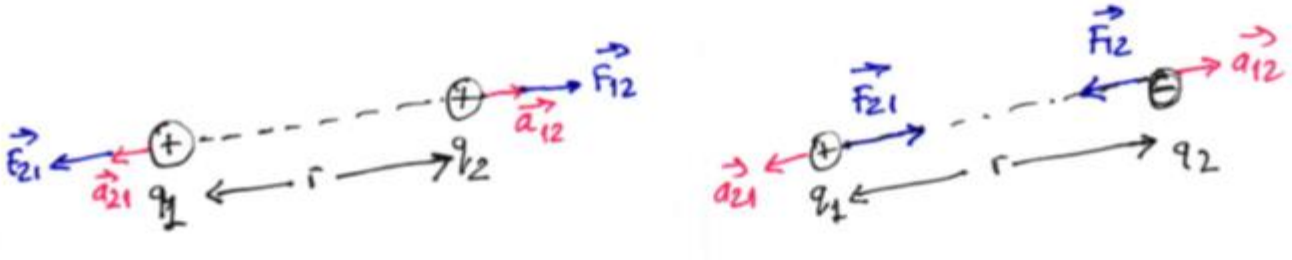
$$\vec{E} = \lim_{q \to 0} \frac{\vec{F}}{q} \quad [N/C,\ V/m]$$ Electric Field intensity ($\vec{E}$); the force per unit charge

Coulomb's Law

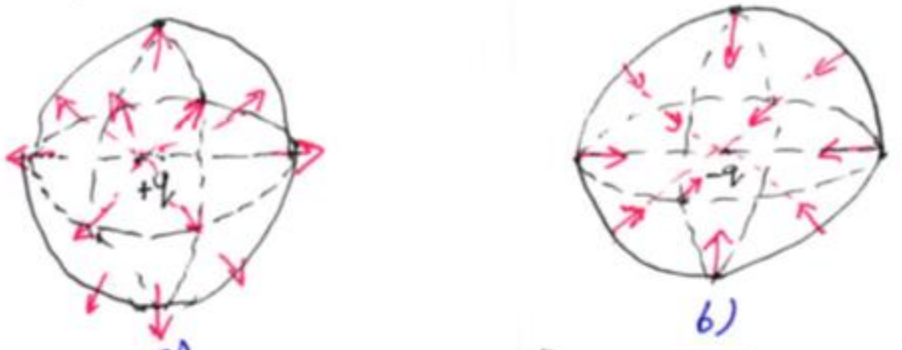


Force between opposite signed charges is attractive while like-signed charges repel each other.

$$\vec{F}_{12} = k \frac{q_1 q_2}{r^2} \vec{a}_{12} \; ; \; k \triangleq \text{Coulomb constant } \left(1/4\pi\varepsilon_0 \right)$$

Using definition of $\vec{E}$, one may formulate the electric field intensity due to a single charge;

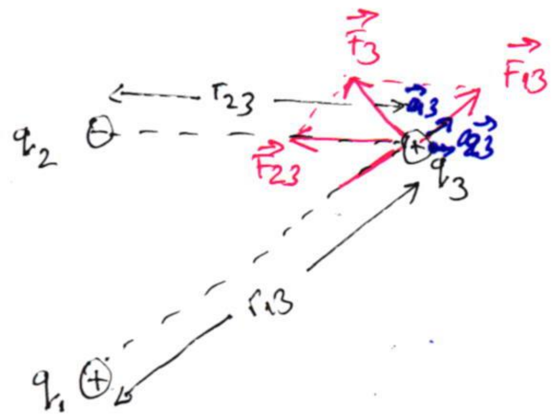
$$\vec{E} = \frac{q}{4\pi\varepsilon_0 r^2} \cdot \vec{a}_r \; ; \; \vec{a}_r \triangleq \text{unit vector in radial direction}$$



a) b)

An illustration of $\vec{E}$ vectors for a single charge
a) positive charge b) negative charge

- Principle of Superposition (more than two charges)



$$\vec{F}_3 = \vec{F}_{13} + \vec{F}_{23} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_3}{r_{13}^2} \cdot \vec{a}_{13} + \frac{q_1 q_2}{r_{23}^2} \vec{a}_{23} \right)$$

Similarly total $\vec{E}$ on q_3 is $\vec{E}_{13} + \vec{E}_{23}$

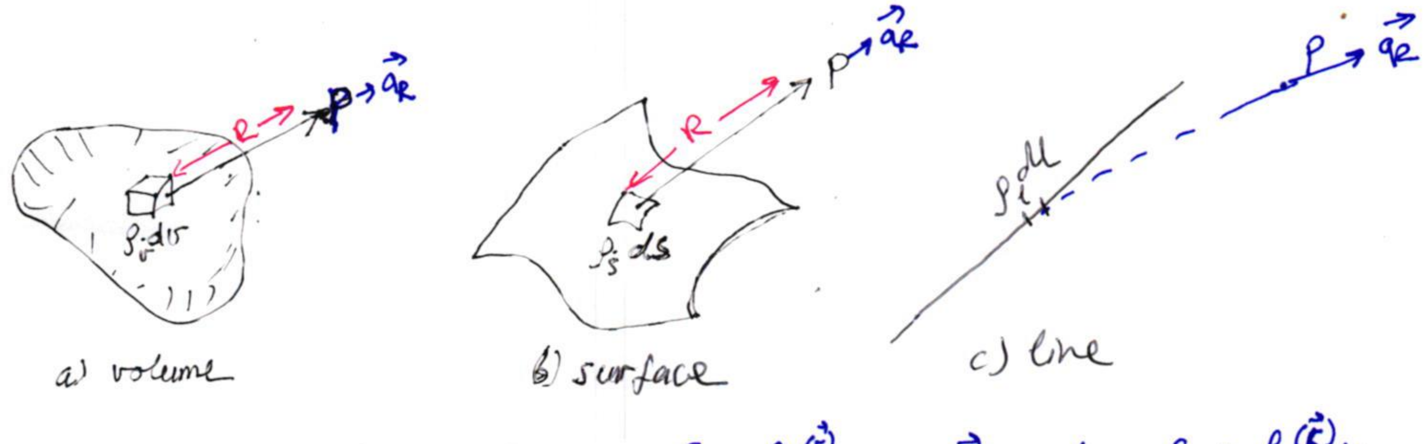
In the presence of N charges, one may write;

$$\vec{F}_j = \sum_{\substack{i=1 \\ i \neq j}}^{N} \vec{F}_{ij}$$

Similarly, the total $\vec{E}$ at a given point can be calculated via principle of superposition

$$\vec{E} = \sum_{i=1}^{N} \vec{E}_i = \sum_{i=1}^{N} \frac{1}{4\pi\varepsilon_0} \frac{q_i}{r_i^2} \cdot \vec{a}_i$$

- Continuous Distribution of charges

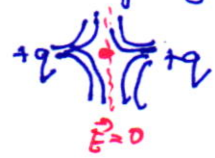


$$\vec{E}_P = \frac{1}{4\pi\varepsilon_0} \int_v \vec{a}_R \frac{\rho_v(\vec{r})}{R^2} dv \; ; \quad \vec{E}_P = \frac{1}{4\pi\varepsilon_0} \int_S \vec{a}_R \frac{\rho_s(\vec{r})}{R^2} ds \; ; \quad \vec{E}_P = \frac{1}{4\pi\varepsilon_0} \int_L \vec{a}_R \frac{\rho_L(\vec{r})}{R^2} dl$$

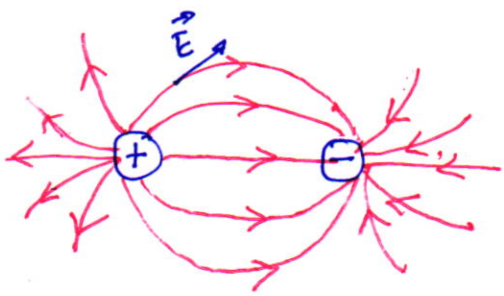
$$\rho_v(\vec{r}) = \lim_{\Delta v_i \to 0} \frac{\Delta q_i}{\Delta v_i} \; , \quad Q = \sum_{i=1} \Delta q_i = \int_v \rho_v(\vec{r}) dv$$

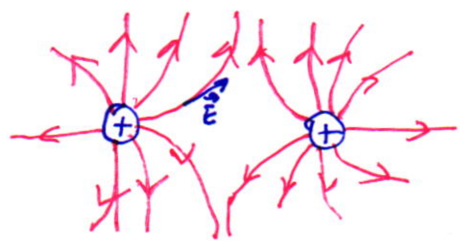
Electric Field Lines

- Field lines are radially outward and inward for + and − charges, respectively
- Very close to a charge, field lines are dominated by it. Hence, in this region lines are radial and spherically symmetric
- In the far field, pattern seems to be as of a single charge $Q = \sum_i^N Q_i$, unless total $Q = 0$. So, field lines become radial and spherically symmetric.
- There may exist null point in which $\vec{E} = 0$



- Direction of $\vec{E}$ is tangential to the field lines
- Field lines should never cross





a.k.a "dipole"

<u>A mathematical approach:</u> Let the differential length on field lines be $\vec{dl}$ which is;

$$\vec{dl} = \sum_{i=1}^{3} \underbrace{\vec{a_i}}_{\text{unit vector}} \cdot \underbrace{da_i}_{\text{differential increment}} \cdot h_i \rightarrow \text{metric coefficient}$$

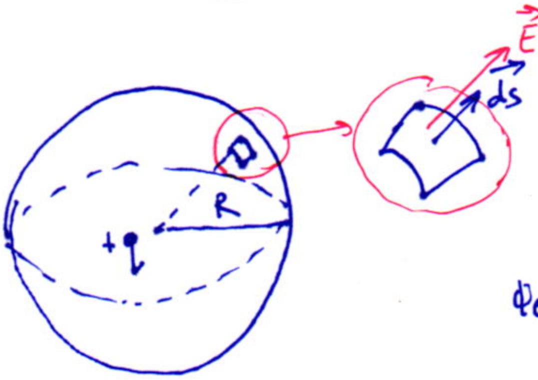
Since $\vec{E}$ and $\vec{dl}$ are parallel as stated above, $\vec{E} \times \vec{dl} = 0$. Therefore

$$\vec{E} \times \vec{dl} = \begin{vmatrix} \vec{a_1} & \vec{a_2} & \vec{a_3} \\ E_1 & E_2 & E_3 \\ h_1 \cdot da_1 & h_2 da_2 & h_3 da_3 \end{vmatrix} = 0 \Rightarrow \frac{h_1 \cdot da_1}{E_1} = \frac{h_2 \cdot da_2}{E_2} = \frac{h_3 \cdot da_3}{E_3}$$

In cartesian coordinates $\frac{dx}{E_x} = \frac{dy}{E_y} = \frac{dz}{E_z}$

In cylindrical coordinates $\frac{d\rho}{E_\rho} = \frac{\rho d\phi}{E_\phi} = \frac{dz}{E_z}$

GAUSS'S LAW



Net electric field flux through ds;

$$d\phi_e = \vec{E}.\vec{ds}$$

Summing over all sphere

$$\phi_e = \sum_i d\phi_e = \oint_S \vec{E}.\vec{ds} = |\vec{E}|.4\pi R^2 = \boxed{\frac{q}{\varepsilon_0}}$$

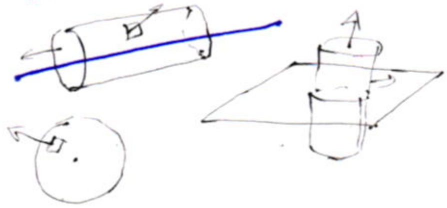
In the above configuration, a sphere is chosen. In fact, for an arbitrarily chosen shape, same result would be obtained. The net flux through any closed surface is proportional to the net charge enclosed is the Gauss's Law. Namely;

$$\phi_E = \oint_S \vec{E}\cdot\vec{ds} = \frac{\sum Q}{\varepsilon_0}$$ ⟶ Total Charges in the Gauss Surface

Gauss's Law provides a utility to determine electric field, but requires symmetry of the system. Some common applications are listed below.

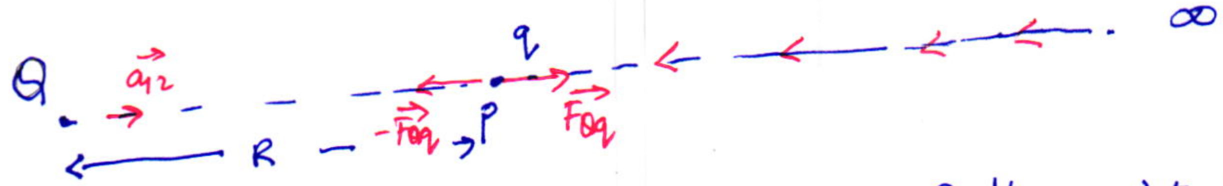
Symmetry	System	Gaussian Surface
Cylindrical	Infinite Rod	Coaxial Cylinder
Planar	Infinite Plane	Pillbox
Spherical	Sphere	Sphere

Illustration



Examples (will be explained in the classroom)

ELECTRIC POTENTIAL



Let one brings the test charge q from ∞ to Point P. At any point, the force acting on test charge q due to the Q was found as;

$$F_{Qq} = q \cdot \frac{Q}{4\pi\varepsilon_0 r^2} \vec{a}_{12}$$

So, to move test charge q, one should apply force in $-\vec{a}_{12}$ direction with magnitude F_{Qq}. Total work done by force will be;

$$W = \int_{\infty}^{P} F_{Qq} \cdot -\vec{a}_{12} \cdot \underbrace{\vec{dl}}_{+\vec{a}_{12}\,dr} = \int_{\infty}^{R} \frac{-Qq}{4\pi\varepsilon_0 r^2} \cdot dr = \frac{Qq}{4\pi\varepsilon_0} \int_{\infty}^{R} \frac{-dr}{r^2} = \frac{Qq}{4\pi\varepsilon_0 R} \quad *$$

work done ~~per~~ on unit charge (W/q) in an electric field is defined as electric potential

Here the electric potential of Q at point P is;

$$V_P = \frac{Q}{4\pi\varepsilon_0 R} \quad [J/C,\ V]$$ Note that it is scalar.

