Three main topics will be dealt on vector analysis;

1- Vector algebra - addition, substraction, multiplication of vectors

2- Orthogonal coordinate systems - Cartesian, cylindirical & spherical coordinates

3- Vector Calculus - differentiation and integration of vectors; line, surface and volume integrals

"del" operator; gradient, divergence and curl operations

vector calculus pertains to the differentiation and integration of vectors.

2.2 Vector addition & substraction

ADDITION

$\vec{A} = |\vec{A}|\, \vec{a}_A$

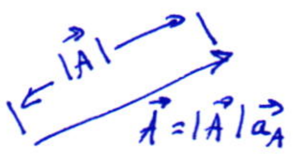
$|\vec{A}|$ → has the unit and dimension of $\vec{A}$

$\vec{a}_A$ → dimensionless unit vector with a unity magnitude having the direction of $\vec{A}$

$\vec{a}_A = \dfrac{\vec{A}}{|\vec{A}|}$

1. Parallelogram Rule

2. Head-to-tail Rule



Commutative Law $\vec{A}+\vec{B} = \vec{B}+\vec{A}$ Associative Law: $(\vec{A}+\vec{B})+\vec{C} = \vec{A}+(\vec{B}+\vec{C})$

SUBT

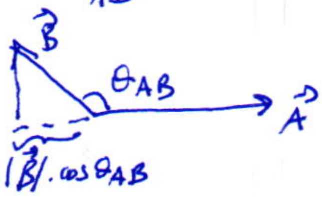
$\vec{A}-\vec{B} = \vec{A} + (-\vec{B})$ $\quad -\vec{B} = |\vec{B}| \cdot (-\vec{a}_B)$

2.3 PRODUCTS of VECTORS

$k\vec{A} = \vec{a}_A\,(k|\vec{A}|)$, where k is a positive scalar

2.3.1 Scalar or Dot Product

$\vec{A}\cdot\vec{B} \triangleq |\vec{A}||\vec{B}|\cos\theta_{AB}$ where $\theta_{AB} < \pi$ radians $(180°)$ is the smaller angle between $\vec{A}$ & $\vec{B}$



$\vec{A}\cdot\vec{A} = |\vec{A}|^2$, $|\vec{A}| = \sqrt{\vec{A}\cdot\vec{A}}$

Commutative Law $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$ Distributive Law $\vec{A}(\vec{B}+\vec{C}) = \vec{A}\vec{B}+\vec{A}\vec{C}$

2-3.2 Vector or Cross Product

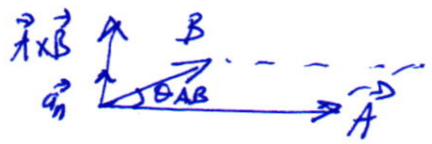
Denoted by $\vec{A}\times\vec{B}$ is a vector perpendicular to the plane containing $\vec{A}$ & $\vec{B}$
Its magnitude is $|\vec{A}||\vec{B}|\sin\theta_{AB}$ where θ_{AB} is the smaller angle between $\vec{A}$ & $\vec{B}$
and its direction follows that of the thumb of the right hand when the fingers rotate from $\vec{A}$ to $\vec{B}$

$$\vec{A}\times\vec{B} \triangleq \vec{a_n}\,(|\vec{A}||\vec{B}|\sin\theta_{AB})$$

$$\vec{B}\times\vec{A} = -\vec{A}\times\vec{B}$$

$$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$$ distributive law

$$\vec{A}\times(\vec{B}\times\vec{C}) \neq (\vec{A}\times\vec{B})\times\vec{C}$$ NOT associative



$$\vec{A}\cdot(\vec{B}\times\vec{C}) = -\vec{A}\cdot(\vec{C}\times\vec{B}) = -\vec{B}\cdot(\vec{A}\times\vec{C}) = -\vec{C}\cdot(\vec{A}\times\vec{B})$$

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \vec{C}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{C}\times\vec{A})$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}\cdot(\vec{A}\cdot\vec{C}) - \vec{C}\cdot(\vec{A}\cdot\vec{B})$$

