DE-2013

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Ch 1.1:
Basic Mathematical Models; Direction Fields

- **Differential equations** are continuous and deterministic equations of derivatives expressing trend (rate, slope) of variation!s!.. in terms of tangents (direction fields), instantaneous values of variations at the point of interest (at an interval). derivatives.
- Physical phenomena involving rates of variations:
  - Motion of fluids, heat dissipation, wave equations, mcehanical, electrical, chemical, biological magnetic, or even state dynamics, Weathercast reports are the best examples of direction fields.

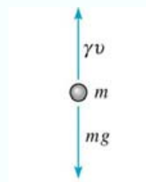
Example 1: Free Fall

- Free fall of a mass, m with initial v0=0, with no additional force introduced on m. Newton's 2nd Law:
- v=v0+a t. x= a.t^2/2.
- The force gravity exerts on mass m, is *F=ma = mg*, with no friction, therefore a=g. Substitute a into the equation.
- Formulating with a differential equation describing the same motion.
- Variables: time *t*, velocity *v*
- *F = ma = m*(dv/d*t*) ←net force
- Force exerted on the mass due to the gravity: *F = mg* ←downward force
- Force of air resistance (friction): $F = \gamma v$ ←upward force
- Then first equation
- Taking $g = 9.8$ m/sec^2, $m = 10$ kg, $\gamma = 2$ kg/sec.
- Acceleration is slowing down to zero with the increase of velocity.
- Units of the friction ????.

$$m\frac{dv}{dt} = mg - \gamma v$$

$$v' = a = 9.8 - 0.2v$$

$$v' + (\gamma / m)v = g / m$$



Example: Direction Field.

Plotting estimates of slopes wo solving the equations.

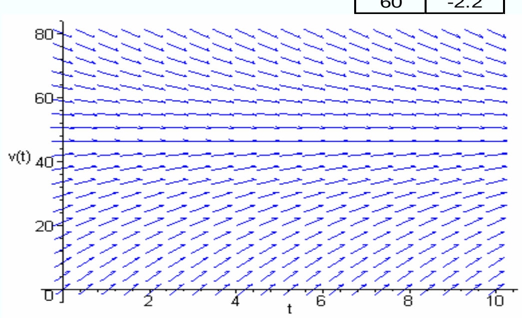
- Arrows give tangent lines to solution curves, and indicate where and how much increasing & decreasing.
  - Slope ~ Δv/Δt
  - v' is independent of *t*. Slope is the same along the axis t.
  - Horizontal solution curves: **equilibrium solutions**. Set $v' = 0$ and solve v.
  - Increment and decrement v..
- In matlab
  - syms v %in matlab
  - v=dsolve('Dv=9.8-.2*v', 'x')
  - v=49+exp(-1/5*x)*C1
  - C1 is arbitrary coefficients, (dfield7, Below is the Maple solution.) Thursday.

$$v' = (-\gamma / m)v + g / m$$

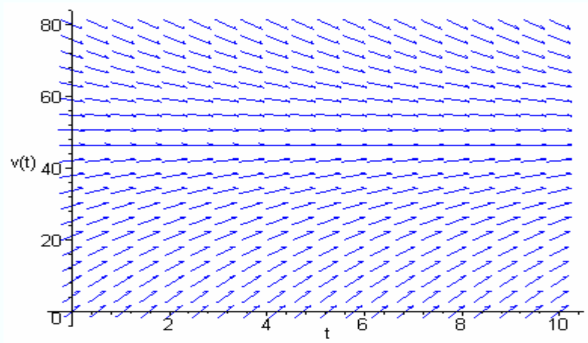
$$v' = 9.8 - 0.2v$$

$$v' = 0 \Leftrightarrow v = \frac{9.8}{0.2} = 49$$

v	v'
0	9.8
5	8.8
10	7.8
15	6.8
20	5.8
25	4.8
30	3.8
35	2.8
40	1.8
45	0.8
50	-0.2
55	-1.2
60	-2.2



- Is speed going to increase incessantly?
- What are the interplaying actors? a, v, t, x.