Using same concept, one may define potential difference between two points as;

$$V_2 - V_1 = -\int_{P_1}^{P_2} \vec{E} \cdot \vec{dl}$$

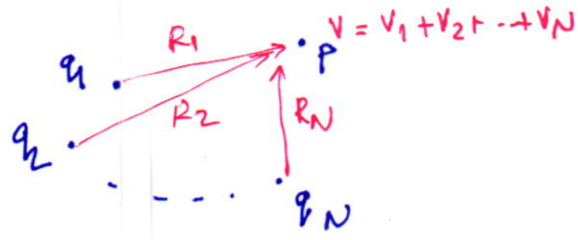
Also, it is consistent with previous postulates. That is, using $\vec{E} = -\nabla V$ and gradient theorem, one may say;

$$V_2 - V_1 = +\int_{P_1}^{P_2} \nabla V \cdot \vec{dl} = V(P_2) - V(P_1) = V_2 - V_1$$

* Note that, unless opposite is specified $V(\infty) \to 0$

The electric potential of a system, consisting N discrete charges can be determined using superposition principle,

$$V_P = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q}{R_i}$$



ELECTRIC POTENTIAL (continued)

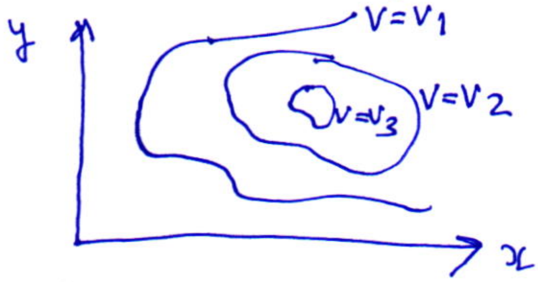
The potential of a continuous charge system can be determined in terms of integrating the relation, that is;

$$V = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho}{R}\, d\xi \; ; \text{ where } d\xi \text{ is the differential for volume, surface or line}$$

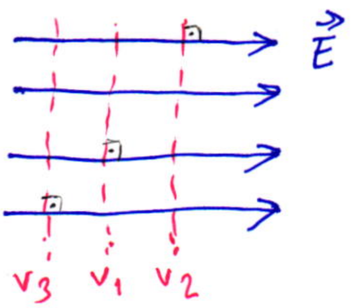
Since V is a scalar, it is more suitable to obtain $\vec{E}$ using $\vec{E} = -\nabla V$

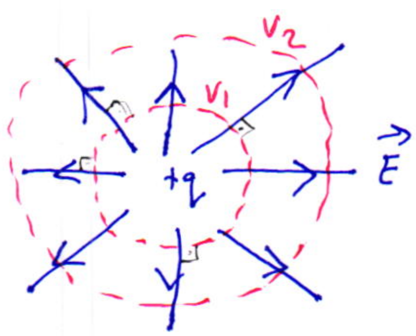
Equipotentials

The curves drawn by constant $V(r)$ are called equipotential curves.

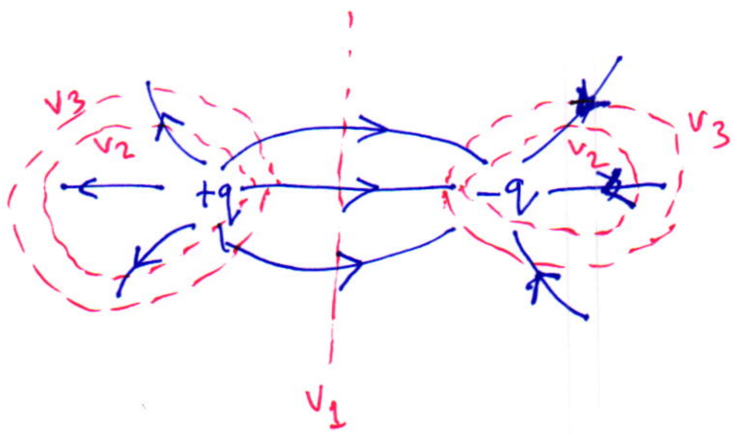


In 3-D case, we obtain equipotential surfaces for which $V(r)$ is constant. Using the relation $\vec{E} = -\nabla V$, one may say $\vec{E}$ is perpendicular to V curves.





V_1, V_2, V_3 constant

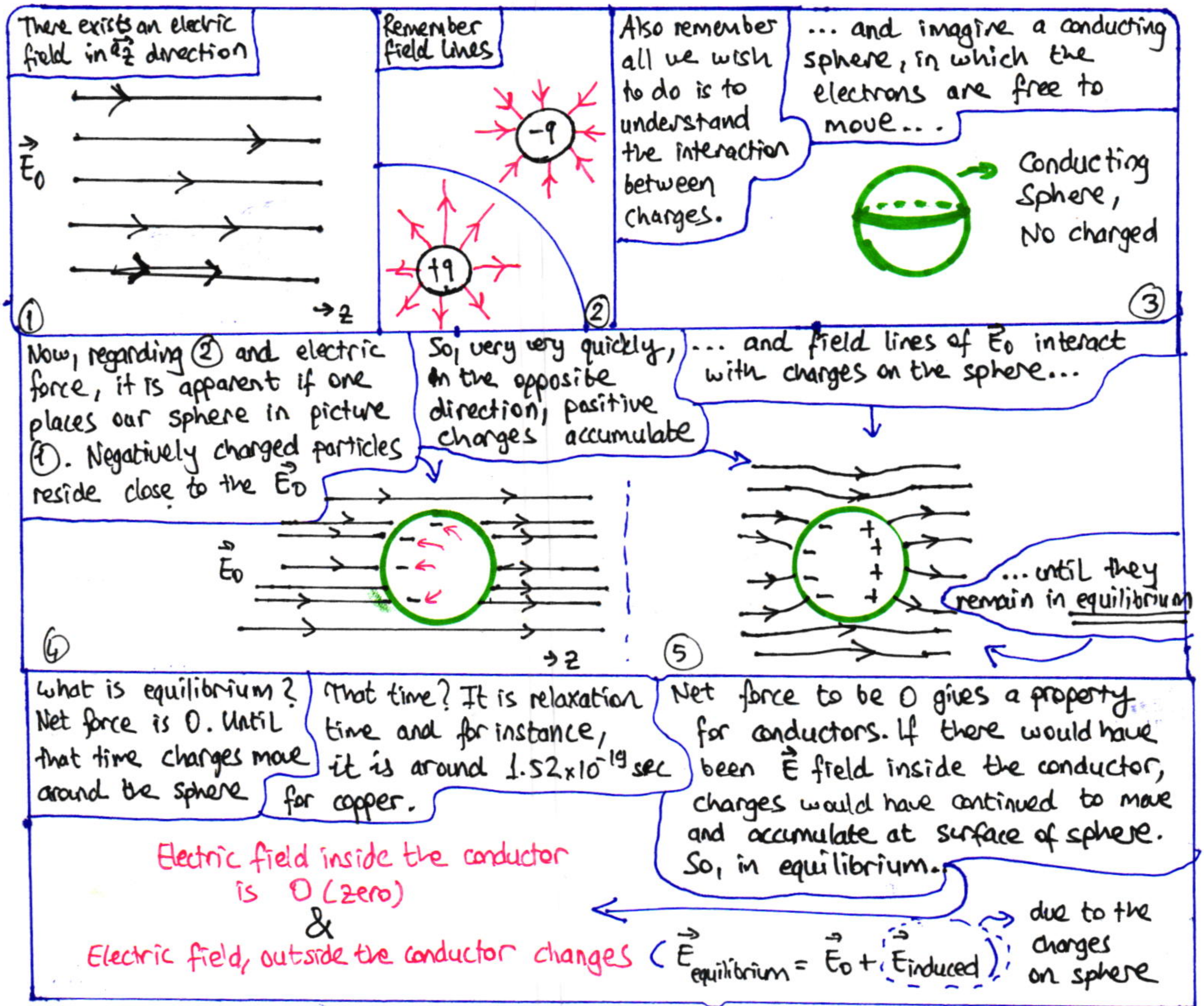


Note that $\vec{E} \perp V$
Therefore no work is done to move a charge along an equipotential curve
(that is $-\int \vec{E}\, d\vec{l} = 0$)

CONDUCTORS in STATIC ELECTRIC FIELD

Inside a conductor, electrons are free to move.

Therefore, if a conductor is placed in a static electric field, charges start to move. Recalling the field lines of charges, one may easily predict this mobility. Look at the cartoon below.

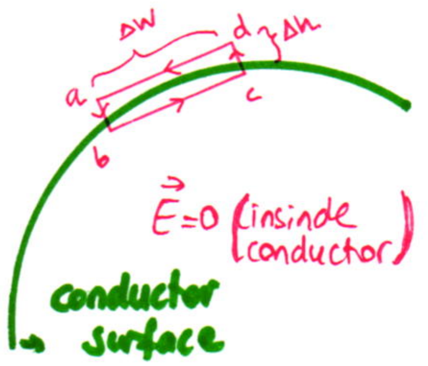


This story is real for any shaped conductor, but charge distribution around the conductor varies. Charge density increases around the sharp parts. This is why lightining arresters are fabricated as they are.

conductors in static electric field (continued)

Note that, in equilibrium, since $\vec{E}=0$ inside the conductor, all the charges must be distributed over the surface of conductors.

If so, then tangential field of $\vec{E}$ must be zero on the surface. If it would not be like that, tangential component would have forced charges to move. And this is what disequilibrium is ($\Sigma \vec{F}_{net} \neq 0$). Let's analyze mathematically;



* Since $\vec{E}=0$ inside the conductor, $\vec{E}$ on bc line is 0 (zero)
* If we take $\Delta h \to 0$, $\vec{E}$ vanishes on ab and dc
* Remember, in electrostatics, $\nabla \times \vec{E} = 0$
* ...and stokes's theorem $\int \nabla \times \vec{E} \, ds = \oint \vec{E} \cdot d\vec{l}$

That path is abcda

* $\oint \vec{E} \cdot d\vec{l} = \hat{t} E . \Delta w \Big|_{ad} + \underbrace{\ldots}_{\text{zero as mentioned above}}$

$\hat{t}$: unit vector in tangential direction

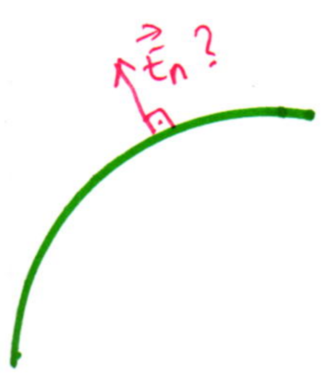
* So, one may say →

$\boxed{\text{Tangential component on the surface of a conductor is 0 (for } \vec{E} \text{ field!!)} \quad \vec{E}_t = 0}$

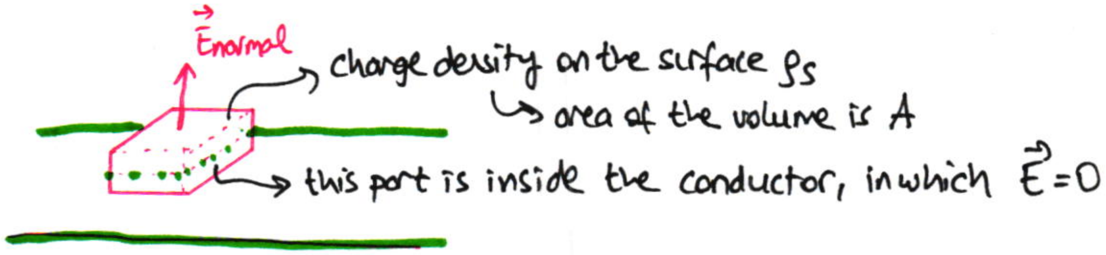
If so, then one may say; inside the conductor every point is at the same potential. You may infere from the fact that $\vec{E} = -\nabla V$

$\boxed{\text{A conductor is equipotential!!!}}$

What about normal component?



To answer this question, imagine the gauss' law for a pill-box, cylinder or any other shape.