2.4 ORTHOGONAL COORDINATE SYSTEMS ($\vec{a}_{u_1}, \vec{a}_{u_2}, \vec{a}_{u_3} \triangleq$ base vectors)

$$\vec{a}_{u_1}\times\vec{a}_{u_2} = \vec{a}_{u_3}$$
$$\vec{a}_{u_2}\times\vec{a}_{u_3} = \vec{a}_{u_1}$$
$$\vec{a}_{u_3}\times\vec{a}_{u_1} = \vec{a}_{u_2}$$

$$\vec{a}_{u_1} \rightarrow \vec{a}_{u_2} \rightarrow \vec{a}_{u_3} \rightarrow \vec{a}_{u_1}$$

$$\vec{A} = A_{u_1}\cdot\vec{a}_{u_1} + A_{u_2}\vec{a}_{u_2} + A_{u_3}\cdot\vec{a}_{u_3}$$

$$\vec{dl} = \vec{a}_{u_1}(h_1\cdot du_1) + \vec{a}_{u_2}(h_2\cdot du_2) + \vec{a}_{u_3}(h_3\cdot du_3)$$ $h_i \triangleq$ metric coefficient

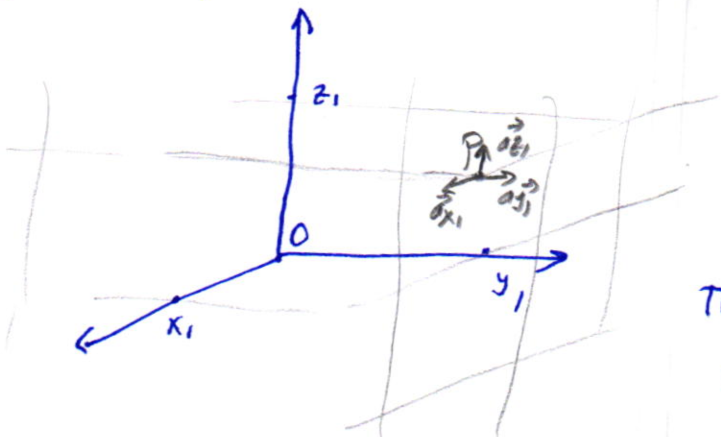
$$dv = h_1\cdot h_2\cdot h_3\, du_1\cdot du_2\cdot du_3$$

$$\vec{ds}_1 = \vec{a}_{u_1}(h_2 h_3\, du_2 du_3)\;;\quad \vec{ds}_2 = \vec{a}_{u_2}(h_1 h_3\, du_1 du_3)\;;\quad \vec{ds}_3 = \vec{a}_{u_3}(h_1 h_2\, du_1 du_2)$$

2-4.1 Cartesian Coordinates

$(u_1, u_2, u_3) = (x, y, z)$

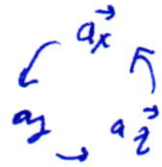
A point $P(x_1, y_1, z_1)$ in Cartesian coordinates is the intersection of three planes specified by $x = x_1$, $y = y_1$, $z = z_1$



$\vec{a_x} \times \vec{a_y} = \vec{a_z}$

$\vec{a_y} \times \vec{a_z} = \vec{a_x}$

$\vec{a_z} \times \vec{a_x} = \vec{a_y}$



The position vector to the point $P(x_1, y_1, z_1)$ is

$$\vec{OP} = \vec{a_x} x_1 + \vec{a_y} y_1 + \vec{a_z} z_1$$

A vector $\vec{A}$ in Cartesian coordinates can be written as;

$$\vec{A} = A_x \cdot \vec{a_x} + A_y \cdot \vec{a_y} + A_z \cdot \vec{a_z}$$

$$\vec{A} \cdot \vec{B} = A_x \cdot B_x + A_y \cdot B_y + A_z \cdot B_z$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \vec{a_x} \cdot (A_y \cdot B_z - A_z B_y) - \vec{a_y} (A_x \cdot B_z - A_z \cdot B_x) + \vec{a_z} (A_x \cdot B_y - A_y \cdot B_x)$$

