# $\vec{E}$ field of a propagating electromagnetic wave in free-space is given $\vec{E} = A\cos(w(t - z/c))\vec{a_y}$. Determine $\vec{H}$ for this wave?

(c: speed of wave, t: time, z: position in z-axis, w: angular frequency)

$$\nabla \times \vec{E} + \frac{d\vec{B}}{dt} = 0 \Rightarrow \frac{d\vec{H}}{dt} = -\frac{1}{\mu_0}\nabla \times \vec{E} \qquad (\vec{B} = \mu_0 \vec{H})$$

↳ free-space

where
$$\nabla \times \vec{E} = \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ d/dx & d/dy & d/dz \\ 0 & E_y & 0 \end{vmatrix} = -\vec{a_x}\frac{dE_y}{dz}$$

↳ $\vec{E}$ has only y component

$$\Rightarrow \nabla \times \vec{E} = -\vec{a_x}\frac{d}{dz}\left(A\cos(w(t - z/c))\right) = -\frac{Aw}{c}\sin\left(w\left(t - \frac{z}{c}\right)\right)\vec{a_x}$$

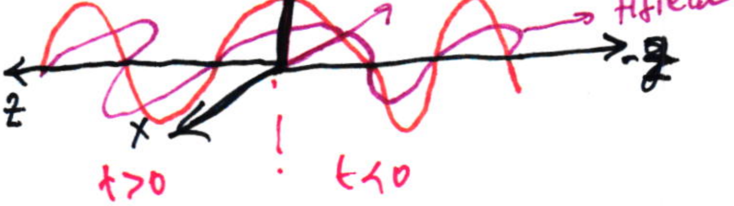
$$\Rightarrow \frac{d\vec{H}}{dt} = \frac{Aw}{\mu_0 c}\sin(w(t - z/c))\vec{a_x}$$

$$\Rightarrow \vec{H} = \frac{Aw}{\mu_0 C}\int \sin(w(t - z/c))\,dt\,\vec{a_x}$$

$$= \frac{Aw}{\mu_0 c}\,\frac{-1}{w}\left[\cos(w(t - z/c)) + \underset{\sim}{C}\right]\vec{a_x}$$

↳ integral constant which may represent static field if exists which can be ignored

$$\Rightarrow \boxed{\vec{H} = \frac{-A}{\mu_0 c}\cos(w(t - z/c))\,\vec{a_x}}$$




A simple representation of propagating wave

# $\vec{E}$ field of a propagating wave in free-space is given as

$$\vec{E} = \frac{100}{\rho} \sin(\alpha z) \cos(\omega t)\, \vec{a_\rho} \ [V/m]$$ in cylindrical coordinate system.

Determine $\vec{H}$ and $\alpha$ ? ($\alpha$: constant). No sources exist in the medium ($\rho_\sigma \equiv 0$, $\vec{J} \equiv 0$)

$$\nabla \times \vec{E} + \frac{d\vec{B}}{dt} = 0, \quad \nabla \times \vec{E} = \frac{1}{\rho} \begin{vmatrix} \vec{a_\rho} & \rho \vec{a_\phi} & \vec{a_z} \\ d/d\rho & d/d\phi & d/dz \\ E_\rho & \rho \underbrace{E_\phi}_{0} & \underbrace{E_z}_{0} \end{vmatrix}$$

$$\Rightarrow \nabla \times \vec{E} = \frac{1}{\rho}\left(-\rho \vec{a_\phi}\left[-\frac{dE_\rho}{dz}\right]\right) = +\vec{a_\phi}\, \frac{100\alpha}{\rho} \cos(\alpha z) \cos\omega t$$

$$\frac{d\vec{B}}{dt} = \mu_0 \frac{d\vec{H}}{dt} = -\nabla \times \vec{E} = \frac{d\vec{H}}{dt} = \frac{-1}{\mu_0} \frac{100\alpha}{\rho} \cos(\alpha z) \int \cos(\omega t) dt\, \vec{a_\phi}$$

$$\Rightarrow \vec{H} = -\frac{100\alpha}{\mu_0 \omega \rho} \cos(\alpha z) \sin(\omega t)\, \vec{a_\phi}$$