Tangential component are 0 (zero). Now, let's utilize divergence theorem.

$$\int \nabla \cdot \vec{E}\, dv = \frac{\rho_s . A}{\varepsilon_0}$$ → total charge enclosed by the volume / ε_0

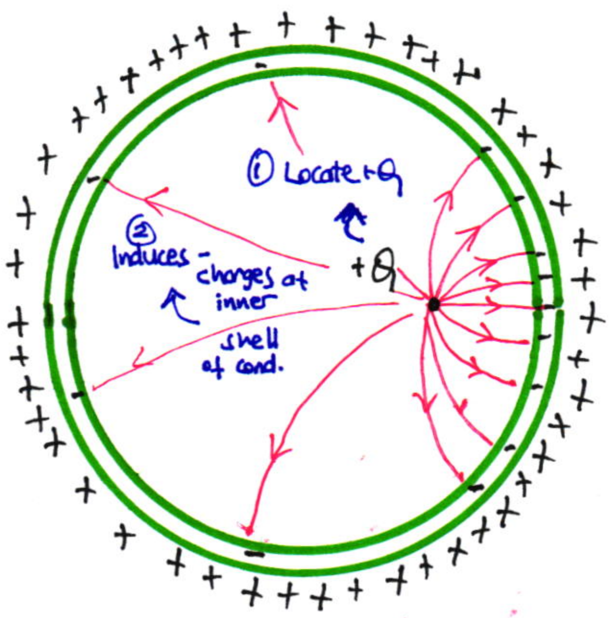
and remember all excess charge is on the surface, which is related to ρ_s

$= \int \vec{E} \cdot d\vec{s}$ → tangential components and $\vec{E}$ inside the conductor is 0 (zero). Hence

$= E_{normal} . A . \vec{a_n}$ → unit vector outward the surface, normally

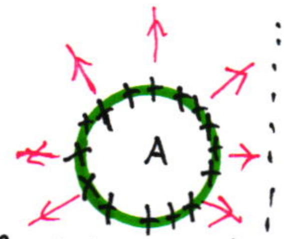
$= \rho_s . A / \varepsilon_0 \Rightarrow \boxed{\vec{E}_n = \frac{\rho_s}{\varepsilon_0}\vec{a_n}}$

Some examples on electrostatic induction

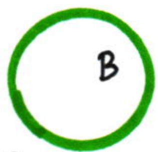


If one locates +Q (or -Q) inside a conductor (at any point), it will induce -Q (totally) on the inside shell. If you choose a Gauss surface inside the conductor, since $\vec{E}=0$, it is reasonable that inner shell to be charged with a total amount of -Q. But note that, it is not uniformly distributed. Close to the inducing chare +Q, induced charges (-) form a dense distribution.

Also, due to the conservation of charges and no charge can exist in the conductor, the outer shell induces with positive charges. This time, +Q appears uniformly distributed on the surface (since it is not connected with -Q or Q inside and must be distributed)

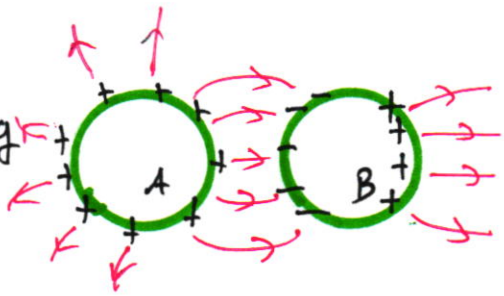


Let A be charged positively

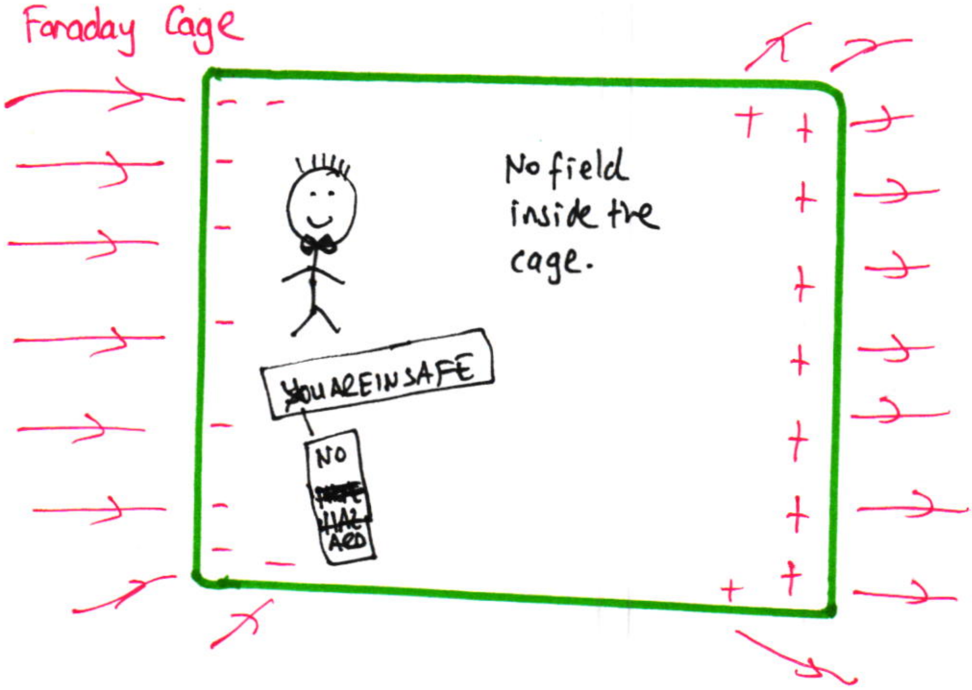


B is at any other location

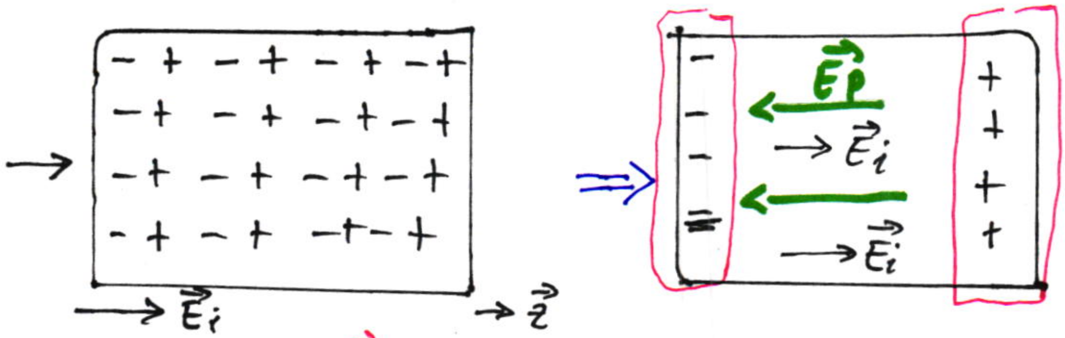
Now let bring B close to A



Faraday Cage



DIELECTRICS in STATIC ELECTRIC FIELD



if we apply an $\vec{E}$ field in $\vec{a_z}$ direction, molecules in dielectric will be oriented as above

+ & - charges can be considered as a source for internal Electric field. Let's say $\vec{E}_p$. As per superposition principle, the total field inside is the vectoral sum of $\vec{E}_i$ & $\vec{E}_p$ that is, $\vec{E} = \vec{E}_i + \vec{E}_p$

For linear dielectrics, it is observed that, $\vec{E}_p$ is in equilibrium with total field inside, in a linear relation;

$$\boxed{\vec{E}_p = -\chi \vec{E}}$$

Here, negative sign is due to the direction and $\chi > 0$ (otherwise, $\vec{E}$ should have been created out of nothing)

Therefore, one may formulate

$$\boxed{\vec{E}_i = (1+\chi)\vec{E}}$$

Here, $\vec{E}_i$ can be considered as the incident field in vacuum, while $\vec{E}$ is the field inside the dielectric. χ is a relation constant with respect to the material property and called as "electric susceptibility".

Note that, in the presence of a dielectric medium, Electric field decreases w.r.t the material property, i.e., electric susceptibility χ. This is the core of dielectric loaded capacitor idea (As we will see later, decreased $\vec{E}$ fields increase the capacity) dielectrics in electrostatic field (continued, 1)

Now, let's consider free charges inside dielectric. Using Gauss' law;

$$\int \varepsilon_0 \vec{E}_i \cdot d\vec{s} = \sum Q_{free} = \int \rho \cdot dv$$

This holds, because Q_{free} exist whether dielectric media exists or not.

Replacing $\vec{E}_i$ with that of the field relation inside dielectric;

$$\boxed{\int \varepsilon_0 (1+\chi) \vec{E} d\vec{s} = \int \rho \, dv}$$

Later on, for clarity of terminology, we will call $(1+\chi)$ as the "relative permittivity" ε_r.

Also, we will express $\varepsilon_0 \cdot \varepsilon_r \cdot \vec{E}$ as "electric flux density" $\vec{D}$ $[C/m^2]$.

Using new concepts, we may write;

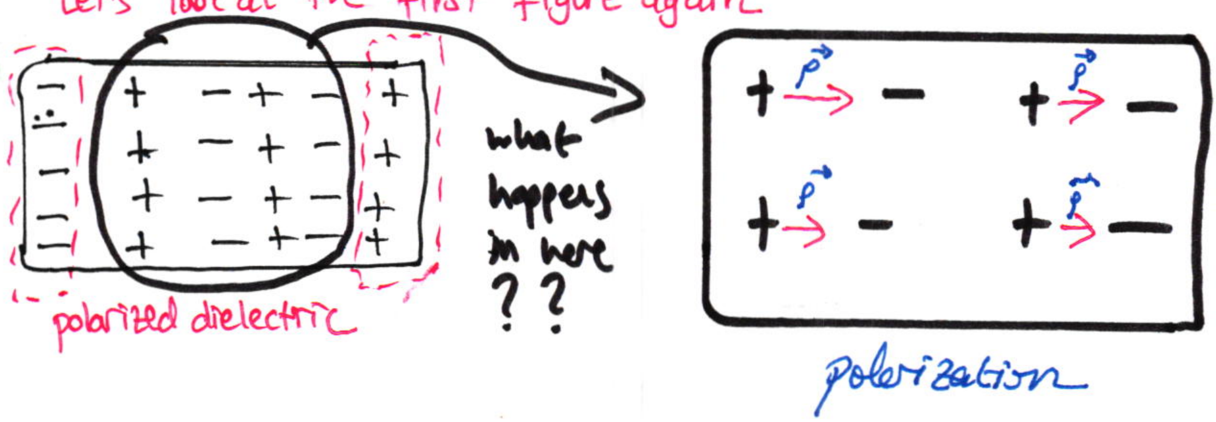
$$\boxed{\int \vec{D} \cdot d\vec{s} = \int \rho \cdot dv}$$

which simply gives

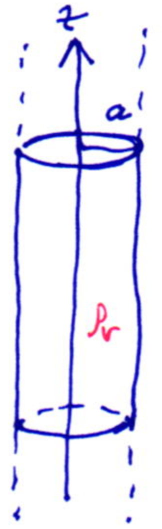
$$\boxed{\nabla \cdot \vec{D} = \rho}$$

Note that, ρ is "free charge" in the medium (Not correlated with polarized charges)

Let's look at the first figure again



$\vec{E}$, gauss

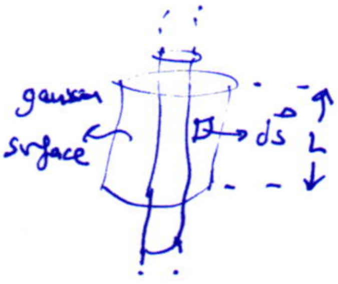


An infinitely long cylinder with radius a has a uniform charge distribution ρ_v (C/m^3). Determine $\vec{E}$ field in whole space? (ε_0 is the permittivity of whole space)

answer

it is convenient to utilize gauss' law due to the symmetry of the problem by a cylindrical gaussian surface.