Since x, y, z are lengths themselves, all three metric coefficients are unity, that is $h_1 = h_2 = h_3 = 1$. Expressions for differential length, differential area and differential volume are;

diff. length

$$\vec{dl} = \vec{a_x} dx + \vec{a_y} dy + \vec{a_z} dz$$

diff. areas

$$\vec{ds_x} = \vec{a_x} \, dy \cdot dz$$

$$\vec{ds_y} = \vec{a_y} \, dx \, dz$$

$$\vec{ds_z} = \vec{a_z} \, dx \, dy$$

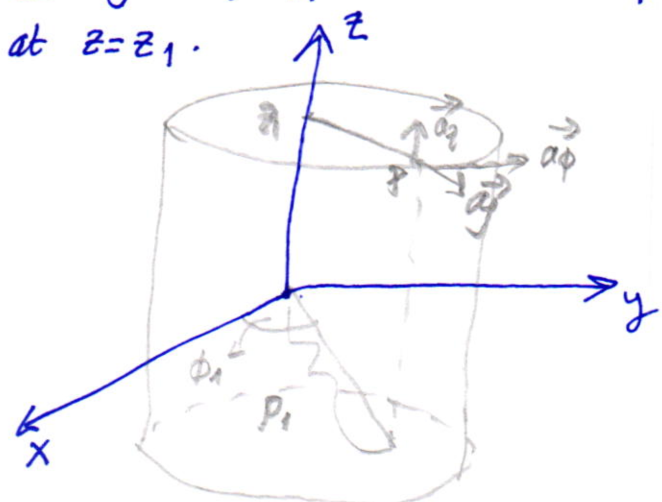
diff. vol

$$dv = dx \, dy \, dz$$

2-4.2 Cylindirical Coordinates

$(u_1, u_2, u_3) = (\rho, \phi, z)$

A point $P(\rho_1, \phi_1, z_1)$ in cylindirical coordinates is the intersection of a circular cylindirical surface $\rho = r_1$, a half-plane containing the z-axis and making an angle $\phi = \phi_1$ with the x-z plane and a plane parallel to the xy plane at $z = z_1$.



$$\vec{a_\rho} \times \vec{a_\phi} = \vec{a_z}$$
$$\vec{a_\phi} \times \vec{a_z} = \vec{a_\rho}$$
$$\vec{a_z} \times \vec{a_\rho} = \vec{a_\phi}$$

A vector $\vec{A}$ in cylindirical coordinates can be written as;

$$\vec{A} = A_\rho \cdot \vec{a_\rho} + A_\phi \cdot \vec{a_\phi} + A_z \cdot \vec{a_z}$$

Two of the three coordinates, ρ and z (u_1 and u_3) are themselves lengths, hence $h_1 = h_3 = 1$. However ϕ is an angle requiring a metric coefficient $h_2 = \rho$ to convert $d\phi$ to dl.

diff. length

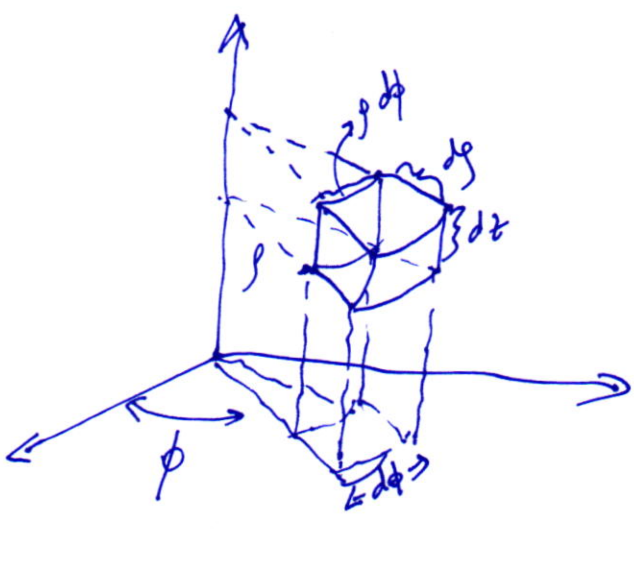
$$\vec{dl} = \vec{a_\rho} \cdot d\rho + \vec{a_\phi}\, \rho \cdot d\phi + \vec{a_z} \cdot dz$$