Example 2: Mice and Owls (1 of 2)

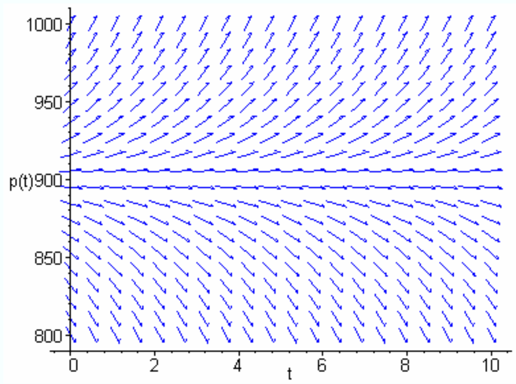
- Consider a mouse population that reproduces at a rate proportional to the current population, with a rate constant equal to 0.5 mice/month (assuming no owls present).
- When owls are present, they eat the mice. Suppose that the owls eat 15 per day (average). Write a differential equation describing mouse population in the presence of owls. (Assume that there are 30 days in a month.)
- Solution:

$$\frac{dp}{dt} = 0.5p - 450$$

Question: Direction Field

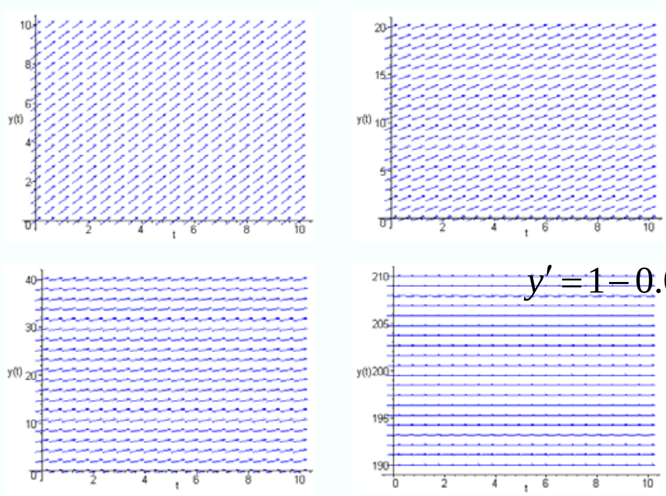
- Discuss solution curve behavior, and find equilibrium soln.

$$p' = 0.5p - 450$$



Question on Direction Field

- Discuss solution curve behavior, and find equilibrium soln.



$$y' = 1 - 0.005y$$

Exam question

- Discuss solution behavior and dependence on the initial value $y(0)$ for the nonlinear DE given below, using the corresponding direction field.

$$y' = (y-1)*(y+2)$$

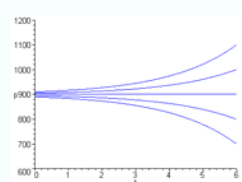
Ch 1.2: Solutions of some DEs Using Calculus

- The general form of DEs of free fall or owl/mice $y' = a\,y - b.$
- The method to solve numerically, as follows, k is a constant.

$$\frac{dp}{dt} = 0.5(p-900) \;\Rightarrow\; \frac{dp/dt}{p-900} = 0.5 \;\Rightarrow\; \int \frac{dp}{p-900} = \int 0.5dt$$

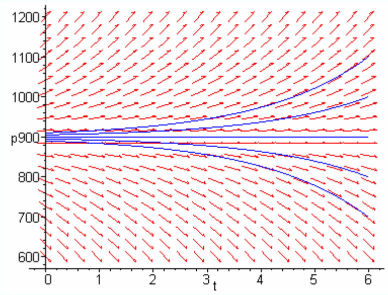
$$\Rightarrow\; \ln|p-900| = 0.5t + C \;\Rightarrow\; |p-900| = e^{0.5t+C}$$