Due to the cylindrical symmetry, space can be subdivided to 2 regions in terms of $\vec{E}$ field. $0<\rho<a$ & $\rho>a$



$0<\rho<a$

$\int \vec{D}\,\vec{ds} = \sum Q_{inside}$ $\quad \vec{ds} = \rho \cdot d\phi\, dz\, \vec{a_\rho}$, $\vec{D} = D_\rho \vec{a_\rho}$

$$\Rightarrow \int \vec{D}\,\vec{ds} = \int_0^L \int_0^{2\pi} \rho\, D_\rho\, d\phi\, dz = 2\pi L \rho D_\rho$$

$$= \sum Q_{inside} = \int \rho_v dv = \rho_v \cdot \underbrace{\pi\rho^2 L}_{\text{volume of closed gaussian surface}}$$

$$\Rightarrow D_\rho = \frac{\rho_v \cdot \pi\rho^2 L}{2\pi L \rho} = \rho_v \cdot \rho/2$$

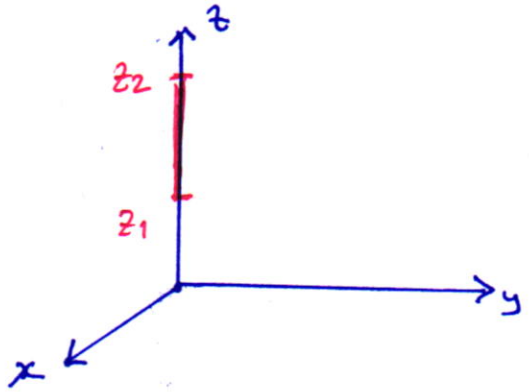
Recall that on the top and bottom surfaces, fields are cancelled due to the symmetry in vertical axis (i.e., $D_z = 0$)

$$\Rightarrow E_\rho = D_\rho/\varepsilon_0 = \frac{\rho_v \cdot \rho}{2\varepsilon_0} \; [V/m] \quad \text{for } 0<\rho<a$$

$\rho>a$

Similarly, $\int \vec{D}\,\vec{ds} = 2\pi L \rho D_\rho$ and $\sum Q_{inside} = \rho_v \cdot \pi a^2 \cdot L$

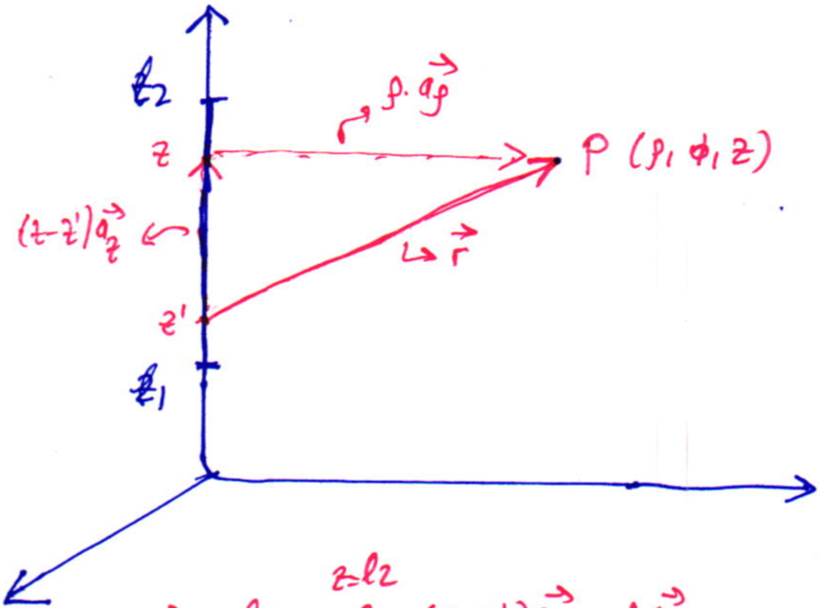
$$\Rightarrow D_\rho = \frac{\rho_v \cdot a^2}{2\rho} \quad \& \quad E_\rho = D_\rho/\varepsilon_0 = \frac{\rho_v a^2}{2\rho\varepsilon_0} \; [V/m]$$



A charged line with ρ_ℓ density is located at z-axis between z_1 & z_2 positions.
Determine $\vec{E}$ field at any given point P.

__answer__

Due to the cylindrical symmetry, one may not regard the $\vec{E}$ field component in ϕ direction. Obviously in ϕ direction $\vec{E}$ is invariant.



position vector
$$\vec{r} = (z-z')\vec{a_z} + \rho \vec{a_\rho}$$
$$\vec{E} = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho_\ell \cdot \vec{a_r}}{r^2} d\ell$$
$$d\ell = dz'$$

$$\Rightarrow \vec{E} = \frac{\rho_\ell}{4\pi\varepsilon_0} \int_{z=\ell_1}^{z=\ell_2} \frac{(z-z')\vec{a_z} + \rho\vec{a_\rho}}{\left[(z-z')^2+\rho^2\right]^{3/2}} dz'$$ let's arrange integral

$$\Rightarrow \vec{E} = \vec{a_\rho}\frac{\rho_\ell \cdot \rho}{4\pi\varepsilon_0} \int_{\ell_1}^{\ell_2} \frac{dz'}{\left[(z-z')^2+\rho^2\right]^{3/2}} + \frac{\vec{a_z}\,\rho_\ell}{4\pi\varepsilon_0}\int_{\ell_1}^{\ell_2} \frac{(z-z')\,dz'}{\left[(z-z')^2+\rho^2\right]^{3/2}}$$

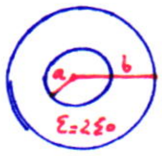
Integrals are derived in next page

① $$\frac{1}{\rho^2} \cdot \frac{z'-z}{\left[\rho^2+(z-z')^2\right]^{1/2}} \Bigg|_{z'=\ell_1}^{z'=\ell_2}$$

② $$\frac{1}{\left[\rho^2+(z-z')^2\right]^{1/2}} \Bigg|_{z'=\ell_1}^{z'=\ell_2}$$

$$\Rightarrow \vec{E} = \left\{ \vec{a_\rho}\left[\frac{\ell_2-z}{\left[\rho^2+(z-\ell_2)^2\right]^{1/2}} - \frac{\ell_1-z}{\left[\rho^2+(z-\ell_1)^2\right]^{1/2}}\right]\frac{1}{\rho} + \vec{a_z}\left[\frac{1}{\left[\rho^2+(z-\ell_2)^2\right]^{1/2}} - \frac{1}{\left[\rho^2+(z-\ell_1)^2\right]^{1/2}}\right]\right\}$$
$$\times\ \rho_\ell / 4\pi\varepsilon_0$$

$\vec{E}, V$, work



A $1\mu C$ charge is uniformly distributed on a conducting sphere with radius a ($a = 1\,m$).
A dielectric material is loaded between the outer ~~conducting~~ shell with radius b and conducting sphere with radius a ($b = 2m$)

a) Determine $\vec{E}$ field everywhere?

b) Determine V everywhere?

c) How much work must be performed to bring a $Q = 2\mu C$ charge from $A(2,0,0)$ to $B(0,0,1)$?

---

a) $\underline{r < a}$

Since inside the conductor $\vec{E}$ field does not exist $\quad \vec{E}_I = 0$

$\underline{a < r < b}$

$\underbrace{\int \vec{D}\, d\vec{s}}_{4\pi r^2} = \sum Q_{inside}$ where $\vec{D} = 2\varepsilon_0 \vec{E}_{II}$

$\Rightarrow \vec{E}_{II} = \dfrac{1\mu C}{2\varepsilon_0 4\pi r^2} \vec{a_r}$ (note that $\vec{E}$ is radially outward)

(units $C/(F/m)\cdot m^2 = \dfrac{C}{Fm} = \dfrac{C}{\frac{C}{V}\cdot m} = V/m$; $1\ \text{Farad} = \dfrac{1C}{1V}$)

$\underline{r < b}$

$\underbrace{\int \vec{D}\, d\vec{s}}_{4\pi r^2} = \sum Q_{inside}$ where $\vec{D} = \varepsilon_0 \vec{E}_{III}$

$\Rightarrow \vec{E}_{III} = \dfrac{1\mu C}{\varepsilon_0 4\pi r^2} \vec{a_r}$

b) $V = -\int \vec{E}\, d\vec{l} \quad (\vec{E} = -\nabla V)$ $\qquad V_I$ is constant (conductor is an equipotential volume)

$V_{II} = -\int \vec{E}_{II}\, d\vec{r} = -\int \dfrac{10^{-6}}{8\pi\varepsilon_0 r^2} dr = -\dfrac{10^{-6}}{8\pi\varepsilon_0}\left(-\dfrac{1}{r}\right) + C_1 \rightarrow$ indefinite integral constant

$V_{III} = -\int \vec{E}_{III}\, d\vec{r} = -\int \dfrac{10^{-6}}{4\pi\varepsilon_0 r^2} dr = \dfrac{10^{-6}}{4\pi\varepsilon_0 r} + C_2$

we know that, $V(\infty)=0$ and $V_{III}(\infty)=0 \rightarrow C_2=0$

Also at $r=b$, $V_{II}=V_{III}$ (Potential is continuous)

$$\Rightarrow \frac{10^{-6}}{8\pi\varepsilon_0}\frac{1}{b}+C_1=\frac{10^{-6}}{4\pi\varepsilon_0 b}\Big|_{b=2m} \rightarrow C_1=\frac{10^{-6}}{8\pi\varepsilon_0}\cdot\frac{1}{2}$$

Similarly, at $r=a$, $V_I=V_{II}$

$$V_I=\frac{10^{-6}}{8\pi\varepsilon_0}+\frac{10^{-6}}{16\pi\varepsilon_0}=\frac{3\,10^{-6}}{16\pi\varepsilon_0}\,V$$

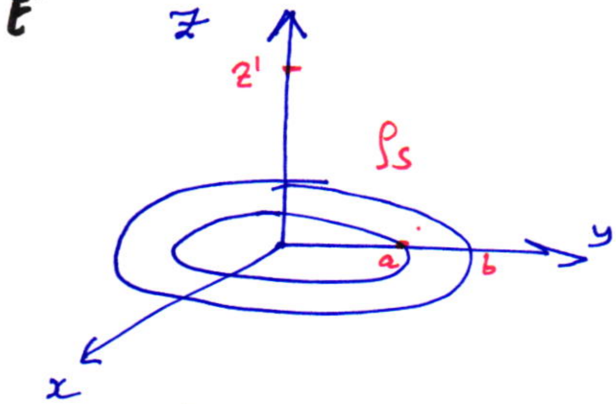
$$V_{II}=\frac{10^{-6}}{8\pi\varepsilon_0 r}+\underbrace{\frac{10^{-6}}{16\pi\varepsilon_0}}_{C_1}, \quad V_{III}=\frac{10^{-6}}{4\pi\varepsilon_0 r}$$

c) Point A is on the sphere with radius b while Point B is on the sphere with radius a. Note that work to be performed is independent on the path. Also potential was work performed for unit charge.

Namely;

$$W_{AB}=q(V_B-V_A)=2\,10^{-6}\left(\frac{3\,10^{-6}}{16\pi\varepsilon_0}-\frac{10^{-6}}{8\pi\varepsilon_0}\right) \text{ Joule}$$

$\vec{E}$



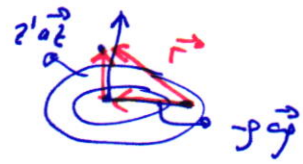
A ring shaped charge system with uniform charge density ρ_s is depicted on the left. ρ_s is distributed for $[\rho, \phi, z] = [[a, b], [0, 2\pi], 0]$

Determine $\vec{E}$ field at $(0, 0, z')$?

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\vec{a_r}\rho_s \, . \, ds}{r^2}$$

Let's insert into the equation

$$\vec{r} = -\rho \cdot \vec{a_\rho} + z' \cdot \vec{a_z}$$



$$\vec{a_r} = \frac{\vec{r}}{|\vec{r}|}$$

$$ds = \rho \, d\phi \, d\rho$$

$$\vec{E} = \frac{\rho_s}{4\pi\varepsilon_0}\int \frac{\overbrace{(-\rho\vec{a_\rho} + z'\vec{a_z})}^{}\rho \, d\rho \, d\phi}{\underbrace{(\rho^2+z'^2)^{1/2}}_{\vec{a_r}}\underbrace{(\rho^2+z'^2)}_{r^2}}$$

Due to the cylindrical symmetry, no contribution to the $\vec{E}$ field exists due to the ρ-component. So, one may eliminate the terms within $\vec{a_\rho}$ direction

$$\Rightarrow \vec{E} = \frac{\rho_s}{4\pi\varepsilon_0}\int_{\phi=0}^{2\pi}\int_{\rho=a}^{b} \frac{\vec{a_z}\, z' \rho \, d\rho \, \overset{2\pi}{d\phi}}{(\rho^2+z'^2)^{3/2}} = \frac{\rho_s z'}{2\varepsilon_0}\int_{\rho=a}^{\rho=b} \frac{\rho \, d\rho \; \vec{a_z}}{(\rho^2+z'^2)^{3/2}}$$

→ Recall Type ② integral

$$-\frac{1}{(z'^2+\rho^2)^{1/2}}\Bigg|_a^b$$

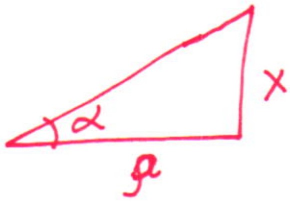
$$\boxed{\Rightarrow \vec{E} = \frac{\rho_s z'}{2\varepsilon_0}\left[\frac{\sqrt{z'^2+b^2} - \sqrt{z'^2+a^2}}{\sqrt{(z'^2+a^2)(z'^2+b^2)}}\right]\vec{a_z}}$$

Type 1 integral

$$\int \frac{dz'}{\left[(z-z')^2+\rho^2\right]^{3/2}} = ?$$

Let's write $z-z'=x$

$-dz'=dx$



$\tan\alpha = x/\rho = \frac{\sin\alpha}{\cos\alpha}$

$$\Rightarrow x = \rho\frac{\sin\alpha}{\cos\alpha} \Rightarrow dx = \rho\,\frac{\cos^2\alpha-(-\sin^2\alpha)}{\cos^2\alpha}\,d\alpha$$

