Three main topics will be dealt on vector analysis;

1- Vector algebra - addition, substraction, multiplication of vectors

2- Orthogonal coordinate systems - Cartesian, cylindirical & spherical coordinates

3- Vector Calculus - differentiation and integration of vectors; line, surface and volume integrals "del" operator; gradient, divergence and curl operations

vector calculus pertains to the differentiation and integration of vectors.

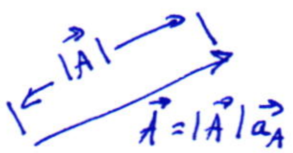
2.2 Vector addition & substraction

ADDITION

$\vec{A} = |\vec{A}| \vec{a}_A$ — $|\vec{A}|$ → has the unit and dimension of $\vec{A}$; $\vec{a}_A$ → dimensionless unit vector with a unity magnitude having the direction of $\vec{A}$

$\vec{a}_A = \dfrac{\vec{A}}{|\vec{A}|}$

1. Parallelogram Rule
2. Head-to-tail Rule



Commutative Law $\vec{A}+\vec{B} = \vec{B}+\vec{A}$ Associative Law: $(\vec{A}+\vec{B})+\vec{C} = \vec{A}+(\vec{B}+\vec{C})$

SUBT

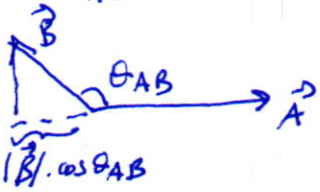
$\vec{A}-\vec{B} = \vec{A}+(-\vec{B})$ $\quad -\vec{B} = |\vec{B}| \cdot (-\vec{a}_B)$

2.3 PRODUCTS of VECTORS

$k\vec{A} = \vec{a}_A (k|\vec{A}|)$, where k is a positive scalar

2.3.1 Scalar or Dot Product

$\vec{A}\cdot\vec{B} \triangleq |\vec{A}||\vec{B}| \cos\theta_{AB}$ where $\theta_{AB} < \pi$ radians (180°) is the smaller angle between $\vec{A}$ & $\vec{B}$



$\vec{A}\cdot\vec{A} = |\vec{A}|^2$, $|\vec{A}| = \sqrt{\vec{A}\cdot\vec{A}}$

Commutative Law $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$ Distributive Law $\vec{A}(\vec{B}+\vec{C}) = \vec{A}\vec{B}+\vec{A}\vec{C}$

2-3.2 Vector or Cross Product

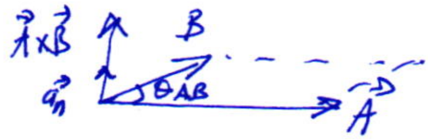
Denoted by $\vec{A}\times\vec{B}$ is a vector perpendicular to the plane containing $\vec{A}$ & $\vec{B}$
Its magnitude is $|\vec{A}||\vec{B}|\sin\theta_{AB}$ where θ_{AB} is the smaller angle between $\vec{A}$ & $\vec{B}$
and its direction follows that of the thumb of the right hand when the fingers rotate from $\vec{A}$ to $\vec{B}$

$$\vec{A}\times\vec{B} \triangleq \vec{a_n}\,(|\vec{A}||\vec{B}|\sin\theta_{AB})$$

$$\vec{B}\times\vec{A} = -\vec{A}\times\vec{B}$$

$$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$$ distributive law

$$\vec{A}\times(\vec{B}\times\vec{C}) \neq (\vec{A}\times\vec{B})\times\vec{C}$$ NOT associative



$$\vec{A}\cdot(\vec{B}\times\vec{C}) = -\vec{A}\cdot(\vec{C}\times\vec{B}) = -\vec{B}\cdot(\vec{A}\times\vec{C}) = -\vec{C}\cdot(\vec{A}\times\vec{B})$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{C}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{C}\times\vec{A})$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{A}\cdot\vec{C}) - \vec{C}\cdot(\vec{A}\cdot\vec{B})$$