Thumb rule is, if $\vec{E}$ & $\vec{H}$ are related to an EM wave, they should satisfy Maxwell's Equations. Then $\nabla \cdot \vec{D} = \underbrace{\rho_\sigma}_{\text{no source}} = 0$, $\nabla \cdot \vec{B} = 0$ & $\nabla \times \vec{H} - \frac{d\vec{D}}{dt} = \underbrace{\vec{J}}_{\text{no source}} = 0$

To determine $\alpha$, we may check these relations. If one tries $\nabla \cdot \vec{D} = \nabla \cdot \vec{B} = 0$, no knowledge about $\alpha$ can be obtained. Let's utilize Ampere-Maxwell equation.

$$\nabla \times \vec{H} = \frac{d\vec{D}}{dt} = \varepsilon_0 \frac{100}{\rho} \sin(\alpha z) \sin(\omega t)(-\omega) \cdot \vec{a_\rho} \quad \left(\equiv \varepsilon_0\, d\vec{E}/dt\right)$$

$$= \frac{1}{\rho} \begin{vmatrix} \vec{a_\rho} & \rho \vec{a_\phi} & \vec{a_z} \\ d/d\rho & d/d\phi & d/dz \\ 0 & \rho \cdot A_\phi & 0 \end{vmatrix} = \frac{1}{\rho}\left[\vec{a_\rho} \frac{-d(A_\phi \rho)}{dz} + \vec{a_z} \overbrace{\frac{d(\rho A_\phi)}{d\phi}}^{0}\right]$$

$$\Rightarrow \nabla \times \vec{H} = -\frac{\vec{a_\rho}}{\rho} \frac{100\alpha^2}{\mu_0 \omega} \sin(\alpha z) \sin(\omega t)$$

$$= -\omega \varepsilon_0 \frac{100}{\rho} \sin(\alpha z) \sin(\omega t)\, \vec{a_\rho} = \varepsilon_0 \frac{d\vec{E}}{dt}$$

$$\alpha^2 = \omega^2 \varepsilon_0 \mu_0 \rightarrow \alpha = \omega \sqrt{\varepsilon_0 \mu_0}$$

# In vacuum, $\vec{B}$ is observed in cylindrical coordinates as given below;

$$\vec{B} = \begin{cases} B_0 \cdot \sin \omega t \, \vec{a_z}, & \rho < a \\ 0, & \rho > a \end{cases}$$

Determine $\vec{E}$ field? Determine the source?

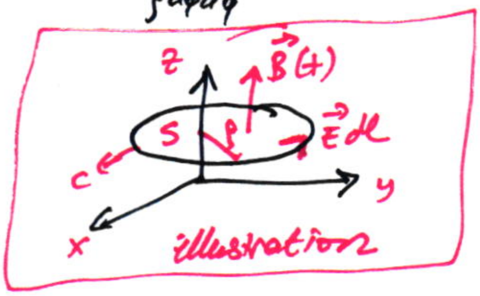
Using Faraday Law, $\oint_C \vec{E}\,\vec{dl} = -\frac{d}{dt}\int_S \vec{B}\,\vec{ds}$. C: contour attached to open surface S


For $\rho < a$

$\vec{E}\cdot\vec{a_\phi} = E_\phi$

$$\oint \underbrace{\vec{E}\,\vec{dl}}_{\rho d\phi \vec{a_\phi}} = E_\phi \cdot \rho \, 2\pi = -\frac{d}{dt}\int \vec{B}\,\underbrace{\vec{ds}}_{\rho d\rho d\phi \vec{a_z}}$$

$\vec{a_z} \rightarrow \vec{B} \parallel \vec{ds}$




$$= -\frac{d}{dt}\int_{\rho=0}^{\rho}\int_{\phi=0}^{2\pi} B_0 \cdot \rho \cdot \sin \omega t \, d\rho \, d\phi$$

$$E_\phi \, \rho \cdot 2\pi = \pi \rho^2 B_0 \cdot -d/dt(\sin \omega t) = -\pi \rho^2 B_0 \, \omega \cos(\omega t)$$