diff. areas

$$\vec{ds_\rho} = \vec{a_\rho}\, \rho\, d\phi\, dz$$
$$\vec{ds_\phi} = \vec{a_\phi}\, d\rho \cdot dz$$
$$\vec{ds_z} = \vec{a_z}\, \rho \cdot d\rho \cdot d\phi$$

diff. vol

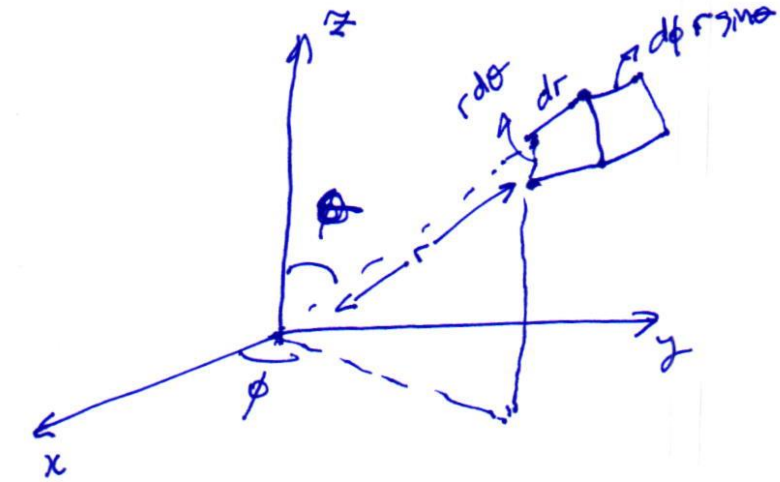
$$dv = \rho\, d\rho \cdot d\phi \cdot dz$$

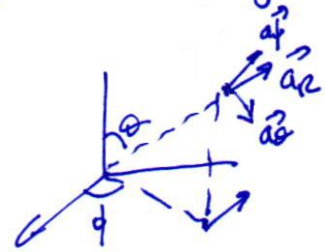


2-4.3 Spherical Coordinates

$(u_1, u_2, u_3) = (R, \theta, \phi)$

A point $P(r_1, \theta_1, \phi_1)$ in spherical coordinates is specified as the intersection of the following three surfaces; a spherical surface centered at the origin with a radius $r = r_1$; a right circular cone with its apex at the origin





$\vec{a_r} \times \vec{a_\theta} = \vec{a_\phi}$

$\vec{a_\theta} \times \vec{a_\phi} = \vec{a_r}$

$\vec{a_\phi} \times \vec{a_r} = \vec{a_\theta}$

$\vec{A} = A_r \cdot \vec{a_r} + A_\theta \cdot \vec{a_\theta} + A_\phi \cdot \vec{a_\phi}$

$\vec{dl} = dr \cdot \vec{a_r} + r\,d\theta\, \vec{a_\theta} + r\sin\theta\, d\phi \cdot \vec{a_\phi}$

$\vec{ds_r} = r\,d\theta \cdot r\sin\theta\, d\phi \cdot \vec{a_r}$

$\vec{ds_\theta} = r\sin\theta\, d\phi\, dr\, \vec{a_\theta}$

$\vec{d\phi} = dr \cdot r\,d\theta \cdot \vec{a_\phi}$

$dv = R^2 \sin\theta\, d\theta\, d\phi\, dr$

$\vec{A} = A_\rho \cdot \vec{a_\rho} + A_\phi \cdot \vec{a_\phi} + A_z \vec{a_z}$ in cartesian coordinates?

