$$



$$Y(t) = -\frac{1}{2}e^{2t} + \frac{3}{17}\cos t - \frac{5}{17}\sin t + \frac{10}{13}e^t \cos 2t + \frac{2}{13}e^t \sin 2t$$

Pay Attension

$$y'' + 4y = 3\cos 2t$$

$$Y(t) = A\sin 2t + B\cos 2t$$
$$\Rightarrow Y'(t) = 2A\cos 2t - 2B\sin 2t,\ Y''(t) = -4A\sin 2t - 4B\cos 2t$$

- **Failure:** Substituting these derivatives into ODE:

$$(-4A\sin 2t - 4B\cos 2t) + 4(A\sin 2t + B\cos 2t) = 3\cos 2t$$
$$(-4A + 4A)\sin 2t + (-4B + 4B)\cos 2t = 3\cos 2t$$
$$0 = 3\cos 2t$$

- Thus no particular solution exists of the form $Y(t) = A\sin 2t + B\cos 2t$

$$Y(t) = At\sin 2t + Bt\cos 2t$$
$$Y'(t) = A\sin 2t + 2At\cos 2t + B\cos 2t - 2Bt\sin 2t$$
$$Y''(t) = 4A\cos 2t - 4B\sin 2t - 4At\sin 2t - 4Bt\cos 2t$$

$$4A\cos 2t - 4B\sin 2t = 3\cos 2t \Rightarrow A = 3/4,\ B = 0$$

$$\Rightarrow Y(t) = \frac{3}{4}t\sin 2t$$