$\Rightarrow E_\phi = \frac{-\rho}{2} B_0 \, \omega \cos(\omega t)$. In other words $\boxed{\vec{E} = -\rho \frac{B_0 \, \omega}{2} \cos(\omega t) \vec{a_\phi}}$

For $\rho > a$

$$\oint \vec{E}\,\vec{dl} = E_\phi \cdot 2\pi\rho = -d/dt \int_{\rho=0}^{a}\int_{\phi=0}^{2\pi} \vec{B}\,\vec{ds} = -\pi a^2 B_0 \cos \omega t \cdot \omega$$

for $\rho > a$, $\vec{B} = 0$

$$\Rightarrow \boxed{E_\phi = \frac{-\omega a^2}{2\rho} B_0 \cos \omega t}$$

Maxwell eq. to determine sources ($\rho_v$, $\vec{J}$)

$\nabla \cdot \vec{D} = \rho_v \Rightarrow \nabla \cdot \vec{E} = \rho_v / \varepsilon_0 = 0 \rightarrow \boxed{\rho_v = 0}$

$\hookrightarrow \frac{dE_\phi}{d\phi} = 0$

$$\nabla \times H - \frac{d\vec{D}}{dt} = \vec{J} \; ; \; \nabla \times \vec{H} = \frac{1}{\rho}\begin{vmatrix} \vec{a_\rho} & \rho \vec{a_\phi} & \vec{a_z} \\ \cdot & \cdot & \cdot \\ 0 & 0 & H_z \end{vmatrix} = 0 \Rightarrow \vec{J} = -\frac{d\vec{D}}{dt} = -\varepsilon_0 \frac{d\vec{E}}{dt}$$

$$\Rightarrow \vec{J} = \begin{cases} -\varepsilon_0 \rho \, B_0 \omega^2 / 2 \, \sin(\omega t) \vec{a_\phi}, & \rho < a \\ -\varepsilon_0 a^2 \omega^2 B_0 / 2\rho \, \sin(\omega t) \vec{a_\phi}, & \rho > a \end{cases}$$

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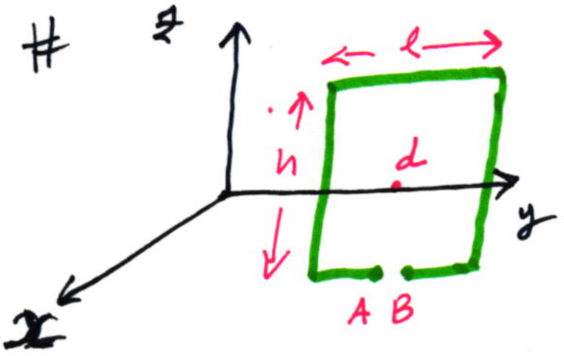



An antenna is located in zy plane, symmetrically w.r.t $y=d$ to z & y axis in an electromagnetic field $\vec{B} = B_0 \cos(t - y/c)\vec{a_x}$. Determine the e.m.f between A and B points and the maximum value of it ($l << 2c$)

$$\mathcal{E}(t) = \text{E.m.f} = -\frac{d}{dt}\int_S \vec{B}\,\vec{ds}$$ . Let's choose $\vec{B} \parallel \vec{ds}$ so $\vec{ds} = dy\,dz\,\vec{a_x}$

$$\Rightarrow \mathcal{E}(t) = -\frac{d}{dt}\int_{y=d-\frac{l}{2}}^{d+\frac{l}{2}}\int_{z=-h/2}^{h/2} B_0 \cos(t - y/c)\,\vec{a_x}\cdot dy\,dz\,\vec{a_x}$$

$$= B_0 \cdot h \; -\frac{d}{dt}\int \cos(t - y/c)\,dy$$

$$= cB_0 h \frac{d}{dt}\left(\sin(t - y/c)\Big|_{y=d-l/2}^{d+l/2}\right) = chB_0 \frac{d}{dt}\left[\sin\left(t - \frac{d+l}{2c}\right) - \sin\left(t - \frac{d-l}{2c}\right)\right]$$