$A_x = A_\rho \cos\phi - A_\phi \sin\phi$

$A_y = A_\rho \sin\phi + A_\phi \cos\phi$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_\rho \\ A_\phi \\ A_z \end{bmatrix}$$

$x = \rho\cos\phi \,,\; y = \rho\sin\phi \,,\; z = z$

$\rho = \sqrt{x^2+y^2} \,,\; \phi = \tan^{-1} y/x \,,\; z = z$

$\vec{A} = \vec{a_R} A_R + \vec{a_\theta} A_\theta + \vec{a_\phi} A_\phi$ in cartesian coordinates

$A_x = A_R \sin\theta\cos\phi + A_\theta \cos\theta\cos\phi - A_\phi \sin\phi$

$A_y = A_R \sin\theta \sin\phi + A_\theta \cos\theta \sin\phi + A_\phi \cos\phi$

$A_z = A_R \cos\theta - A_\theta \sin\theta$

$x = R\sin\theta\cos\phi \,,\; y = R\sin\theta\sin\phi \,,\; z = R\cos\theta$

$R = \sqrt{x^2+y^2+z^2} \,,\; \theta = \tan^{-1} \dfrac{\sqrt{x^2+y^2}}{z} \,,\; \phi = \tan^{-1} \dfrac{y}{x}$

TRANSFORMATION BETWEEN COORD. SYS.

GRADIENT OF A SCALAR FIELD

$$dF = \left(\frac{dF}{d\ell_1}\right) \cdot d\ell_1 + \left(\frac{dF}{d\ell_2}\right) d\ell_2 + \left(\frac{dF}{d\ell_3}\right) d\ell_3$$

$$= \underbrace{\left(\frac{dF}{d\ell_1}\vec{a_1} + \frac{dF}{d\ell_2}\vec{a_2} + \frac{dF}{d\ell_3}\vec{a_3}\right)}_{\nabla F} \cdot \underbrace{\left(\vec{a_1} \cdot d\ell_1 + d\ell_2 \cdot \vec{a_2} + d\ell_3 \cdot \vec{a_3}\right)}_{\vec{d\ell}}$$

$$= \nabla F \cdot \vec{d\ell} = |\nabla F| \cdot |d\ell| \cdot \cos\theta$$

↳ The vector that represents both the magnitude and the direction of the maximum space rate of increase of a scalar as the gradient of that scalar.

$$\boxed{\begin{aligned} \vec{d\ell} &= \vec{a_1}\, d\ell_1 + \vec{a_2}\, d\ell_2 + \vec{a_3}\, d\ell_3 \\ &= \vec{a_1}\,(h_1 \cdot da_1) + \vec{a_2}\,(h_2\, da_2) + \vec{a_3}\,(h_3\, da_3) \qquad h_i: \text{metric coefficient} \end{aligned}}$$

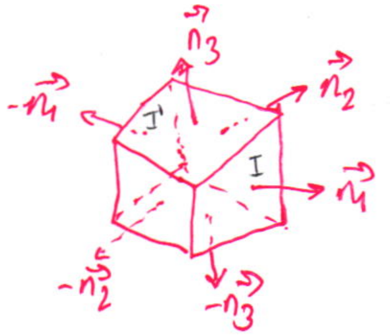
$$\Rightarrow \nabla F = \frac{1}{h_1}\frac{dF}{da_1} \cdot \vec{a_1} + \frac{1}{h_2}\frac{dF}{da_2} \cdot \vec{a_2} + \frac{1}{h_3}\frac{dF}{da_3}\vec{a_3}$$

$$\int_A^B \nabla F \cdot \vec{d\ell} = F(B) - F(A) \rightarrow \text{fundamental theorem of gradient}$$

DIVERGENCE of a VECTOR FIELD

$$\operatorname{div} \vec{D} = \lim_{\Delta v \to 0} \frac{\oint_s \vec{D}\cdot d\vec{s}}{\Delta v}$$

"Net outward flux of $\vec{F}$ per unit volume as volume about the point tends to zero"



$$\vec{F}\cdot d\vec{s} = \vec{F}\cdot\vec{n}\cdot ds$$

For I and I' surfaces:

$\vec{n}_I = -\vec{n}_{I'}$

$ds_{I'} = h_2 \cdot h_3 \cdot da_2\, da_3$

$ds_I = (h_2 + dh_2)(h_3 + dh_3)\, da_2\, da_3$

$\vec{F}_I = F_I(a_1 + da_1, a_2, a_3) \cdot \vec{a}_{F_I}$

$\vec{F}_{I'} = F_{I'}(a_1, a_2, a_3)\, \vec{a}_{F_{I'}}$

Then, for surface I and I' $\vec{F}\cdot\vec{n}\cdot ds / dV =$

$$\Big[\, \overbrace{F_I(a_1 + da_1, a_2, a_3)(h_2+dh_2)(h_3+dh_3)\,da_2 \cdot da_3}^{\text{surface } I \;\; ds_I} \cdot \vec{n}_1 \;-\; F_I(a_1, a_2, a_3)(h_2 \cdot h_3 \cdot da_2\, da_3)\, \vec{n}_1 \,\Big]$$

$\boxed{h_1 da_1 \; h_2 da_2 \; h_3 da_3} \rightarrow$ approximate infinitesimally small volume

$$\simeq \frac{d(F_I\, h_2 h_3)}{h_1 h_2 h_3\, da_1}$$

Through similar process, one may obtain a general expression for the divergence of a vector such that

$$\nabla\cdot\vec{D} = \operatorname{div}\vec{D} = \frac{1}{h_1 h_2 h_3}\left[\frac{d(h_2 h_3 D_1)}{da_1} + \frac{d(h_1 h_3 D_2)}{da_2} + \frac{d(h_1 h_2 D_3)}{da_3}\right]$$

Divergence Theorem

$$(\nabla \cdot \vec{D})_j \cdot \Delta v_j = \oint_{S_j} \vec{D}\, \vec{ds}$$

A volume V is subdivided into N, small differential volume, of which Δv_j is typical.

$$\underbrace{\lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} (\nabla \cdot \vec{D})_j \, \Delta v_j \right]}_{} = \lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} \oint_{S_j} \vec{D} \cdot \vec{ds} \right]$$

by definition, it is volume integral of $\nabla \cdot \vec{D}$, namely $= \oint_{v} \nabla \cdot \vec{D} \cdot dv$

Hence

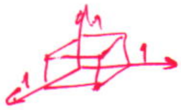
$$\oint_{v} \nabla \cdot \vec{D}\, dv = \underbrace{\lim_{\Delta v_j \to 0} \left[\sum_{j=1}^{N} \int_{S_j} \vec{D} \cdot \vec{ds} \right] = \oint_{S} \vec{D} \cdot \vec{ds}}_{}$$

Since inward and outward flux cancel each other through the volume except the surronding surface

$$\boxed{\int_{v} \nabla \cdot \vec{D}\, dv = \oint_{S} \vec{D} \cdot \vec{ds}}$$

Requires the vector field $\vec{D}$, as well as its first derivatives exist and be continuous both in v and on S.

Ex $\vec{D} = x^2 \vec{a_x} + xy \vec{a_y} + yz \vec{a_z}$

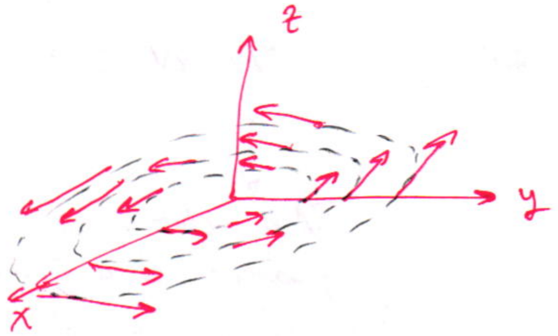


$\int \nabla \cdot \vec{D}\, dv = \int \vec{D} \vec{ds} = 2$

CURL of a VECTOR FIELD

$$\nabla\times\vec{A} \triangleq \lim_{\Delta s\to 0} \frac{1}{\Delta S}\, \vec{a}_n \oint_C \vec{A}\, d\vec{l}$$

$$\nabla\times\vec{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} \vec{a}_1 h_1 & \vec{a}_2 h_2 & \vec{a}_3 h_3 \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$