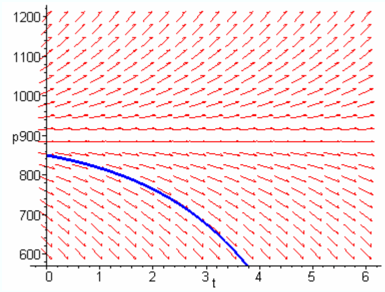
$$\Rightarrow\; p-900 = \pm e^{0.5t}e^{C} \;\Rightarrow\; p = 900 + ke^{0.5t},\; k = \pm e^{C}$$



$$p' = 0.5p - 450 \quad \Rightarrow \quad p = 900 + ke^{0.5t},$$

- Thus we have infinitely many solutions to our equation, since k is an arbitrary constant.
- Graphs of solutions (**integral curves**) for several values of k.
- The equilibrium solution is at $k = 0$, and fr any $k \neq 0$, function diverges from equilibrium towards explosion or extinction.
- **IVP:** For $p(0) = 850$, $k = -50$..





IVP

$$\frac{dy}{dt} = a\left(y - \frac{b}{a}\right) \quad \Rightarrow \quad \frac{dy/dt}{y - b/a} = a \quad \Rightarrow \quad \int \frac{dy}{y - b/a} = \int a\,dt$$

$$\Rightarrow \quad \ln|y - b/a| = a\,t + C \quad \Rightarrow \quad |y - b/a| = e^{at+C}$$

$$\Rightarrow \quad y - b/a = \pm e^{at} e^{C} \quad \Rightarrow \quad y = b/a + ke^{at}, \;\; k = \pm e^{C}$$

$$y = \frac{b}{a} + ke^{at}$$

$$y(0) = y_0 = \frac{b}{a} + ke^{0} \quad \Rightarrow \quad k = y_0 - \frac{b}{a}$$

$$y = \frac{b}{a} + \left[y_0 - \frac{b}{a}\right]e^{at}$$

Summary.

- Recall: To find Equilibrium Solution, set $y' = 0$ & solve for y:

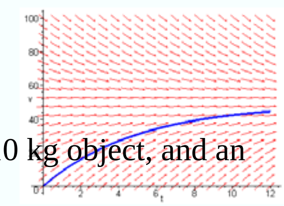
$$y' = ay - b \overset{set}{=} 0 \quad \Rightarrow \quad y(t) = \frac{b}{a}$$

- The solution to IVP is:

$$y = \frac{b}{a} + \left[y_0 - \frac{b}{a} \right] e^{at}$$

- And the behavior:
  - If $y_0 = b/a$, then y is constant, with $y(t) = b/a$, *equilibrium solution.*
  - If any $y_0 \neq b/a$ and $a = 0$, Eqlrm sln shifts with the difference.
  - ….. Draw your conclusions.

Free Fall Equation



- Recall equation of free fall descent of 10 kg object, and an air resistance coefficient $\gamma = 2$ kg/sec:

$$dv/dt = 9.8 - 0.2v$$

- Suppose object is dropped from a distance $s(0)$ 300 m. above ground.
  - (a) Find velocity at any time t.
  - (b) How long it takes until it hits ground and how fast will it be moving then?
- For part (a), we need to solve the initial value problem

$$v' = 9.8 - 0.2v, \quad v(0) = 0$$

- Using result from previous slide, solution (a)

$$y = \frac{b}{a} + \left[y_0 - \frac{b}{a} \right] e^{at} \Rightarrow v = \frac{9.8}{0.2} + \left[0 - \frac{9.8}{0.2} \right] e^{-.2t} \Rightarrow v = 49\left(1 - e^{-.2t}\right)$$

Ch 1.3: Classification of DEs

- Ordinary Differential Equations: The unknown function depends on a single independent variable. The equations discussed are ODEs.
- Partial Differential Equations: The unknown function depends on more the one independent variables.
- Systems of Differential Equations: Classification depending on the number of unknown functions that are involved in the equation. If there is a single unknown function, then one equation is, if there are more, then a system of equations is required. Predator-prey equations have the form where $u(t)$ and $v(t)$ are the respective populations of prey and predator species. The constants a, c, α, γ depend on the particular species being studied.
- Stochastic of Differential Equations: If the driving force is random.