$$= chB_0\left[\cos\left(\overbrace{t - \frac{d+l}{2c}}^{x}\right) - \cos\left(\overbrace{t - \frac{d-l}{2c}}^{y}\right)\right]$$

$$= chB_0\left[-2\sin\left(\frac{x+y}{2}\right)\cdot \sin\left(\frac{x-y}{2}\right)\right]$$

$$= -2chB_0 \sin\left(t - \frac{d}{c}\right)\sin\left(-\frac{l}{2c}\right)$$

$$= 2chB_0 \sin(t - d/c)\sin(l/2c)$$ $\quad l/2c << 1 \rightarrow \sin\left(\frac{l}{2c}\right) \approx \frac{l}{2c}$

$$\approx 2chB_0 \sin(t - d/c)\frac{l}{2c} \simeq B_0 \cdot \underbrace{S}_{\text{area of antenna}} \cdot \sin(t - d/c)$$

Maximum value of $\mathcal{E}(t) \rightarrow B_0 \cdot S$ [Volt]

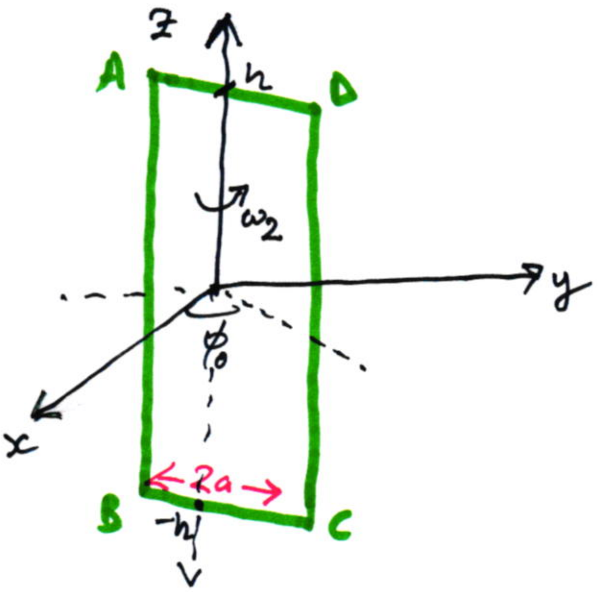
If both $\vec{B}$ and $S$ vary by time; the induced e.m.f becomes

$$\boxed{\mathcal{E}(t) = -\frac{d}{dt}\int_S \vec{B}\,\vec{ds} = -\int_S \frac{d\vec{B}}{dt}\,\vec{ds} + \oint_C (\vec{v}\times\vec{B})\,\vec{dl}}$$

Since $\nabla\times\vec{E} = -\frac{d\vec{B}}{dt}$, the first term of the right side of above equation may be written as $-\int \frac{d\vec{B}}{dt}\,\vec{ds} = \int \nabla\times\vec{E}\,\vec{ds} = \oint_C \vec{E}\,\vec{dl}$ for which $\vec{E}$ is produced by time varying $\vec{B}$. Hence;

$$\boxed{\mathcal{E}(t) = \oint_C [\vec{E} + (\vec{v}\times\vec{B})]\,\vec{dl}}$$

is the generalized form of Faraday Law (moving surface + time-varying $\vec{B}$)




Consider time-varying $\vec{B} = B_0 \cos \omega_1 t\, \vec{a_x}$ and a circulating rectangular loop in $\vec{a_\phi}$ direction with angular speed $\omega_2$. No charge density $\rho_v$ exists.

Let's analyze this system. $\vec{E}$ is observed that oscillating in $\vec{a_y}$ direction.

$\vec{E}$?, $\vec{J}$?, $\mathcal{E}(t)$?

$\phi(t)$?