Plot of $\vec{A} = -y\vec{a}_x + x\vec{a}_y$

$$\nabla\times\vec{A} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = 2\vec{a}_z$$

Stokes's Theorem

For a very small differential area Δs_j, bounded by a contour C_j, the definition of $\nabla\times\vec{A}$ leads to

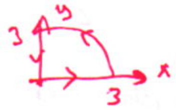
$$(\nabla\times\vec{A})_j\,(\vec{\Delta s}_j) = \oint_{C_j} \vec{A}\cdot d\vec{l}$$

$$\lim_{\Delta v_j\to 0} \sum_{j=1}^{N} (\nabla\times\vec{A})_j(\vec{\Delta s}_j) = \lim_{\Delta v_j\to 0} \sum_{j=1}^{N} \oint_{C_j} \vec{A}\cdot d\vec{l}$$

$$\boxed{\int_S (\nabla\times\vec{A})\, d\vec{s} = \oint_C \vec{A}\, d\vec{l}}$$

"the surface integral of the curl of a vector field over an open surface is equal to the closed line integral of the vector along the contour bounding the surface"

Ex $\vec{F} = xy\vec{a}_x - 2x\vec{a}_y$



$$\nabla\times\vec{F} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy & -2x & 0 \end{vmatrix} = -(2+x)\vec{a}_z$$

$$\int_S \nabla\times\vec{F}\, d\vec{s} = \int_0^3\int_0^{\sqrt{9-y^2}} (\nabla\times\vec{F})\,\vec{a}_z\, dx\, dy$$

$$= \int_0^3 \left[\int_0^{\sqrt{9-y^2}} -(2+x)\, dx\right] dy$$

$$= -9(1+\pi/2)$$

$$\oint \vec{F}\, d\vec{l}$$

TWO NULL IDENTITIES

Identity I $\nabla \times (\nabla V) \equiv 0$ The curl of the gradient of any scalar field is identically zero

$\downarrow$

" If a vector field is curl-free, then it can be expressed as the gradient of a scalar field."

if $\nabla \times \vec{E} = 0 \rightarrow \vec{E} = -\nabla V$ ↳ vector potential (electric scalar potential)

$\downarrow$

" an irrotational (e conservative) vector field can always be expressed as the gradient of a scalar field."

Identity II $\nabla \cdot (\nabla \times \vec{A}) \equiv 0$ The divergence of the curl of any vector field is identically zero.

$\downarrow$

If a vector field is divergenceless, then it can be express as the curl of another vector field

If $\nabla \cdot \vec{B} = 0 \rightarrow \vec{B} = \nabla \times \vec{A}$ ↳ magnetic vector potential

Proof of Identity I

$$\int_S [\nabla \times (\nabla V)]\, ds = \oint_C \nabla V\, d\vec{l}$$ → which starts and ends at the same point, hence no gradient occurs

Proof of Identity II

$$\int_v [\nabla \cdot (\nabla \times \vec{A})]\, dv = \oint_S \nabla \times \vec{A}\, d\vec{s} = \underbrace{\oint_{C_1} \vec{A}\, d\vec{l} + \oint_{C_2} \vec{A}\, d\vec{l}}_{\text{cancel each other}}$$

HELMHOLTZ THEOREM

Static Fields (Time Invariant)

In static case, $\vec{E}$ & $\vec{D}$ and $\vec{H}$ & $\vec{B}$ are independent pairs. $\vec{E}$ and $\vec{H}$ may exist together in the space and may be called as electromagnetostatic field. For instance, an electrostatic field creats current in a conducting medium. This current creates magnetostatic field.