2.4 ORTHOGONAL COORDINATE SYSTEMS ($\vec{a}_{u1}, \vec{a}_{u2}, \vec{a}_{u3} \triangleq$ base vectors)

$$\vec{a}_{u1}\times\vec{a}_{u2} = \vec{a}_{u3}$$
$$\vec{a}_{u2}\times\vec{a}_{u3} = \vec{a}_{u1}$$
$$\vec{a}_{u3}\times\vec{a}_{u1} = \vec{a}_{u2}$$

$$\vec{a}_{u1} \rightarrow \vec{a}_{u2} \rightarrow \vec{a}_{u3} \rightarrow \vec{a}_{u1}$$

$$\vec{A} = A_{u1}\cdot\vec{a}_{u1} + A_{u2}\,\vec{a}_{u2} + A_{u3}\cdot\vec{a}_{u3}$$

$$\vec{dl} = \vec{a}_{u1}\,(h_1\cdot du_1) + \vec{a}_{u2}\,(h_2\cdot du_2) + \vec{a}_{u3}\,(h_3\cdot du_3)$$ $h_i \triangleq$ metric coefficient

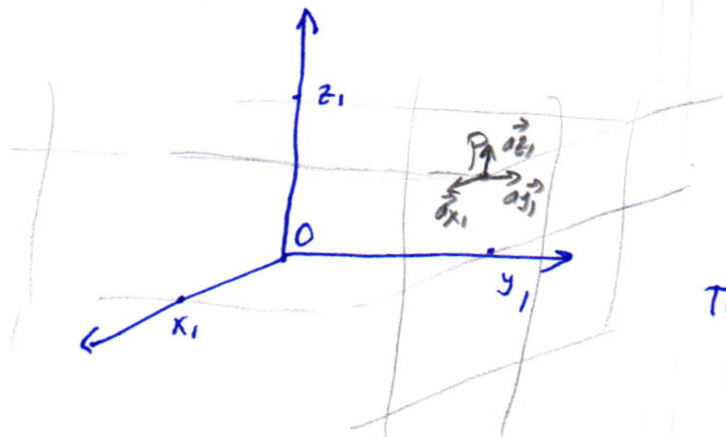
$$dv = h_1\cdot h_2\cdot h_3\, du_1\cdot du_2\cdot du_3$$

$$\vec{ds}_1 = \vec{a}_{u1}\,(h_2 h_3\, du_2 du_3)\ ;\quad \vec{ds}_2 = \vec{a}_{u2}\,(h_1 h_3\, du_1 du_3);\quad \vec{ds}_3 = \vec{a}_{u3}\,(h_1 h_2\, du_1 du_2)$$

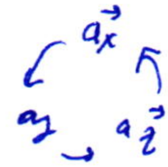
2-4.1 Cartesian Coordinates

$(u_1, u_2, u_3) = (x, y, z)$

A point $P(x_1, y_1, z_1)$ in Cartesian coordinates is the intersection of three planes specified by $x = x_1$, $y = y_1$, $z = z_1$



$\vec{a}_x \times \vec{a}_y = \vec{a}_z$
$\vec{a}_y \times \vec{a}_z = \vec{a}_x$
$\vec{a}_z \times \vec{a}_x = \vec{a}_y$



The position vector to the point $P(x_1, y_1, z_1)$ is

$$\vec{OP} = \vec{a}_x x_1 + \vec{a}_y y_1 + \vec{a}_z z_1$$

A vector $\vec{A}$ in Cartesian coordinates can be written as;

$$\vec{A} = A_x \cdot \vec{a}_x + A_y \cdot \vec{a}_y + A_z \cdot \vec{a}_z$$

$$\vec{A} \cdot \vec{B} = A_x \cdot B_x + A_y \cdot B_y + A_z \cdot B_z$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \vec{a}_x \cdot (A_y \cdot B_z - A_z B_y) - \vec{a}_y (A_x \cdot B_z - A_z \cdot B_x) + \vec{a}_z (A_x \cdot B_y - A_y \cdot B_x)$$

Since x, y, z are lengths themselves, all three metric coefficients are unity, that is $h_1 = h_2 = h_3 = 1$. Expressions for differential length, differential area and differential volume are;

diff. length
$$\vec{dl} = \vec{a}_x dx + \vec{a}_y dy + \vec{a}_z dz$$

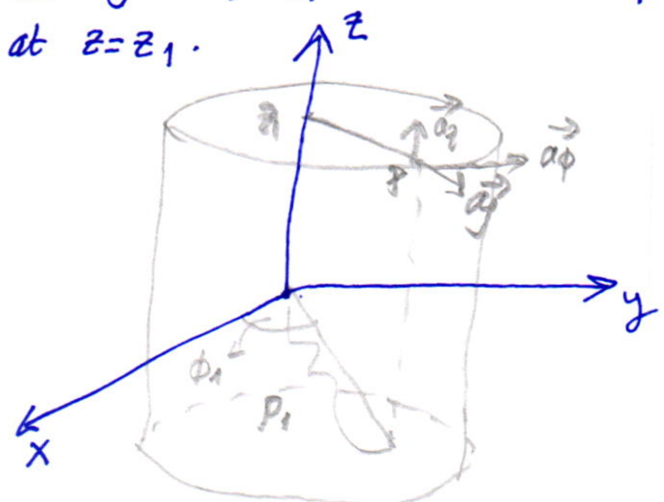
diff. areas
$$\vec{ds}_x = \vec{a}_x \, dy \cdot dz$$
$$\vec{ds}_y = \vec{a}_y \, dx \, dz$$
$$\vec{ds}_z = \vec{a}_z \, dx \, dy$$