[At $t=0$, $\phi=\phi_0$]

To determine $\vec{E}$ field, one may utilize $\nabla \cdot \vec{D} = \rho_v$ & $\nabla \times \vec{E} + d\vec{B}/dt = 0$

$\vec{E}$ is observed to be oscillating in $\vec{a_y}$ direction, besides $\rho_v \equiv 0$. Hence

$$\nabla \cdot \vec{E} = \frac{dE_x}{dx} + \frac{dE_y}{dy} + \frac{dE_z}{dz} = 0 \quad \text{where } E_x = E_z = 0 \text{. So, } \frac{dE_y}{dy} = 0$$

On the other hand; $\nabla \times \vec{E} = d\vec{B}/dt = -\omega_1 B_0 \sin(\omega_1 t)\vec{a_x}$

$$\nabla \times \vec{E} = \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ d/dx & d/dy & d/dz \\ 0 & E_y & 0 \end{vmatrix} = \vec{a_x}\left(-dE_y/dz\right) + \vec{a_z}\underbrace{\frac{dE_y}{dx}}_{0} = +\omega_1 B_0 \sin(\omega_1 t)\vec{a_x}$$

$$\Rightarrow -dE_y/dz \cdot \vec{a_x} = +\omega_1 B_0 \sin(\omega_1 t)\vec{a_x}$$

$$\Rightarrow \boxed{E_y = -\omega_1 B_0 z \sin(\omega_1 t)} + \underbrace{C(t)}$$

integral constant varying by time with no position dependence It may be ignored in calculations.

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To determine $\vec{J}$, one may utilize $\nabla \times \vec{H} - \frac{d\vec{D}}{dt} = \vec{J}$

$\nabla \times \vec{H} = 0$ ($\vec{H}$ or $\vec{B}$ is independent of position)

$$\frac{-d\vec{D}}{dt} = -\varepsilon_0 \frac{d\vec{E}}{dt} = \boxed{\varepsilon_0 \omega_1^2 B_0 z \cos(\omega_1 t)\vec{a_y} = \vec{J}}$$

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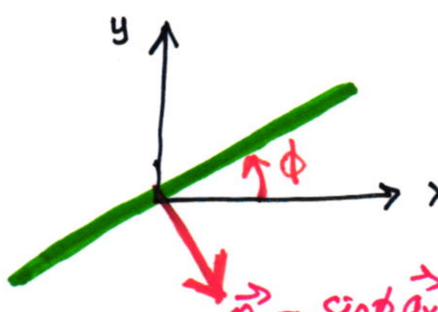
Let's calculate the magnetic flux through the loop as a function of time, that is $\phi(t)$

$\phi(t) = \int \vec{B}\,\vec{ds}$.




$\vec{n} = \sin\phi\,\vec{a_x} - \cos\phi\,\vec{a_y}$

↳ normal of $\vec{ds}$

$$\int \vec{B}\,\vec{ds} = \int B_0 \cos\omega_1 t\,\vec{a_x} \cdot (\sin\phi\,\vec{a_x} - \cos\phi\,\vec{a_y})\,ds$$

$$= B_0 \cdot \cos\omega_1 t \sin\phi \cdot A$$

↳ surface area of Loop

$\phi = \phi_0 + \omega_2 t$

So,

$$\boxed{\phi(t) = B_0 \cos(\omega_1 t)\sin(\phi_0 + \omega_2 t) \cdot A}$$

$[T.m^2$ or $Wb]$

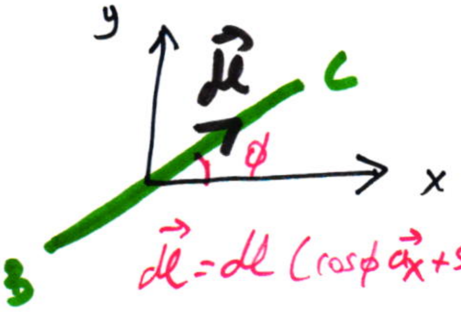
Let's calculate $\oint \vec{E}\,\vec{dl}$ on the loop. which is the circulation of $\vec{E}$ through the surface.