Let's arrange w.r.t. above identities

$$\int\frac{dz'}{\left[(z-z')^2+\rho^2\right]^{3/2}} = \int\frac{-dx}{\left[x^2+\rho^2\right]^{3/2}} = -\rho\int\frac{d\alpha}{\cos^2\alpha\left[\underbrace{\rho^2\left(1+\frac{\sin^2\alpha}{\cos^2\alpha}\right)}_{x^2+\rho^2}\right]^{3/2}}$$

$$= -\frac{1}{\rho^2}\int\frac{d\alpha}{\cos\alpha^{-1}}$$

$$= -\frac{1}{\rho^2}\sin\alpha = -\frac{1}{\rho^2}\frac{x}{\sqrt{x^2+\rho^2}}$$

Type 2 integral

$$\int\frac{(z-z')\,dz'}{\left[(z-z')^2+\rho^2\right]^{3/2}} = ?$$

Let $x = z-z'$

$\Rightarrow dx = -dz'$

$$\Rightarrow -\int\frac{x\,dx}{\underbrace{(x^2+\rho^2)}_{u}{}^{3/2}} = ?$$

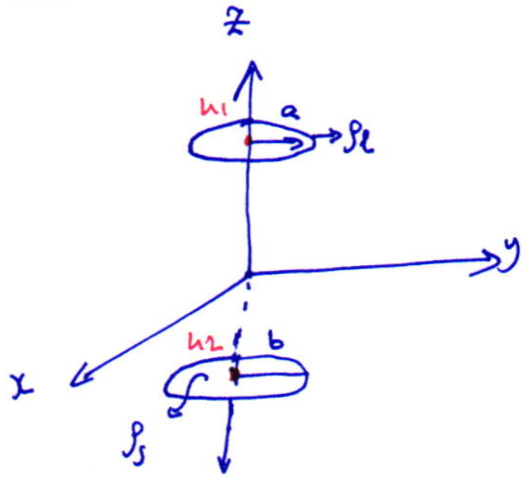
$u \Rightarrow du = 2x\,dx$

$$\Rightarrow -\int\frac{du/2}{u^{3/2}} = ? = \frac{1}{\sqrt{u}}$$

$$\Rightarrow -\int\frac{x\,dx}{(x^2+\rho^2)^{3/2}} = \frac{1}{\sqrt{x^2+\rho^2}}$$

$$= \frac{1}{\sqrt{(z-z')^2+\rho^2}}$$

$\vec{E}$ & V



1) line charged (with ρ_ℓ [C/m]) circle and a surface charged disk (with ρ_s [C/m²]) are located at $(0,0,h_1)$ and $(0,0,-h_2)$, respectively.

a) Determine the $\vec{E}$ field at origin?

b) If $\vec{E}=0$, ρ_s/ρ_ℓ ? (at origin)

c) If $\vec{E}=0$ at origin, determine the potential there?

For ring with ρ_ℓ charge distribution

$d\ell = a\,d\phi$

$\vec{R}_1 = -a\vec{a_\rho} - h_1\vec{a_z}$, $R_1 = \sqrt{a^2+h_1^2}$

For disc with ρ_s charge distribution

$ds = \rho\,d\rho\,d\phi$

$\vec{R}_2 = -\rho\vec{a_\rho} + h_2\vec{a_z}$

↳ since disc has a surface, each discrete point in the integral varies with ρ (why it's not "b" as in ring)

$R_2 = \sqrt{\rho^2+h_2^2}$

$\vec{E}_{tot} = \vec{E}_1 + \vec{E}_2$ (superposition principle)

$$\vec{E}_1 = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho_\ell \vec{a_r}\,d\ell}{r^2} = \frac{\rho_\ell}{4\pi\varepsilon_0}\int_{\phi=0}^{2\pi} \frac{(-a\vec{a_\rho} - h_1\vec{a_z})\,a\,d\phi}{(a^2+h_1^2)^{3/2}}$$

we may cancel $\vec{a_\rho}$ component as to cylindirical symmetry or you may expand $\vec{a_\rho}$ in cartesian coordinates and evaluate mathematically $\int_0^{2\pi}\cos\phi\,d\phi = 0$

$$= \frac{-\rho_\ell a}{4\pi\varepsilon_0}\int_0^{2\pi} \frac{h_1\vec{a_z}\,d\phi}{(a^2+h_1^2)^{3/2}} = \frac{-a\rho_\ell h_1}{2\varepsilon_0(a^2+h_1^2)^{3/2}}\vec{a_z}$$

cancelled

$$\vec{E}_2 = \frac{1}{4\pi\varepsilon_0}\int_S \frac{\rho_s\vec{a_r}\,ds}{r^2} = \frac{\rho_s}{4\pi\varepsilon_0}\int_{\phi=0}^{2\pi}\int_{\rho=0}^{b} \frac{(-\rho\vec{a_\rho} + h_2\vec{a_z})}{(\rho^2+h_2^2)^{3/2}}\rho\,d\rho\,d\phi$$

$$= \frac{\rho_s h_2}{4\pi\varepsilon_0}\cdot 2\pi \int_{\rho=0}^{b} \frac{\rho\,d\rho\,\vec{a_z}}{(\rho^2+h_2^2)^{3/2}}$$

→ recall Type 1 integral

$$\hookrightarrow \left.\frac{-1}{\sqrt{\rho^2+h_2^2}}\right|_0^b = \frac{1}{h_2} - \frac{1}{\sqrt{b^2+h_2^2}}$$

$$\Rightarrow \vec{E}_2 = \frac{\rho_s h_2}{2\varepsilon_0}\left[\frac{1}{h_2} - \frac{1}{\sqrt{b^2+h_2^2}}\right]\vec{a_z} \qquad \vec{E}_{tot} = \vec{E}_1 + \vec{E}_2 = \cdots$$

b) $\vec{E}_{tot} = 0 \quad \vec{E}_1 + \vec{E}_2 = 0$

$$\Rightarrow \vec{a_z}\left(\frac{-a\rho_\ell h_1}{2\varepsilon_0 (a^2+h_1^2)^{3/2}} + \frac{\rho_s h_2}{2\varepsilon_0}\left[\frac{1}{h_2} - \frac{1}{\sqrt{b^2+h_2^2}}\right]\right) = 0$$

$$\Rightarrow \frac{\rho_\ell}{\rho_s} = \frac{\cancel{2\varepsilon_0 (a^2+h_1^2)^{3/2}} \cdot h_2\left[\frac{1}{h_2} - \frac{1}{\sqrt{b^2+h_2^2}}\right]}{a h_1 \, \cancel{2\varepsilon_0 (a^2+h_1^2)^{3/2}}}$$

$=$

c) $$V_1 = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho_\ell \, d\ell}{r} = \frac{\rho_\ell}{4\pi\varepsilon_0}\int_\phi \frac{a\,d\phi}{\sqrt{a^2+h_1^2}} = \frac{a\rho_\ell}{2\varepsilon_0\sqrt{a^2+h_1^2}}$$

$$V_2 = \frac{1}{4\pi\varepsilon_0}\iint_{\rho\phi} \frac{\rho_s \overbrace{ds}^{\rho d\rho d\phi}}{r} = \frac{\rho_s}{2\varepsilon_0}\int_{\rho=0}^{b} \frac{\rho\,d\rho}{\sqrt{\rho^2+h^2}} = \frac{\rho_s}{2\varepsilon_0}\left(\sqrt{b^2+h_2^2} - h_2\right)$$

$$u = \rho^2 + h^2 \rightarrow du = 2\rho d\rho$$

$$\int \frac{du/2}{\sqrt{u}} = \frac{1}{2}\sqrt{u}\cdot 2 \Big|_{u=\sqrt{h^2}}^{u=\sqrt{\rho^2+h^2}}$$

$V_{tot} = V_1 + V_2$

WORK

How much work is performed to bring a $5\mu C$ charge from origin to $(2, \pi/4, \pi/2)$ point if a static electric field is defined as $\vec{E} = 5e^{-R/4}\vec{a_r} + \frac{10}{R\sin\theta}\vec{a_\phi}$?

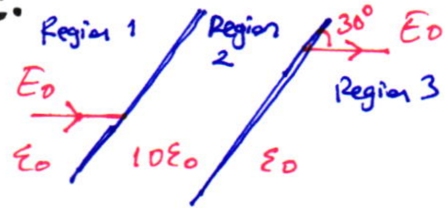
$$W = -\int_1^2 \underbrace{\vec{F}\,\vec{dl}}_{} \quad \text{where } \vec{F} = Q\vec{E}$$

in spherical coordinates : $\vec{dl} = dR\,\vec{a_R} + R\,d\theta\,\vec{a_\theta} + R\sin\theta\,d\phi\,\vec{a_\phi}$

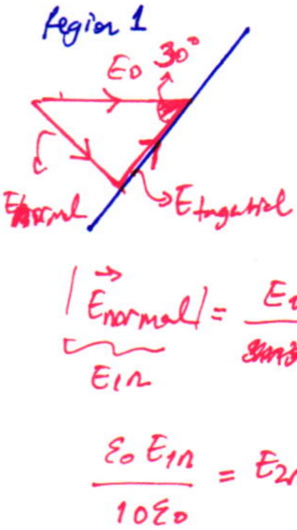
$$\Rightarrow W = -\int_{R=0}^{R=2} \underbrace{5\,10^{-6}}_{Q}\,\underbrace{5e^{-R/4}}_{E_R}\,dR + -\int_{\phi=0}^{\phi=\pi/2} 5\,10^{-6}\,\frac{10R\sin\theta}{R\sin\theta}\,d\phi$$

$$\Rightarrow W = -5\,10^{-6}\left\{-5\cdot4\,e^{-R/4}\Big|_0^2 + 10\,\phi\Big|_0^{\pi/2}\right\} \simeq -118\,\mu J$$

B.C.



Determine the components (tangential & normal) of $\vec{E}$ field in Region 2, as well as the magnitude ?



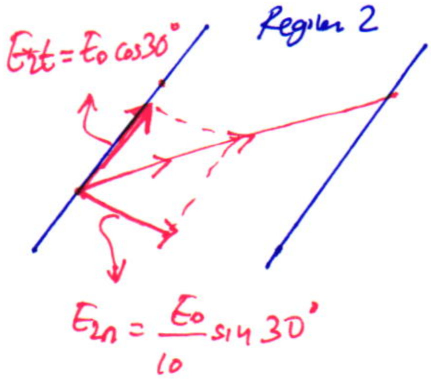
Under normal conditions, no surface charge exists on dielectric-dielectric interface. Since $D_{1n} = D_{2n} \rightarrow \varepsilon_0 E_{1n} = 10\varepsilon_0 E_{2n}$

Also tangential components are continuous on the interface

That is $E_{1t} = E_{2t}$. Let's use them

$$\underbrace{|\vec{E}_{normal}|}_{E_{1n}} = E_0\sin 30°, \quad \underbrace{|\vec{E}_{tangential}|}_{E_{1t}} = E_0\cos 30°$$

$$\frac{\varepsilon_0 E_{1n}}{10\varepsilon_0} = E_{2n} = \frac{E_0\sin 30°}{10} \;; \quad E_{1t} = E_{2t} = E_0 . \cos 30°$$

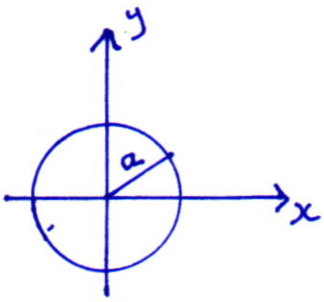


$$|E_2| = \sqrt{|E_{2n}|^2 + |E_{2t}|^2}$$

$$= E_0\sqrt{\frac{\sin^2 30}{100} + \cos^2 30}$$

$$\simeq \frac{E_0}{20}\,17.35$$

8



A ring shape charge system is distributed on a circle with radius a as;

$$\rho_{ring} = \begin{cases} \rho_\ell , & x>0 \\ -\rho_\ell , & x<0 \end{cases}$$

Determine electric field intensity $\vec{E}$ at origin

<u>answer</u>

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho_{ring}\cdot\vec{a_r}}{r^2}\,d\ell$$

where $\vec{a_r}$ is in the opposite direction of $\vec{a_\rho}$

$d\ell = a.d\phi$

↳ metric coefficient

$r = a$

Positioning

~~Direction~~ of $\vec{a_r}$ (i.e., $-\vec{a_\rho}$) is ϕ-dependent, hence it is more convenient to express position vector in cartesian coordinates. That is;

$$\vec{a_\rho} = \vec{a_x}\cos\phi + \vec{a_y}\sin\phi$$

Our integral line is constituted of three regions (invariant) in terms of integral variables. That is;

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\left[\int_{\phi=0}^{\pi/2}\frac{\rho_\ell\,\vec{a_r}}{r^2}\overbrace{a.d\phi}^{d\ell} + \int_{\phi=\pi/2}^{3\pi/2}\frac{-\rho_\ell\cdot\vec{a_r}}{r^2}\overbrace{a.d\phi}^{d\ell} + \int_{\phi=3\pi/2}^{2\pi}\frac{\rho_\ell\cdot\vec{a_r}}{r^2}\overbrace{a\,d\phi}^{d\ell}\right]_{r=a}$$

If we arrange integral; (note that $\vec{a_r} = -\vec{a_\rho}$)

$$\vec{E} = \frac{\rho_\ell}{4\pi\varepsilon_0}\frac{a}{a^2}\left[\int_{\phi=0}^{\pi/2}-(\vec{a_x}\cos\phi + \vec{a_y}\sin\phi)\,d\phi + \int_{\phi=\pi/2}^{3\pi/2}-(-(\vec{a_x}\cos\phi + \vec{a_y}\sin\phi))\,d\phi + \int_{\phi=3\pi/2}^{2\pi}-(\vec{a_x}\cos\phi + \vec{a_y}\sin\phi)\,d\phi\right]$$