diff. vol.
$$dv = dx \, dy \, dz$$

2-4.2 Cylindirical Coordinates

$(u_1, u_2, u_3) = (\rho, \phi, z)$

A point $P(\rho_1, \phi_1, z_1)$ in cylindirical coordinates is the intersection of a circular cylindirical surface $\rho = r_1$, a half-plane containing the z-axis and making an angle $\phi = \phi_1$ with the x-z plane and a plane parallel to the xy plane at $z = z_1$.



$$\vec{a_\rho} \times \vec{a_\phi} = \vec{a_z}$$
$$\vec{a_\phi} \times \vec{a_z} = \vec{a_\rho}$$
$$\vec{a_z} \times \vec{a_\rho} = \vec{a_\phi}$$

A vector $\vec{A}$ in cylindirical coordinates can be written as;

$$\vec{A} = A_\rho \cdot \vec{a_\rho} + A_\phi \cdot \vec{a_\phi} + A_z \cdot \vec{a_z}$$

Two of the three coordinates, ρ and z (u_1 and u_3) are themselves lengths, hence $h_1 = h_3 = 1$. However ϕ is an angle requiring a metric coefficient $h_2 = \rho$ to convert $d\phi$ to dl.

diff. length

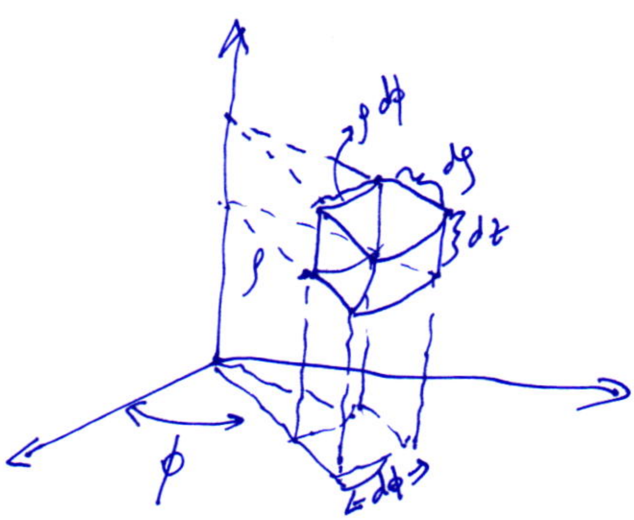
$$\vec{dl} = \vec{a_\rho} \cdot d\rho + \vec{a_\phi}\, \rho \cdot d\phi + \vec{a_z} \cdot dz$$

diff. areas

$$\vec{ds_\rho} = \vec{a_\rho}\, \rho\, d\phi\, dz$$
$$\vec{ds_\phi} = \vec{a_\phi}\, d\rho \cdot dz$$
$$\vec{ds_z} = \vec{a_z}\, \rho \cdot d\rho \cdot d\phi$$

diff. vol

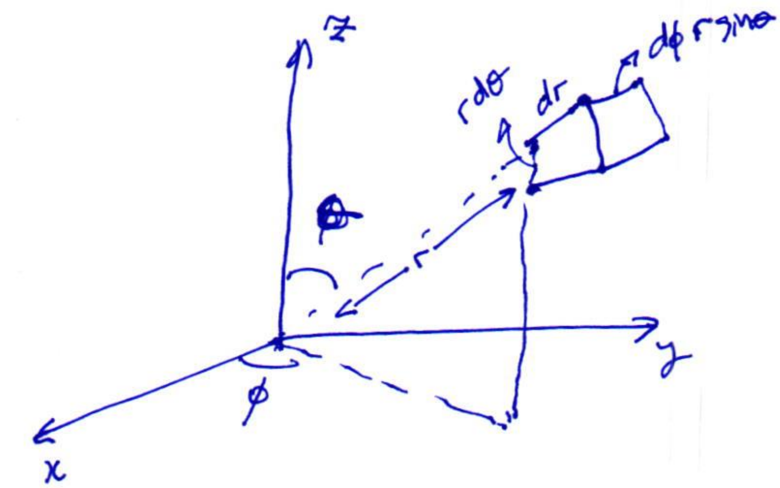
$$dv = \rho\, d\rho \cdot d\phi \cdot dz$$