$$\frac{dv}{dt} = 9.8 - 0.2v, \quad \frac{dp}{dt} = 0.5p - 450$$

$$L\frac{d^2Q(t)}{dt^2} + R\frac{dQ(t)}{dt} + \frac{1}{C}Q(t) = E(t)$$

$$\alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t} \quad \text{(heat eqn)}$$

$$a^2 \frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial^2 u(x,t)}{\partial t^2} \quad \text{(wave eqn)}$$

$$du/dt = a\,u - \alpha\,uv$$

$$dv/dt = -cv + \gamma\,uv$$

Order of Differential Equations

- The order of a differential equation is the order of the highest derivative that appears in the equation.
- Examples:

$$y' + 3y = 0$$

$$y'' + 3y' - 2t = 0$$

$$\frac{d^4y}{dt^4} - \frac{d^2y}{dt^2} + 1 = e^{2t}$$

$$u_{xx} + u_{yy} = \sin t$$

- We will be studying differential equations for which the highest derivative can be isolated:

$$y^{(n)}(t) = f\left(t, y, y', y'', y''', \ldots, y^{(n-1)}\right)$$

Linear & Nonlinear Differential Equations

- An ODE $F\left(t, y, y', y'', y''', \ldots, y^{(n)}\right) = 0$ is **linear** if *F* is linear in the variables y, y', y'', y''', …, $y^{(n)}$.

- The general linear ODE has the form

$$a_0(t)y^{(n)} + a_1(t)y^{(n-1)} + \cdots + a_n(t)y = g(t)$$

- Determine whether the equations below are linear or nonlinear. And linearization?

(1) $y' + 3y = 0$ (2) $y'' + 3e^y y' - 2t = 0$ (3) $y'' + 3y' - 2t^2 = 0$

(4) $\dfrac{d^4y}{dt^4} - t\dfrac{d^2y}{dt^2} + 1 = t^2$ (5) $u_{xx} + uu_{yy} = \sin t$ (6) $u_{xx} + \sin(u)u_{yy} = \cos t$

$$\frac{d^2\Theta}{dt^2} + \frac{g}{L}\sin\Theta = 0 \qquad\qquad \frac{d^2\Theta}{dt^2} + \frac{g}{L}\Theta = 0$$

- Note that linear DEs accommodate no products of the function *y(t)*, and its derivatives

  and neither the function or its derivatives occur to any power other than the first power.
- Solutions to Differential Equations:
- Implicit and explicit solutions, are the solutions in the form of y, and y(t), respectively. For example, $yy' = t$, $y(2)=1$, $y^2=t^2-3$ is implicit while $y(t)=-+\text{sqrt}(t^2-3)$ is explicit, check IV to determine which one is correct.
- A solution $\phi(t)$ to an ordinary differential equation

$$y^{(n)}(t) = f\left(t, y, y', y'', \ldots, y^{(n-1)}\right)$$

- satisfies the equation:

$$\phi^{(n)}(t) = f\left(t, \phi, \phi', \phi'', \ldots, \phi^{(n-1)}\right)$$

- Example: Verify the following solutions of the ODE

$$y'' + y = 0; \;\; y_1(t) = \sin t, \;\; y_2(t) = -\cos t, \;\; y_3(t) = 2\sin t$$

- $yy' = -x$, $y(2)=1$, $y^2+x^2=1$ is implicit while $y(t)dy/dt=-x(t)$ $y(t) = \sin(t)$, $x(t)=\cos(t)$, is explicit.

Solutions to Differential Equations

- Three important questions in the study of differential equations:
  - Is there a solution? (Existence)
  - If there is a solution, is it unique? (Uniqueness)
  - If there is a solution, how do we find it?
    (Analytical Solution, Numerical Approximation, etc)