Recall $\int\cos\phi\,d\phi = \sin\phi + C$

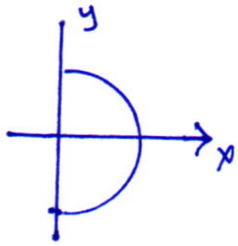
$\int\sin\phi\,d\phi = -\cos\phi + C$

$$\vec{E} = \frac{\rho_\ell}{4\pi\varepsilon_0 a}\left\{ \left[\vec{a_x}(-\sin\phi) + \vec{a_y}\cos\phi\right]\Big|_{\phi=0}^{\pi/2} + \left[\vec{a_x}\sin\phi + \vec{a_y}\cos\phi\right]\Big|_{\phi=\pi/2}^{3\pi/2} \right.$$

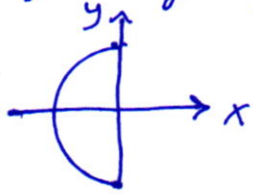
$$\left. + \left[\vec{a_x}(-\sin\phi) + \vec{a_y}(-\cos\phi)\right]\Big|_{\phi=3\pi/2}^{2\pi} \right.$$

$$= \frac{\rho_\ell}{4\pi\varepsilon_0 a}\left[\overbrace{-\vec{a_x} - \vec{a_y} + -2\vec{a_x} + 0a_y - \vec{a_x} + \vec{a_y}}^{-4\vec{a_x}}\right] = \frac{\rho_\ell}{\pi\varepsilon_0 a}(-\vec{a_x}) \quad [V/m \text{ or } N/C]$$

Also, using direction of $\vec{E}$, one could rearrange geometry as;



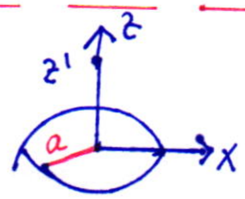
$$\rho_{ring} = \begin{cases} 0, & x<0 \\ 2\rho_\ell, & x>0 \end{cases}$$ or



$$\rho_\ell = \begin{cases} -2\rho_\ell, & x<0 \\ 0, & x>0 \end{cases}$$

---

v



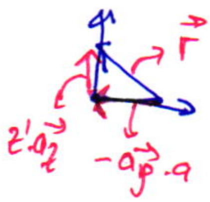
A ring shaped charge system with charge density ρ_ℓ is located and centered at xy plane and origin, respectively. Determine the potential at z'

answer

$$v = \frac{1}{4\pi\varepsilon_0}\int_{\phi=0}^{2\pi} \frac{\rho_\ell}{r}\,\overbrace{d\ell}^{a.d\phi}$$

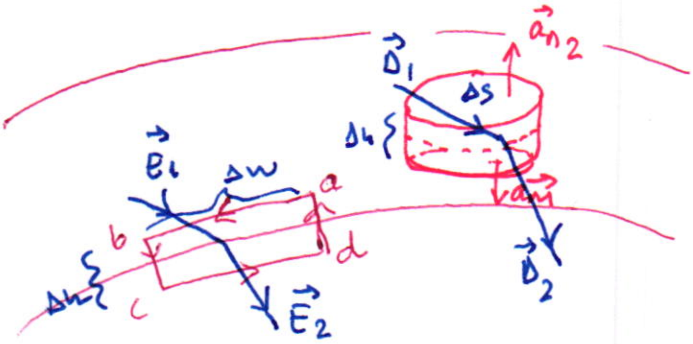
$r = |\vec{r}|$ where $\vec{r} = -\vec{a_\rho}.a + \vec{a_z}.z'$

$\Rightarrow r = \sqrt{a^2 + z'^2}$



$$v = \frac{\rho_\ell}{4\pi\varepsilon_0}\int_{\phi=0}^{2\pi} \frac{a}{\sqrt{a^2+z'^2}}\,d\phi = \frac{\rho_\ell . a}{2\varepsilon_0\sqrt{a^2+z'^2}} \quad [V \text{ or } J/C]$$

Boundary Conditions for Electrostatic Fields



Consider the interface between two media. Recall that

$$\oint \vec{E}\, d\vec{l} = 0$$

$$\int \vec{D}\, d\vec{s} = \rho_v$$

Tangential Let's analyze $\int \vec{E}\, d\vec{l}$ on abcda path. As $\Delta h \to 0$ and ab & cd lines tend to a point like line we have;

$$\int \vec{E}\, d\vec{l} = E_{1t} \cdot \Delta w - E_{2t}\, \Delta w = 0,$$ here bc & ad paths do not contribute to the integral

Hence, one may write;

$$\boxed{E_{1t} = E_{2t}}$$

Since the $\vec{E}$ field vanishes inside a conductor, no tangential component exist on the conductor surface even in the presence of charges. Also, this equality says that the tangential components of $\vec{E}$ fields are continuous.

Normal Let's analyze $\int \vec{D}\, d\vec{s} = \rho_v$ through the pillbox. As $\Delta h \to 0$, integral becomes

$$(\vec{D}_1 \vec{a}_{n2} + \vec{D}_2 \vec{a}_{n1})\, \Delta s = \underbrace{\rho_v \cdot \Delta s}_{\text{total charge}}$$ and $\vec{a}_{n2} = -\vec{a}_{n1}$ (normal vector is outward ~~the~~)

Then one may write

$$\vec{a}_{n2} (\vec{D}_1 - \vec{D}_2) = \rho_v$$ here ρ_v becomes ρ_s since tending $\Delta h \to 0$ forms volume to ~~a~~ surface

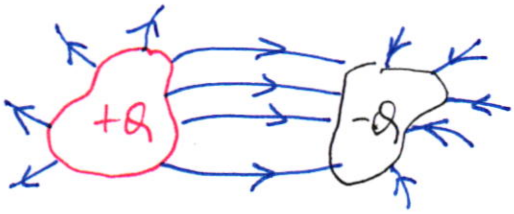
Finally,

$$\boxed{D_{1n} - D_{2n} = \rho_s}$$

Note that, under normal conditions there is no charge discontinuity between dielectric interfaces. Also, since $\vec{E}$ field is 0 inside a conductor, the normal component outward ~~from the~~ surface becomes $D_{1n} = \rho_s$ or $E_{1n} = \rho_s / \varepsilon$ as we obtained in conductors lecture.

CAPACITANCE

Capacitors vary in geometry but are composed of two conductors carrying equal and opposite sign charges.



Basic Configuration of a capacitor

If one charges (as in pyhsics lab experiments) the capacitor system (e.g., conductors) an amount of Q charges moves from one conductor to other conductor and a potential difference ΔV occurs between them.

Experimentally, it is observed that the amount of charge accumulated at the capacitor is linearly related to the potential difference ΔV. This linear dependancy is related with the concept of capacitance. Namely;

$$Q = C\,\Delta V$$

So, we may say that C capacitance is a measure of the capacity of storing charge per applied potential difference. Hence, unit is Q/V [Coulomb/Volt] and which is farad [F] in SI unit system. This system is called as capacitor and conductors are seperated by free-space or a dielectric medium.

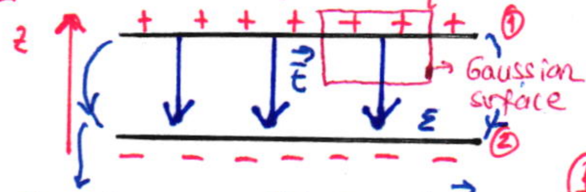
Now, let's analyze a capacitor and determine its capacitance. Our strategy is to calculate potential difference for an amount of Q. So, we may relate $\vec{E}$ field and ΔV as;

$\boxed{\Delta V = V_{12} = -\int \vec{E}\, d\vec{l}}$. Here ΔV is the potential difference between 1 & 2 conductors, if we know geometry and distribution of Q, we may define $\vec{E}$ in terms of charges (gauss' law or other relations). Finally we may calculate capacitance $C = Q/\Delta V$

CAPACITANCE (continued 2)

Let's work with some examples.

1. Parallel-Plate Capacitor



Since, plates are not infinite, all $\vec{E}$ field lines are not straight. These curved lines (fringing fields) are ignored in the analysis.

$\vec{E}$ field will be directed in $-z$ direction and as we derived previously

$$\oint \vec{E}\,d\vec{s} = Q/\varepsilon$$

① Let the cross-sectional (perpendicular to the z direction) of gaussian surface be S and the charge density of parallel plate is Q/S (here the gaussian surface encloses whole plate). Then, $\vec{E}$ will be in the direction $-\vec{a}_z$ (+ charges to − charges) with an amplitude of $Q/\varepsilon S$. So; $\vec{E} = -\vec{a}_z \dfrac{Q}{\varepsilon S}$ [V/m]

② Potential difference between plate 1 & plate 2 is;

$$V_{12} = V_1 - V_2 = -\int_{\infty}^{d} \vec{E}\,d\vec{l} - \left[-\int_{\infty}^{0} \vec{E}\,d\vec{l}\right] = -\int_{0}^{d} \vec{E}\,d\vec{l}$$

Note that $d\vec{l} = \vec{a}_z\,dz$. So;

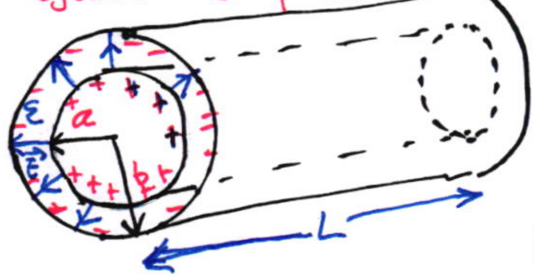
$$V_{12} = -\int_{0}^{d}\left(-\vec{a}_z \frac{Q}{\varepsilon S}\right)(\vec{a}_z\,dz) = \boxed{\frac{Qd}{\varepsilon S} = \Delta V}$$

③ Simply read the linear relation between potential difference and Q in terms of C as;

$$\boxed{C = \frac{Q}{V_{12}} = \varepsilon\frac{S}{d}} \quad [F, C/V]$$

Note that C is a positive scalar and depends only geometric configuration.

2. Cylindrical Capacitor



A dielectric with ε permittivity is loaded between conductors.

A Q (+) is distributed uniformly on the inner cylinder with radius a. As we examined before

① $\oint \vec{E}\,d\vec{s} = Q/\varepsilon$ where $\vec{E}$ is in $\vec{a}_\rho$ direction and $ds = 2\pi r L$

So; $\vec{E} = \vec{a}_\rho \dfrac{Q}{2\pi\varepsilon L r}$ [V/m]

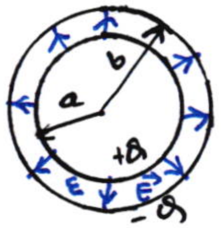
② $$V_{ab} = V_a - V_b = -\int_{\infty}^{a} \vec{E}\,\underbrace{d\vec{l}}_{\vec{a}_\rho d\rho} - \left[-\int_{\infty}^{b} \vec{E}\,\underbrace{d\vec{l}}_{\vec{a}_\rho d\rho}\right] = -\int_{b}^{a} \frac{Q\vec{a}_\rho}{2\pi\varepsilon L r}\cdot \vec{a}_\rho\,d\rho$$

$$V_{ab} = \frac{Q}{2\pi\varepsilon L}\cdot -\left[\ln a - \ln b\right] = \frac{Q}{2\pi\varepsilon L}\ln(b/a)$$

③ $$\boxed{C = \frac{Q}{V_{ab}} = \frac{2\pi\varepsilon L}{\ln(b/a)}} \quad [C/V, F]$$

CAPACITANCE (continued 3)

3. Spherical Capacitor



① $\oint \vec{E}\,d\vec{s} = Q/\varepsilon \rightarrow \vec{E} = \dfrac{Q}{4\pi\varepsilon R^2}\,\vec{a}_R$

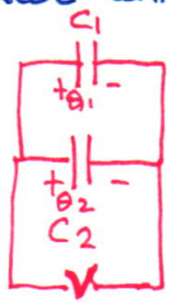
$4\pi\varepsilon R^2 \rightarrow$ Area of the spherical gaussian surface

② $V_{ab} = -\int_b^a \vec{E}\,d\vec{l} = -\int_b^a \dfrac{Q}{4\pi\varepsilon R^2}\,dR = \dfrac{Q}{4\pi\varepsilon}\left(\dfrac{1}{a} - \dfrac{1}{b}\right)$

$d\vec{l} = dR\,\vec{a}_R$

③ $\boxed{C = Q/V = 4\pi\varepsilon\left(\dfrac{ab}{b-a}\right)}$ $[F, C/V]$

Parallel Connection of Capacitors



$\equiv$?