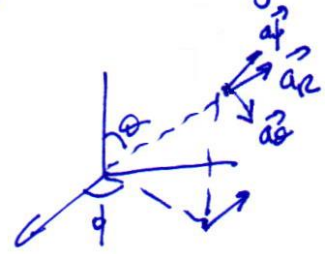


2-4.3 Spherical Coordinates

$(u_1, u_2, u_3) = (R, \theta, \phi)$

A point $P(r_1, \theta_1, \phi_1)$ in spherical coordinates is specified as the intersection of the following three surfaces; a spherical surface centered at the origin with a radius $r = r_1$; a right circular cone with its apex at the origin





$$\vec{a_r} \times \vec{a_\theta} = \vec{a_\phi}$$

$$\vec{a_\theta} \times \vec{a_\phi} = \vec{a_r}$$

$$\vec{a_\phi} \times \vec{a_r} = \vec{a_\theta}$$

$$\vec{A} = A_r \cdot \vec{a_r} + A_\theta \cdot \vec{a_\theta} + A_\phi \cdot \vec{a_\phi}$$

$$\vec{dl} = dr \cdot \vec{a_r} + r\,d\theta\, \vec{a_\theta} + r\sin\theta\, d\phi \cdot \vec{a_\phi}$$

$$\vec{ds_r} = r\,d\theta \cdot r\sin\theta\, d\phi \cdot \vec{a_r}$$

$$\vec{ds_\theta} = r\sin\theta\, d\phi\, dr\, \vec{a_\theta}$$

$$\vec{d\phi} = dr \cdot r\,d\theta \cdot \vec{a_\phi}$$

$$dv = R^2 \sin\theta\, d\theta\, d\phi\, dr$$

$\vec{A} = A_\rho \cdot \vec{a_\rho} + A_\phi \cdot \vec{a_\phi} + A_z \vec{a_z}$ in cartesian coordinates?

$A_x = A_\rho \cos\phi - A_\phi \sin\phi$

$A_y = A_\rho \sin\phi + A_\phi \cos\phi$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$x = \rho\cos\phi \,,\; y = \rho\sin\phi \,,\; z = z$

$\rho = \sqrt{x^2+y^2} \,,\; \phi = \tan^{-1} y/x \,,\; z = z$

$\vec{A} = \vec{a_R} A_R + \vec{a_\theta} A_\theta + \vec{a_\phi} A_\phi$ in cartesian coordinates

$A_x = A_R \sin\theta\cos\phi + A_\theta \cos\theta\cos\phi - A_\phi \sin\phi$

$A_y = A_R \sin\theta \sin\phi + A_\theta \cos\theta \sin\phi + A_\phi \cos\phi$

$A_z = A_R \cos\theta - A_\theta \sin\theta$

$x = R\sin\theta\cos\phi \,,\; y = R\sin\theta\sin\phi \,,\; z = R\cos\theta$

$R = \sqrt{x^2+y^2+z^2} \,,\; \theta = \tan^{-1} \dfrac{\sqrt{x^2+y^2}}{z} \,,\; \phi = \tan^{-1} \dfrac{y}{x}$

GRADIENT of a SCALAR FIELD

$$dF = \left(\frac{dF}{dl_1}\right) \cdot dl_1 + \left(\frac{dF}{dl_2}\right) dl_2 + \left(\frac{dF}{dl_3}\right) dl_3$$

$$= \underbrace{\left(\frac{dF}{dl_1}\vec{a_1} + \frac{dF}{dl_2}\vec{a_2} + \frac{dF}{dl_3}\vec{a_3}\right)}_{\nabla F} \cdot \underbrace{\left(\vec{a_1} \cdot dl_1 + dl_2 \cdot \vec{a_2} + dl_3 \cdot \vec{a_3}\right)}_{\vec{dl}}$$

$$= \nabla F \cdot \vec{dl} = |\nabla F| \cdot |dl| \cdot \cos\theta$$

↳ The vector that represents both the magnitude and the direction of the maximum space rate of increase of a scalar as the gradient of that scalar.