Note that $\vec{E}$ has only y component for which the $\oint \vec{E}\,\vec{dl}$ becomes 0 on AB & CD paths (since $\vec{dl} = dl\,\vec{a_z}$)

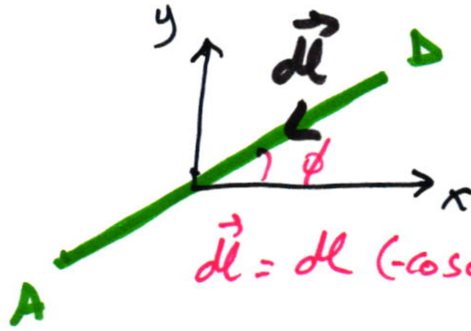
Hence

$$\oint \vec{E}\,\vec{dl} = \overbrace{\int_{AB} \vec{E}\,\vec{dl}}^{0} + \overbrace{\int_{CD} \vec{E}\,\vec{dl}}^{0} + \int_{BC} \vec{E}\,\vec{dl} + \int_{DA} \vec{E}\,\vec{dl} = \int_{BC} \vec{E}\,\vec{dl} + \int_{DA} \vec{E}\,\vec{dl}$$

for the chosen direction of dS;



$\vec{dl} = dl\,(\cos\phi\,\vec{a_x} + \sin\phi\,\vec{a_y})$



$\vec{dl} = dl\,(-\cos\phi\,\vec{a_x} - \sin\phi\,\vec{a_y})$

So;

$$\oint \vec{E}\,\vec{dl} = \int_{\rho=-a}^{a} E\,\vec{a_y}\,d\rho\,(\cos\phi\,\vec{a_x} + \sin\phi\,\vec{a_y}) + \int_{\rho=-a}^{a} E\,\vec{a_y}\,(-\cos\phi\,\vec{a_x} - \sin\phi\,\vec{a_y})\,d\rho$$

$$= E\sin\phi\,2a\Big|_{\substack{z=-h \\ \phi=\phi_0+\omega_2 t}} + -E\sin\phi\,2a\Big|_{\substack{z=h \\ \phi=\phi_0+\omega_2 t}}$$

$$= B_0 h\omega_1 2a \sin(\phi_0+\omega_2 t)\sin(\omega_1 t) + B_0 h\omega_1 \sin(\omega_1 t)\,2a \sin(\phi_0+\omega_2 t)$$

$$= \underbrace{4ah}_{A} B_0 \omega_1 \sin(\omega_1 t)\sin(\phi_0+\omega_2 t)$$

$$\boxed{\oint \vec{E}\,\vec{dl} = A B_0 \omega_1 \sin(\omega_1 t)\sin(\phi_0+\omega_2 t)} \quad [\text{Volt}]$$

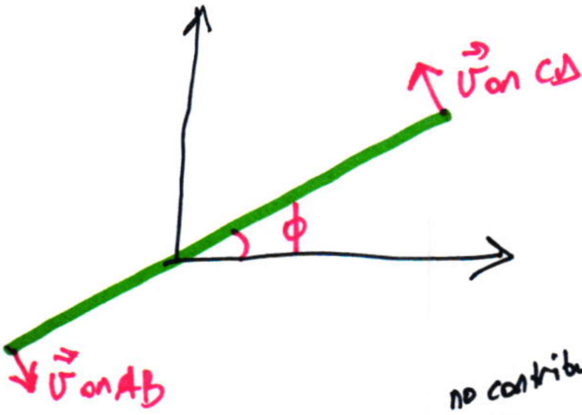
Let's calculate $\oint_C (\vec{v}\times\vec{B})\vec{dl}$. Note that $\vec{v}\times\vec{B} \parallel \vec{a_z}$ $(\vec{a_\phi}\times\vec{a_y})$ So the contour integral

C is composed of AB & CD paths (z direction)

on AB $\vec{dl} = dl\,(-\vec{a_z})$ and on CD, $\vec{dl} = dl\,\vec{a_z}$ for chosen $\vec{ds}$

on AB, $\vec{v} = \omega_2 . a . \vec{a_\phi}$ and on CD, $\vec{v} = \omega_2 . a . (-\sin\phi\,\vec{a_x} + \cos\phi\,\vec{a_y})$