Both capacitors have same potential difference V

$C_1 = \dfrac{Q_1}{V}$, $C_2 = \dfrac{Q_2}{V}$

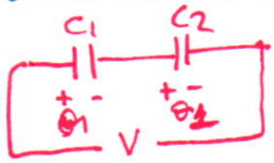
Total amount of charge of the system is $Q_1 + Q_2$

Here $C_{eq} = \dfrac{Q_1 + Q_2}{V} = C_1 + C_2$

Cequivalent of any number of capacitors is

$C_{eq} = C_1 + C_2 + \dots + C_N = \sum_{i=1}^{N} C_i$

Series Connection of Capacitors



Potential difference of capacitors are respectively

$V_1 = \dfrac{Q_1}{C_1}$, $V_2 = \dfrac{Q_2}{C_2}$

Total potential difference and total charge are

$V = V_1 + V_2$, $Q = Q_1 = Q_2$

Then, equivalent capacitor is

$C_{eq} = \dfrac{Q}{V} = \dfrac{Q_1}{\dfrac{Q_1}{C_1} + \dfrac{Q_1}{C_2}} = \dfrac{1}{\dfrac{1}{C_1} + \dfrac{1}{C_2}}$

Any number of capacitors have an equivalent capacity of

$C_{eq} = \left[\sum_{i=1}^{N} \dfrac{1}{C_i}\right]^{-1}$

ELECTROSTATIC ENERGY

How much work would be performed to assemble a group of charges?

Recall electrostatic field is conservative ($\nabla \times \vec{E} = 0$) and hence, can be expressed in terms of a scalar potential ($\nabla \times (\nabla V) = 0$, vector null identity) as $\vec{E} = -\nabla V$

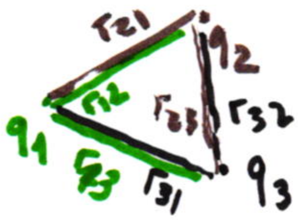
Also, the electric force on a charge q was identified in terms of $\vec{E}$ as $\vec{F} = q\vec{E}$

Besides, the work to be performed in order to bring charge q under $\vec{E}$ field was; (from A to B)

$$W = -\int_A^B \vec{F}\,d\vec{l} = -q\int_A^B \vec{E}\,d\vec{l} = q\underbrace{\int_A^B \nabla V\,dl = q\,[V(B)-V(A)]}_{\text{gradient theorem}}$$

Recall, the potential due to a charge q at any point with distance r is $V = \dfrac{q}{4\pi\varepsilon_0 r}$ ↳ free space

Regarding all these concepts, consider a 3 charges system as below



At first, let locate q_1 and we are not to do work since no $\vec{E}$ field at first stage exist. Now, if we bring q_2, the work to be done is;

$$W_2 = q_2 \underbrace{\frac{q_1}{4\pi\varepsilon_0 r_{12}}}_{\to V_{12}}$$

Now if we bring a third charge to the system; we have to do work against electrostatic fields generated by q_1 and q_2. Namely

$$W_3 = q_3 \underbrace{\frac{q_1}{4\pi\varepsilon_0 r_{13}}}_{V_{13}} + q_3 \underbrace{\frac{q_2}{4\pi\varepsilon_0 r_{23}}}_{\to V_{23}}$$

The entire work to assemble 3 charges is a superposition of all works, that is;

$$W = W_2 + W_3 = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right]$$

If we were to bring N charges together, the work to be performed would be (in the same manner as 3 charge system);

$$W = W_2 + W_3 + \cdots + W_N = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\sum_{j<i}^{N}\frac{q_i q_j}{r_{ij}}$$

Since $r_{ij} = r_{ji}$ and using commutative property ($q_i q_j = q_j q_i$), we may write down

$$W = \frac{1}{2}\,\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{q_i q_j}{r_{ij}}$$

note that ($\frac{1}{2}$) ↳ note that ($j \neq i$)

→ This is the potential energy of charge system. If system were dissolved, this much kinetic energy would be released. Question is where the potential energy is stored

Now, we may write W briefly as;

$$W=\frac{1}{2}\sum_{i=1}^{N} q_i V_i \qquad \text{where} \qquad V_i=\frac{1}{4\pi\varepsilon_0}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{q_j}{r_{ij}}$$

Let's generalize in the presence of continuous charge distribution. $(\Sigma \rightarrow \int)$

$$W=\frac{1}{2}\int \underbrace{\rho_v}_{\text{charge density}}\ \overbrace{V}^{\text{potential generated by the charge density}}\, dv$$

Recall, $\nabla\cdot\vec{D}=\rho_v$ and $\nabla\cdot(\vec{D}V)=\underbrace{V\,\nabla\cdot\vec{D}}_{\text{we need this}}+\vec{D}\cdot\overbrace{\nabla V}^{-\vec{E}}$ (vector identity). So;

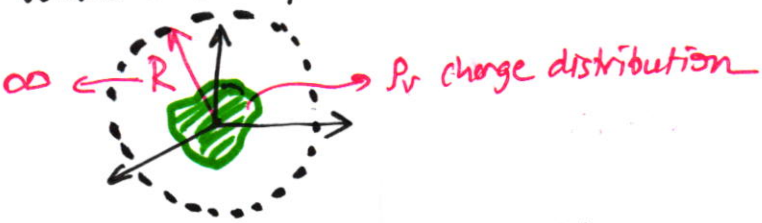
$$W=\frac{1}{2}\int \nabla\cdot\vec{D}\,V\,dv=\frac{1}{2}\left[\underbrace{\int \nabla\cdot(\vec{D}V)\,dv}_{\text{apply gauss' law}}-\int \vec{D}\cdot\underbrace{\nabla V}_{-\vec{E}}\,dv\right]$$

$$\Rightarrow W=\frac{1}{2}\left[\oint_S \vec{D}V\,ds+\int_v \vec{E}\cdot\vec{D}\,dv\right]$$

S: surface bounding volume v

v: the volume enclosing all charges

Now consider volume v is a sphere with radius r which tends to infinity



Consider $\oint_S \vec{D}V\,ds$. $\vec{D}$ and V falls off like $1/r^2$ and $1/r$, respectively $(ds \propto r^2)$

So, in the limit case $\oint_S \vec{D}V\,ds \rightarrow 0$ and can be neglected. Then,

$$W=\frac{1}{2}\int_v \vec{E}\cdot\vec{D}\,dv$$

v: over all space

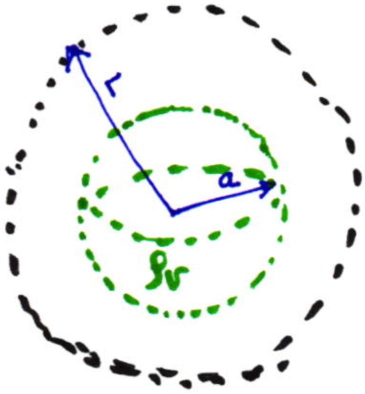
$\rightarrow$ This says that potential energy of the charge system is stored in $\vec{E}$ field $(\vec{D}=\varepsilon_0\vec{E})$

For clarity let's express W in terms of $\vec{E}$ as;

$$\boxed{W=\frac{1}{2}\varepsilon_0\int_v \vec{E}^2\,dv}$$

and since it's volume integral, the energy density is simply $\rightarrow$

$$\boxed{u=\frac{\varepsilon_0 E^2}{2}}$$



Suppose a cloud of charges (with a total amount of Q) uniformly distributed in a sphere with radius a. Let's check the potential energy in the classical sense and with the formula we've derived.

Let there be a small sphere with charge function $q(r)$ at the first stage and let force to bring a dq amount of charge into a spherical shell at $r+dr$ position

The work to be done is;

$$dW = \frac{1}{4\pi\varepsilon_0}\frac{q(r)\,dq}{r}$$

Recall that a uniformly distributed charge in spherical form acts as if the whole charges were in the center. Now, $q(r)$ in terms of ρ_v and dq are

$$q(r) = \frac{4}{3}\pi r^3 \rho_v \,, \quad dq = \underbrace{4\pi r^2}_{\text{surface of shell}} \rho_v \, dr \rightarrow \text{thickness of shell}$$

Then, total amount of work to construct the whole sphere (radius a) is;

$$W = \frac{4\pi}{3\varepsilon_0}\rho_v^2 \int_0^a r^4 dr = \frac{4\pi}{15\varepsilon_0}\rho_v^2 a^5$$

Now, let's derive in terms of $\vec{E}$ $\left(W = \frac{1}{2}\varepsilon_0 \int E^2 dv\right)$

<u>for $r<a$</u>

$$\oint \vec{E}\,d\vec{S} = \frac{1}{\varepsilon_0}\int_0^r \rho_v \overbrace{dv}^{4\pi r^2 dr} \rightarrow \vec{E} = \frac{1}{4\pi\varepsilon_0 r^2}\cdot\rho_v\cdot\frac{4}{3}\pi r^3 = \frac{\rho_v\cdot r}{3\varepsilon_0}\vec{a}_r$$

$$W_I = \frac{1}{2}\int_0^a \varepsilon_0 \cdot \frac{\rho_v^2 r^2}{9\varepsilon_0^2}\cdot \overbrace{\underbrace{4\pi r^2}_{ds} dr}^{dv} = \frac{\varepsilon_0 \rho_v^2}{2\cdot 9\varepsilon_0^2}\cdot 4\pi \int_0^a r^4 dr = \frac{2\pi \rho_v^2}{\varepsilon_0 45} a^5$$

<u>for $r>a$</u>

$$\vec{E} = \frac{1}{4\pi\varepsilon_0 r^2}\cdot\rho_v \frac{4}{3}\pi a^3 = \frac{\rho_v a^3}{3\varepsilon_0 r^2}\vec{a}_r$$

$$W_{II} = \frac{1}{2}\int_a^\infty \varepsilon_0 \frac{\rho_v^2 a^6}{9\varepsilon_0^2 r^4}\cdot \overbrace{\underbrace{4\pi r^2}_{ds} dr}^{dv} = \frac{\rho_v^2 a^6}{18\varepsilon_0}\cdot 4\pi \overbrace{\int_a^\infty \frac{dr}{r^2}}^{1/a} = \frac{\rho_v^2 a^5 2\pi}{9\varepsilon_0}$$

Total work $\rightarrow$ $W_I + W_{II} = \frac{\rho_v^2 a^5 2\pi}{\varepsilon_0}\left(\frac{1}{45}+\frac{1}{9}\right)$ which yields the same result with classical approach.

POISSON'S and LAPLACE EQUATIONS

Recall
$$\left.\begin{aligned}\nabla\cdot\vec{E} &= \rho_v/\varepsilon \\ \nabla\times\vec{E} &= 0 \longrightarrow \vec{E} = -\nabla V\end{aligned}\right\} \boxed{\nabla\cdot(\nabla V) = -\rho_v/\varepsilon}$$
Poisson's equation in electrostatics

$\nabla\cdot\nabla \equiv \nabla^2 \rightarrow$ laplacian operator stands for "the divergence of gradient of ..."

Here, ρ_v and ε are the charge density and dielectric permittivity of simple medium, respectively. ρ may be a function of space coordinates.

Second order poisson's equation holds everywhere while 2^{nd} order derivatives exist.
partial differential

In cartesian coordinates

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} = -\rho_v/\varepsilon \quad [V/m^2]$$

In cylindrical coordinates

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right) + \frac{1}{\rho^2}\frac{d^2V}{d\phi^2} + \frac{d^2V}{dz^2} = -\rho_v/\varepsilon$$

In spherical coordinates

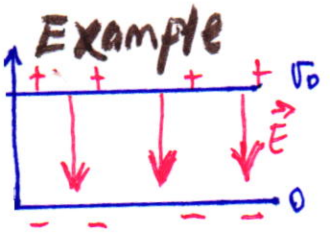
$$\frac{1}{R^2}\frac{d}{dR}\left(R^2\frac{dV}{dR}\right) + \frac{1}{R^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dV}{d\theta}\right) + \frac{1}{R^2\sin\theta}\frac{d^2V}{d\phi} = -\rho_v/\varepsilon$$

If no free charge exists ($\rho_v = 0$) poisson's equation simplifies as;

$\nabla^2 V = 0$ which is known as Laplace's equation

Once the scalar potential is obtained through the solution of partial differential equation, one may determine $\vec{E}$ utilizing $\vec{E} = -\nabla V$.

Example



Potential between plates? Surface density at plates? (fringing effects are ignored)
function

Since no free charges exist between plates, $\nabla^2 V = 0$ also, V varies only in z direction ($dV/dx = dV/dy = 0$). Hence, if we integrate $\nabla^2 V$ two times w.r.t z, we obtain;

$V = Az + B$ where A & B are constants.

Now, utilizing boundary conditions at z=d and z=0

$$\left.\begin{aligned}V(d) &= V_0 = Ad + B \\ V(0) &= 0 = B\end{aligned}\right\} \begin{aligned}B &= 0 \\ A &= V_0/d\end{aligned}$$

So, $\boxed{V(z) = \frac{V_0}{d}z}$

Surface charge densities can be written in terms of $\vec{E}$ field and $\vec{E} = -\nabla V$.