$$\boxed{\begin{aligned} \vec{dl} &= \vec{a_1}\, dl_1 + \vec{a_2}\, dl_2 + \vec{a_3}\, dl_3 \\ &= \vec{a_1}(h_1 \cdot da_1) + \vec{a_2}(h_2 da_2) + \vec{a_3}(h_3 da_3) \qquad h_i : \text{metric coefficient} \end{aligned}}$$

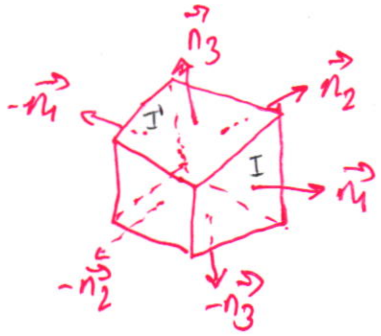
$$\Rightarrow \nabla F = \frac{1}{h_1}\frac{dF}{da_1} \cdot \vec{a_1} + \frac{1}{h_2}\frac{dF}{da_2} \cdot \vec{a_2} + \frac{1}{h_3}\frac{dF}{da_3}\vec{a_3}$$

$$\int_A^B \nabla F \cdot \vec{dl} = F(B) - F(A) \rightarrow \text{fundamental theorem of gradient}$$

DIVERGENCE of a VECTOR FIELD

$$div\ \vec{D} = \lim_{\Delta v \to 0} \frac{\oint_s \vec{D}\cdot d\vec{s}}{\Delta v}$$

"Net outward flux of $\vec{F}$ per unit volume as volume about the point tends to zero"



$$\vec{F}\cdot d\vec{s} = \vec{F}\cdot\vec{n}\cdot ds$$

For I and I' surfaces

$\vec{n}_I = -\vec{n}_{I'}$

$ds_{I'} = h_2 \cdot h_3 \cdot da_2\, da_3$

$ds_I = (h_2 + dh_2)(h_3 + dh_3)\, da_2\, da_3$

$\vec{F}_I = F_I(a_1 + da_1,\ a_2, a_3)\cdot \vec{a}_{F_I}$

$\vec{F}_{I'} = F_{I'}(a_1, a_2, a_3)\,\vec{a}_{F_{I'}}$

Then, for surface I and I' $\vec{F}\cdot\vec{n}\cdot ds \cancel{/} dV =$

$$\left[\overbrace{F_I(a_1 + da_1, a_2, a_3)}^{\text{surface I}}\overbrace{(h_2 + dh_2)(h_3 + dh_3)\, da_2 \cdot da_3}^{ds_I} \cdot \vec{n}_1 - F_I(a_1, a_2, a_3)(h_2 \cdot h_3 \cdot da_2\, da_3)\,\vec{n}_1 \right]$$

$\boxed{h_1 da_1\ h_2 da_2\ h_3 da_3}$ → approximate infinitesimally small volume

$$\simeq \frac{d(F_I h_2 h_3)}{h_1 h_2 h_3\, da_1}$$

Through similar process, one may obtain a general expression for the divergence of a vector such that

$$\nabla\cdot\vec{D} = div\ \vec{D} = \frac{1}{h_1 h_2 h_3}\left[\frac{d(h_2 h_3 D_1)}{da_1} + \frac{d(h_1 h_3 D_2)}{da_2} + \frac{d(h_1 h_2 D_3)}{da_3}\right]$$

Divergence Theorem

$(\nabla \cdot \vec{D})_j \cdot \Delta v_j = \oint_{S_j} \vec{D}\, \vec{ds}$ A volume V is subdivided into N, small differential volume, of which Δv_j is typical.

$$\underbrace{\lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} (\nabla \cdot \vec{D})_j \, \Delta v_j \right]}_{} = \lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} \oint_{S_j} \vec{D} \cdot \vec{ds} \right]$$

by definition, it is volume integral of $\nabla \cdot \vec{D}$, namely $= \oint_{v} \nabla \cdot \vec{D} \cdot dv$

Hence

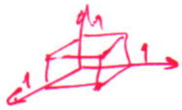
$$\oint_{v} \nabla \cdot \vec{D}\, dv = \underbrace{\lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} \int_{S_j} \vec{D} \cdot \vec{ds} \right] = \oint_{S} \vec{D} \cdot \vec{ds}}_{}$$

Since inward and outward flux cancel each other through the volume except the surronding surface

$$\boxed{\int_{v} \nabla \cdot \vec{D}\, dv = \oint_{S} \vec{D} \cdot \vec{ds}}$$

Requires the vector field $\vec{D}$, as well as its first derivatives exist and be continuous both in v and on S.