$= \omega_2 . a . (\sin\phi\,\vec{a_x} - \cos\phi\,a_y)$




$$\text{So, } \oint(\vec{v}\times\vec{B})\vec{dl} = \int_{AB}\Big[\omega_2 a\,(\sin\phi\,\overbrace{\vec{a_x}}^{\text{no contribution}} - \cos\phi\,\vec{a_y}) \times B_0\cos(\omega_1 t)\vec{a_x}\Big]\vec{dl}(-\vec{a_z})$$

$$+ \int_{CD}\Big[\omega_2 a\,(\overbrace{-\sin\phi\,\vec{a_x}}^{\text{no contribution}} + \cos\phi\,\vec{a_y}) \times B_0\cos(\omega_1 t)\vec{a_x}\Big]\,dl\,(\vec{a_z})$$

Path integral $dl$ contributes multiplier $2h$ $\int_{z=-h}^{h} dl = 2h$

$$\text{Here } \oint[\vec{v}\times\vec{B}]\vec{dl} = -2h\,\omega_2 a\,B_0\cos(\omega_1 t)\cos\phi\,\vec{a_z} + -2h\,\omega_2 a\,B_0\cos(\omega_1 t)\cos\phi\,\vec{a_z}$$

$$= -4ah\,\omega_2 B_0\cos(\omega_1 t)\cos\phi\,\vec{a_z}$$

$$\boxed{\oint(\vec{v}\times\vec{B})\vec{dl} = -A\omega_2 B_0\cos(\omega_1 t)\cos(\phi_0 + \omega_2 t)} \text{ Volt}$$

In total

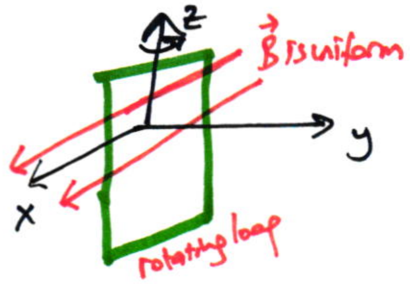
$$\oint_C(\vec{E} + \vec{v}\times\vec{B})\vec{dl} = AB_0\omega_1\sin(\omega_1 t)\sin(\phi_0 + \omega_2 t) - AB_0\omega_2\cos(\omega_1 t)\cos(\phi_0 + \omega_2 t)$$

$$\overset{?}{=} -d\Phi/dt$$

$$\Phi = AB_0\cos(\omega_1 t)\sin(\phi_0 + \omega_2 t) \Rightarrow -\frac{d\Phi}{dt} = \oint_C[\vec{E} + \vec{v}\times\vec{B}]\vec{dl}$$

If $\vec{B}$ is constant in time, that is $\omega_1 = 0$, $\vec{B} = B_0 . \vec{a_x}$

Flux through the surface becomes; $\phi = A B_0 \sin(\phi_0 + \omega_2 t)$




Since $\frac{dB}{dt} = 0$, no $\vec{E}$ field is produced

so $\int \vec{E} d\vec{l} = 0$

$$\mathcal{E}(t) = \frac{-d\phi}{dt} = -A B_0 \omega_2 \cos(\phi_0 + \omega_2 t)$$

$$= \oint (\vec{v} \times \vec{B}) d\vec{l} = -A \omega_2 B_0 \cos(\phi_0 + \omega_2 t)$$

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If loop does not rotate, that is $\omega_2 = 0$ while $\vec{B}$ varies by time

Flux through the surface becomes, $\phi = A B_0 \cos(\omega_1 t) \sin \phi_0$

Since, no moving loop $\oint (\vec{v} \times \vec{B}) d\vec{l} = 0$ but $\int \vec{E} d\vec{l} = A B_0 \omega_1 \sin(\omega_1 t) \sin(\phi_0)$

$$= \frac{-d\phi}{dt} = \mathcal{E}(t)$$

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If both $\vec{B}$ and loop is constant in time, that is $\omega_1 = \omega_2 = 0$

$\int \vec{E} d\vec{l} = \int \vec{v} \times \vec{B} \, dl = 0$ and no e.m.f is generated

$\phi = \int \vec{B} d\vec{s} = A B_0 . \sin \phi_0 \rightarrow -d\phi/dt = 0$ and no e.m.f is produced

~THE END~

Good Luck with your final exams

"Special thanks to Kemal Yüksek for helping in the preparation of lecture notes"