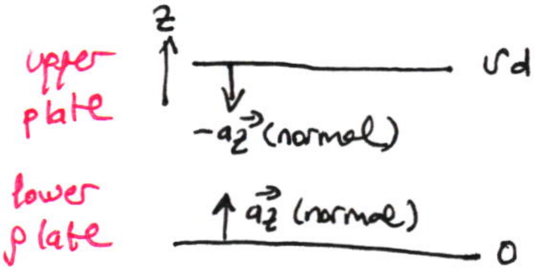
Only z variation exists, hence;

$$\vec{E} = -\nabla V = -\frac{dV}{dz}\vec{a_z} = -\frac{V_0}{d}\vec{a_z}$$

Recall the boundary condition on conductor surface ($E_{1n} = \rho_s/\varepsilon$). For upper plate normal of surface is in $-\vec{a_z}$ direction which is $\vec{a_z}$ for lower plate. Therefore;

$$E_{1n} = \vec{E} \cdot \underbrace{(-\vec{a_z})}_{normal} = \rho_s/\varepsilon \rightarrow \rho_s = \varepsilon \frac{V_0}{d}$$

$$E_{1n} = \vec{E} \cdot \underbrace{(\vec{a_z})}_{normal} = \rho_s/\varepsilon \Rightarrow \rho_s = -\varepsilon \frac{V_0}{d}$$



Example

Method of Images

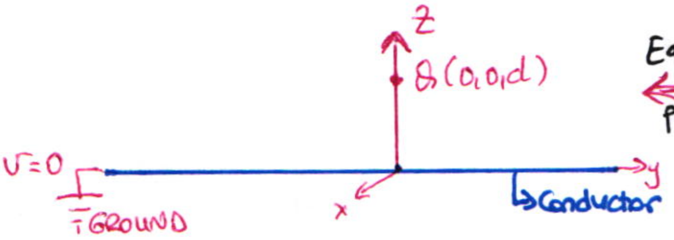


Fig. 1

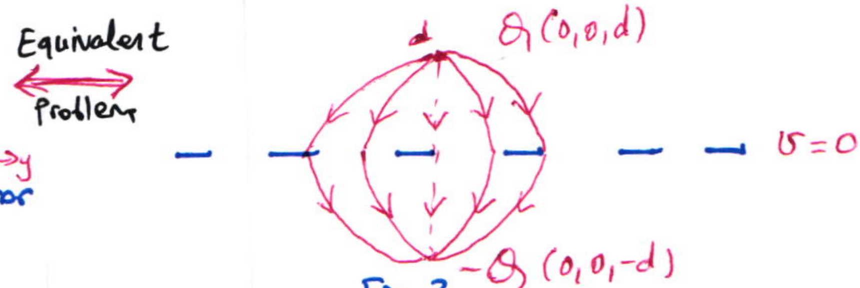


Fig. 2

Consider the problem shown in Fig. 1. A positive Q charge is located at $z=d$ $(x,y=0)$ point and we want to determine $\vec{E}$ field and potential at any point given at $z>0$ half-space. Utilizing Coulomb's law or Gauss' law is impractical for the solution of the considered problem. Another way we examined so far is to solve Poisson's equation under appropriate boundary conditions ($\nabla^2 V = -\rho_s/\varepsilon$, $V(x,y,0)=0$) which is not an easy task.

On the other hand, any solution of a Poisson's (or Laplace) equation which satisfies the same boundary conditions must be the only solution irrespective of the method of the solution. Now, examine Fig. 1 and Fig. 2. In Fig. 1, conductor plane at $z=0$ is grounded. In Fig. 2, a $-Q$ charge is located at $(0,0,-d)$ point which satisfies $V(x,y,0)$ potential to be 0 together with Q charge potential. That is;

$$V(x,y,0) = \left[\frac{Q}{\sqrt{x^2+y^2+d^2}} + \frac{-Q}{\sqrt{x^2+y^2+(-d)^2}}\right]\frac{1}{4\pi\varepsilon_0} = 0$$

Here, $-Q$ charge is the mirror of Q charge and the configuration in Fig. 2 is an easier and alternative way for the solution of the problem considered in Fig. 1. Note that, problem is considered in $z>0$ half-space. So, at any given point at $z>0$, $\vec{E}$ field or V scalar potential can be obtained by solving the problem, depicted in Fig. 2.

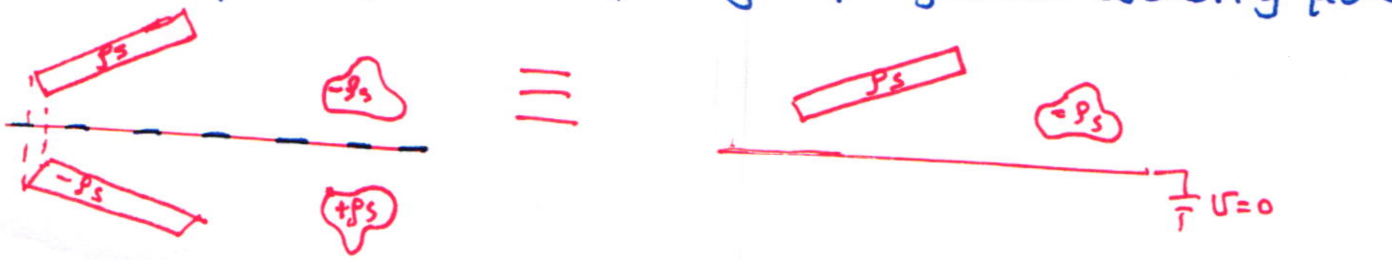
The scalar potential & electric field intensity at any given point at $z>0$ half space now can be determined by;

$$V(x,y,z) = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{-Q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

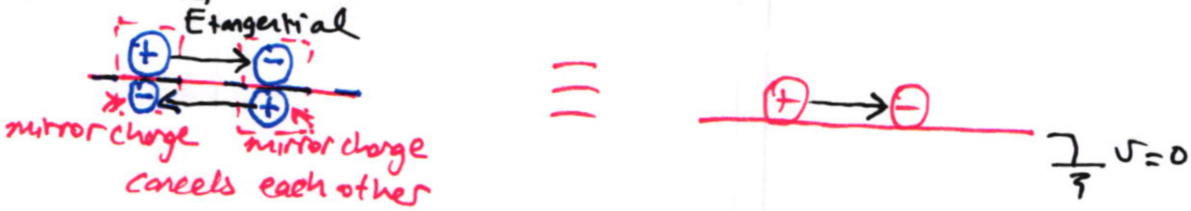
$$\vec{E}(x,y,z) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r_1^2}\vec{a}_{r_1} + \frac{1}{4\pi\varepsilon_0}\frac{-Q}{r_2^2}\vec{a}_{r_2}$$

$r_1^2 = x^2+y^2+(z-d)^2$ $\qquad$ $r_2^2 = x^2+y^2+(z+d)^2$

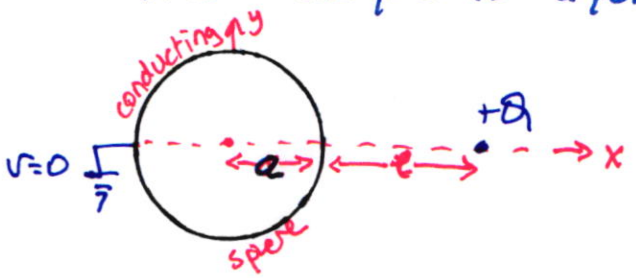
Some examples of method of images for grounded conducting plane



Also, we may bear an analogy for the tangential component of $\vec{E}$ field on conductors.

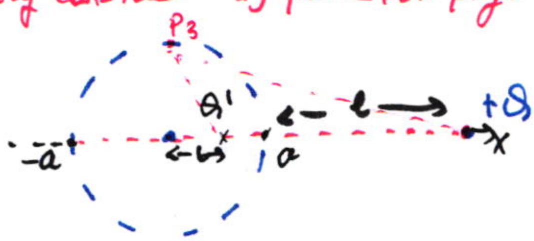


Another example is a point charge and a conducting sphere.



The problem is to determine V or $\vec{E}$ in the space $R^3 - \{x^2+y^2+z^2 \leq a^2\}$. Let's construct the equivalent problem.

Boundary condition is $V=0$ at $x^2+y^2+z^2=a^2$. Due to the symmetry, we may only consider xy plane, simply.



Let's locate Q' at $(b,0,0)$ point. The potential at $P_1(a,0,0)$ and $P_2(-a,0,0)$ are 0 as well as at $P_3(0,a,0)$. The potential at P_3 is;

$$V_{P_3} = \frac{Q'}{4\pi\varepsilon_0\sqrt{b^2+a^2}} + \frac{Q}{4\pi\varepsilon_0\sqrt{a^2+R^2}}\Bigg|_{R=a+\ell} = 0$$

Potentials at P_1 and P_2 are

$$V_{P_1} = \frac{Q'}{4\pi\varepsilon_0(a-b)} + \frac{Q}{4\pi\varepsilon_0(R-a)}\Bigg|_{R=a+\ell}, \quad V_{P_2} = \frac{Q'}{4\pi\varepsilon_0(a+b)} + \frac{Q}{4\pi\varepsilon_0(R+a)}\Bigg|_{R=a+\ell} = 0$$

Utilizing expressions of V_{P_1} and V_{P_2} $\Rightarrow \dfrac{a-b}{R-a} = \dfrac{a+b}{R+a} \Rightarrow b = \dfrac{a^2}{R}$ (position of Q')

Putting b in terms of a & R into V_{P_3}, one may obtain $Q' = -\dfrac{a}{R}Q$ (amount of Q' charge in terms of Q)

Now, one may calculate $\vec{E}$ or V in the defined space $R^3 - \{x^2+y^2+z^2 \leq a\}$ letting whole space is homogeneous in terms of ε.

$\vec{E}, V$, B.C.

$V = e^{-x}\cos y \ , x>0$

$\vec{E}$ field in whole space?

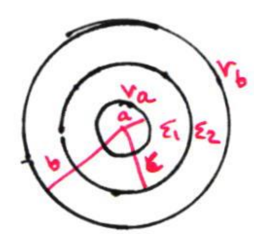
$\rho_s(y,z) = ?$

$V = -\sin x \cdot \cos z \ , x<0$

$$\vec{E} = -\nabla V \Rightarrow \vec{E} = \begin{cases} -\left(-e^{-x}\cos y\ \vec{a_x} - e^{-x}\sin y\ \vec{a_y}\right. , & x>0 \\ -\left(-\cos x\ \cos z\ \vec{a_x} + \sin x\ \sin z\ \vec{a_z}\right. , & x<0 \end{cases}$$

$\vec{D}_{1n} - \vec{D}_{2n} = \varepsilon_1 \vec{E}_{1n} - \varepsilon_2 \vec{E}_{2n} = \rho_s(y,z)$, $x=0$ plane. Note that normal components of $\vec{E}_1$ and $\vec{E}_2$ are x-components on yz ~~plane~~ plane. Hence

$$\varepsilon_1 e^{-x}\cos y - \varepsilon_2 \cos x \cos z \Big|_{x=0} = \varepsilon_1 \cos y - \varepsilon_2 \cos z = \rho_s$$



Two conducting cylinders ($r=a$, $r=b$) are concentric and are at potentials V_a and V_b, respectively.

$W, V, \vec{E}, Q$?

$\nabla^2 V = 0$ ($a<r<b$) Laplace's equation. In cylindrical coordinates;

$$\nabla^2 V = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right) + \frac{1}{\rho^2}\frac{d^2 V}{d\phi} + \frac{d^2 V}{dz^2} = 0$$

Note that, $V(\phi)$ & $V(z) = 0$. Hence $\nabla^2 V = \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right) = 0$

$\Rightarrow$ we may integrate with respect to ρ, 2 times which yields;

$V(\rho) = A\ln\rho + B$ " $\left(\frac{1}{\rho}\frac{d}{d\rho}\left(\frac{d(A\ln\rho + B)}{d\rho}\right)\right) = 0$ " where A & B are coefficients which can be determined via boundary conditions

Boundary conditions $r=a$, $r=b$, $r=c$

$V(a) = V_a$ $V(b) = V_b$ $V_1(c) = V_2(c)$ $D_1(c) = D_2(c)$

$\Rightarrow$ $A_1 \ln a + B_1 = V_a$

$A_2 \ln b + B_2 = V_b$

$A_1 \ln c + B_1 = A_2 \ln c + B_2$

$D_{1n} - D_{2n} = \rho_s = 0$

4 unknowns, 4 equations, sufficient to determine $A_{1,2}$ $B_{1,2}$

$D_{1n} = \varepsilon_1 E_{1n}$ and $\vec{E}_1 = -\nabla V_1$. Note that $\vec{E}_1$ has only ρ component

$D_{2n} = \varepsilon_2 E_{2n}$ and $\vec{E}_2 = -\nabla V_2$. Note that $\vec{E}_2$ has only ρ component

$\Rightarrow \varepsilon_2 - A_2 / c = \varepsilon_1 - A_1 / c$

Once A_1, A_2, B_1, B_2 are determined, one may find V_1, V_2 as well as $\vec{E}_1 = -\nabla V_1$ and $\vec{E}_2 = -\nabla V_2$. Then using gauss' law $\oint \vec{E}\,d\vec{s} = \Sigma Q/\varepsilon_0$ total amount of Q can be determined. (Note that $Q_A = Q_B$)

We will perform tedious calculations at class

$$W_1 = \frac{1}{2}\int_0^L\int_0^{2\pi}\int_a^c \varepsilon_1 E_1^2 \rho\, d\rho\, d\phi\, dz \quad , \quad W_2 = \frac{\varepsilon_2}{2}\int_0^L\int_0^{2\pi}\int_c^b E_2^2 \rho\, d\rho\, d\phi\, dz$$