Ex $\vec{D} = x^2 \vec{a_x} + xy \vec{a_y} + yz \vec{a_z}$

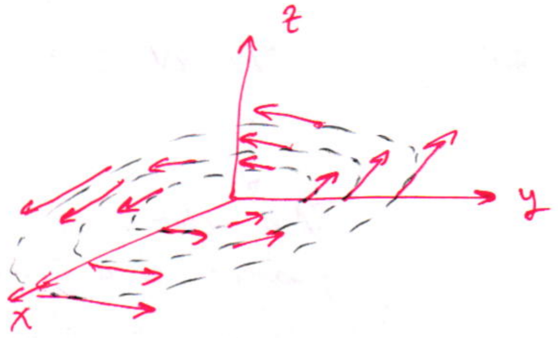


$\int \nabla \cdot \vec{D}\, dv = \int \vec{D}\, \vec{ds} = 2$

CURL of a VECTOR FIELD

$$\nabla \times \vec{A} \triangleq \lim_{\Delta s \to 0} \frac{1}{\Delta S} \vec{a}_n \oint_C \vec{A}\, d\vec{l}$$

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \vec{a}_1 h_1 & \vec{a}_2 h_2 & \vec{a}_3 h_3 \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$



Plot of $\vec{A} = -y\vec{a}_x + x\vec{a}_y$

$$\nabla \times \vec{A} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = 2\vec{a}_z$$

Stokes's Theorem

For a very small differential area Δs_j, bounded by a contour C_j, the definition of $\nabla \times \vec{A}$ leads to

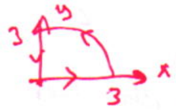
$$(\nabla \times \vec{A})_j (\vec{\Delta s}_j) = \oint_{C_j} \vec{A} \cdot d\vec{l}$$

$$\lim_{\Delta v_j \to 0} \sum_{j=1}^{N} (\nabla \times \vec{A})_j (\vec{\Delta s}_j) = \lim_{\Delta v_j \to 0} \sum_{j=1}^{N} \oint_{C_j} \vec{A} \cdot d\vec{l}$$

$$\boxed{\int_S (\nabla \times \vec{A})\, d\vec{s} = \oint_C \vec{A}\, d\vec{l}}$$

"the surface integral of the curl of a vector field over an open surface is equal to the closed line integral of the vector along the contour bounding the surface"

Ex $\vec{F} = xy\vec{a}_x - 2x\vec{a}_y$



$$\nabla \times \vec{F} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & -2x & 0 \end{vmatrix} = -(2+x)\vec{a}_z$$

$$\int_S \nabla \times \vec{F}\, d\vec{s} = \int_0^3 \int_0^{\sqrt{9-y^2}} (\nabla \times \vec{F})\, \vec{a}_z\, dx\, dy$$

$$= \int_0^3 \left[\int_0^{\sqrt{9-y^2}} -(2+x)\, dx \right] dy$$

$$= -9(1+\pi/2)$$

$$\oint \vec{F}\, d\vec{l}$$

TWO NULL IDENTITIES

Identity I $\nabla \times (\nabla V) \equiv 0$ The curl of the gradient of any scalar field is identically zero

$\downarrow$

" If a vector field is curl-free, then it can be expressed as the gradient of a scalar field. "

if $\nabla \times \vec{E} = 0 \rightarrow \vec{E} = -\nabla V$ ↳ vector potential (electric scalar potential)

$\downarrow$

" an irrotational (ie conservative) vector field can always be expressed as the gradient of a scalar field. "

Identity II $\nabla \cdot (\nabla \times \vec{A}) \equiv 0$ The divergence of the curl of any vector field is identically zero.

$\downarrow$

If a vector field is divergenceless, then it can be express as the curl of another vector field

if $\nabla \cdot \vec{B} = 0 \rightarrow \vec{B} = \nabla \times \vec{A}$ ↳ magnetic vector potential

Proof of Identity I

$$\int_S [\nabla \times (\nabla V)]\, ds = \oint_C \nabla V\, d\vec{l}$$ → which starts and ends at the same point, hence no gradient occurs

Proof of Identity II

$$\int_v [\nabla \cdot (\nabla \times \vec{A})]\, dv = \oint_S \nabla \times \vec{A}\, d\vec{s} = \underbrace{\oint_{c_1} \vec{A}\, d\vec{l} + \oint_{c_2} \vec{A}\, d\vec{l}}_{\text{cancel each other}}$$

HELMHOLTZ THEOREM