Midterm Exam Math 255, by Dr. M. Sakalli, CSE, Marmara Univ. Nov. 09, 2010

Differential Equations. Duration: 2hrs. Wish you all Good luck.

Note 2: All the questions appearing in this page were completely or partially presented (taught) in the class.

1. y''-y=0, ➔ r^2=1, y'=-+1, exp(-t), exp(+t), and their linear summations from the linearity condition. C1y1 and c2y2..

2. Explain (with your reasoning) the order, the degree and the linearity of the equations you are given below.
   a. $L(u) = u_{xx} + u_t$
   b. $L(z) = z^{(3)} - z + k^3 z$
   c. $L(y) = y' - y + y^2$

a) Linear, 2, 1, b) Linear 3, 1, c) nonlinear, 1, 1, but easily solvable.

3. Remember the linearity condition, $L(\alpha u+\beta v) = \alpha L(u) + \beta L(v)$, α and β are constants. Prove your statements above for a linear and a nonlinear one you choose.

a) $L(u+v) = (\alpha u+\beta v)_{xx}+(\alpha u+\beta v)_t = \alpha u_{xx} + \beta v_{xx} + \alpha u_t + \beta v_t$
$= \alpha L(u) + \beta L(v)$ Linearity satisfied.

b) $L(\alpha u+\beta v) = (\alpha u+\beta v)^{(3)} + (k^3 - 1)(\alpha u+\beta v)$
$= \alpha u^{(3)} + \beta v^{(3)} + \alpha(k^3 - 1)u+\beta(k^3 - 1)v$
$= \alpha u^{(3)} + \alpha(k^3 - 1)u + \beta v^{(3)} + \beta(k^3 - 1)v = \alpha L(u) + \beta L(v)$
Linearity satisfied.

c) $L(\alpha u+\beta v) = (\alpha u+\beta v)' - (\alpha u+\beta v) + (\alpha u+\beta v)^2$
$= \alpha u'+\beta v' - \alpha u+\beta v + (\alpha^2 u^2+2\alpha\beta uv+\beta^2 v^2)$ not equal to $\alpha L(u)+ \beta L(v)$
$L(\alpha u) = \alpha u' - \alpha u + \alpha^2 u^2$
$L(\beta u) = \beta u' - \beta u + \beta^2 u^2$
The relation of linearity cannot be satisfied here due to the $2\alpha\beta uv$.

4. $dy/dt - r y = - y^2 / k$, for constants $r>0$, $k>0$ and $y(0) = y_0$;
   a. Find analytical solution and check if the solutions satisfy the relation.
   This is Bernoulli, two ways to solve,
   substitute $z = y^{-1}$ ➔ $z' = - y' y^{-2}$, ➔ $z'z^{-2} = - y'$
   $\mathbf{-z'z^{-2} - r z^{-1} = - z^{-2} / k}$ ➔ $\mathbf{z' + r z = 1 / k}$ ➔ $\mathbf{z' = -(r)(z - 1/rk)}$,
   $\ln|z - 1/rk| = -rt + c$,

$|z - 1/rk| = C\exp(-rt) \Rightarrow z = 1/rk + C\exp(-rt) \Rightarrow y = kr/(1 + krC\exp(-rt))$
$|z - 1/rk| = C\exp(-rt) \Rightarrow z = 1/rk - C\exp(-rt) \Rightarrow y = kr/(1 - krC\exp(-rt))$

The second method without substitution.. reducible to separable form..
step1) $dy/dt - r\,y = -y^2/k$, $dy/dt = y(r - y/k) = r\,y(1 - y/kr)$
$= r\,y(1 - y/K)$, substitute $K = kr$,

Step2) $dy[(A/y + B/(1 - y/K)] = rdt$,
$(A/y + B/(1 - y/K) = 1$
$Ay/K + By = 1$, $A = 1$, $-A/K + B = 0$, $B = 1/K$,
Step3)
$dy[1/y + (1/K)/(1 - y/K)] = rdt$
$\ln|y| - \ln|1 - y/K| = rt + c$
$|Ky/(K - y)| = C\exp(rt)$, substitute $C = C\exp(rt)$,
$Ky = CK - Cy$, $y(K + C) = CK$, $y = CK/(K + C)$, $y = K/(K/C + 1)$,
Substitute back C, and K
$y = K/(1 + (K\exp(-rt))/C)$, $y = kr/(1 + (kr\exp(-rt))/C)$,
$y = r/\{1/k + (r\exp(-rt))/C\}$,
$y = r/\{1/k + rC\exp(-rt)\}$
almost gives the same equation found by solving Bernoulli which was
$y = r/(1/k + rC\exp(-rt))$

b. Plot the direction fields (y versus t) (to justify your analytical solutions) and there you must clearly indicate the equilibrium solutions and mark which ones are stable (converging to a solution point) and (diverging from a point).

Equilibrium points are $y=0$ and $y=rk$ where $y'=0$,
For the values $y>rk$, trend is negative slope, the negativity increases as the y value increases since $r\,y - y^2/k$ grows in negative direction. For the values $0<y<rk$, the y' trend is positive, converging to the point rk, so rk is stable point of convergence, for any $y>0$, but the point $y=0$ is a divergence point, since trend is diverging from this point. $y<0$, y' diverging to $-\inf$.

c. Plot y' vs y of the same equation (very simple), and there indicate the equilibrium points, and mark convergence and diverging directions to the points.

d. Use the same plot to determine the maximum sustainable harvesting (f(y), y) point, (which is defined as the point where the amount consumed is equal to the amount produced)? Derivate this eq, $H = r\,y - y^2 / k$, $d(H)/dy = 0$. Will give the peak point, which is half of the rk, $y_h = rk/2$, and there the maximum sustainable harvesting trend of y' arrives to he value $h = y' = r\,y - y^2 / k = r^2k/2 - r^2k/4 = r^2k/4$.

5. Find the solution of the DE and show the uniqueness of solution at t=0.
$t y' / y = -2 + 4t^2 / y$

$t y' + 2y = 4t^2, \Rightarrow \mu' = 2\mu/t$, $\ln(\mu) = 2\ln(t)$, $\mu = t^2$, $d(uv) = d(t^2 y) = 4t^3$
$d(t^2 y) = t^4 + c \Rightarrow y = t^2 + c / t^2$
Obvious that solution cannot be unique at t=0.