Time-varying magnetic field ($d\vec{H}/dt \neq 0$) creates electric field $\vec{E}$. Similarly a time-varying electric field ($d\vec{E}/dt \neq 0$) creates magnetic field $\vec{H}$. These relations are formulated via Maxwell's Equations. Time-varying fields result in electromagnetic <u>waves</u>

Maxwell's Equations

Differential Form		Integral Form
$\nabla \times \vec{E} + \frac{d\vec{B}}{dt} = 0$	Faraday's Law	$\oint_C \vec{E}.d\vec{l} = -\int_S \frac{d\vec{B}}{dt} d\vec{s} = \frac{-d\phi}{dt}$
$\nabla \times \vec{H} - \frac{d\vec{D}}{dt} = \vec{J}$	Maxwell-Ampere Equation	$\oint_C \vec{H}.d\vec{l} = \int_S (\vec{J} + \frac{d\vec{D}}{dt}).d\vec{s}$
$\nabla . \vec{D} = \rho_v$	Gauss's Law	$\int \vec{D}.d\vec{s} = Q$
$\nabla . \vec{B} = 0$	Gauss's Law	$\int \vec{B}.d\vec{s} = 0$

$\vec{E}$ [V/m] Electric Field Vector

$\vec{H}$ [A/m] Magnetic Field Vector

$\vec{D}$ [C/m^2] Displacement Vector (Electric Flux Density)

$\vec{B}$ [T, Wb/m^2] Magnetic Induction Vector (Magnetic Flux Density)

ρ_v [C/m^3] Charge Density

$\vec{J}$ [A/m^2] Current Density

Faraday's Law: A time-varying magnetic field creates electric field

Maxwell-Ampère Equation: A time-varying electric field results in magnetic field.

Constitutive Relations (Classification of Media)

$$\vec{D} = \varepsilon \vec{E} = \varepsilon_0 \vec{E} + \underbrace{\vec{P}}_{\text{Polarization vector}} \;; \quad \varepsilon = \varepsilon_0 (1 + \underbrace{\chi_e}_{\text{electric susceptibility}}) \vec{E}$$

$$\vec{B} = \mu \vec{H}$$

$$\left.\begin{aligned} \vec{D} &= \bar{\bar{\varepsilon}} \vec{E} + \bar{\bar{\xi}} \vec{H} \\ \vec{B} &= \bar{\bar{\zeta}} \vec{E} + \bar{\bar{\mu}} \vec{H} \end{aligned}\right\} \text{Bi-anisotropic medium}$$

A coupling exists between $\vec{E}$ and $\vec{H}$ due to the medium properties

$$\left.\begin{aligned} \vec{D} &= \bar{\bar{\varepsilon}} \vec{E} \\ \vec{B} &= \bar{\bar{\mu}} \vec{H} \end{aligned}\right\} \text{Anisotropic medium}$$

$\bar{\bar{\varepsilon}}$ and $\bar{\bar{\mu}}$ are tensors. Note that $\vec{B}$ and $\vec{H}$ are no longer parallel.

$$\bar{\bar{\varepsilon}} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{zz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix}$$

If $\vec{\varepsilon}$ and $\vec{\mu}$ are functions of space coordinates → inhomogeneous medium

$$\boxed{\varepsilon(\vec{r}),\ \mu(\vec{r})}$$

If $\vec{\varepsilon}$ and $\vec{\mu}$ are functions of fields → non-linear medium (ferrite materials)

$$\varepsilon(\vec{E}),\ \mu(\vec{H})$$

Simple medium: Linear, isotropic, homogeneous

- Linear: ε, μ are properties of medium, don't depend on $\vec{E}, \vec{H}$ (external fields)
- isotropic: $\vec{D} = \varepsilon \vec{E}$, $\vec{B} = \mu \vec{H}$
- homogeneous: μ, ε ~~doesn~~ don't vary by position

complex permittivity

$$\varepsilon = \underbrace{\varepsilon_0}_{\text{dielectric constant of vacuum } \frac{1}{36\pi}10^{-9}} \underbrace{\varepsilon_r}_{\text{relative dielectric permittivity}} (1 - j \tan\delta) \qquad \tan\delta \to \text{loss tangent} \triangleq \frac{\sigma}{\omega \varepsilon_0 \varepsilon_r}$$

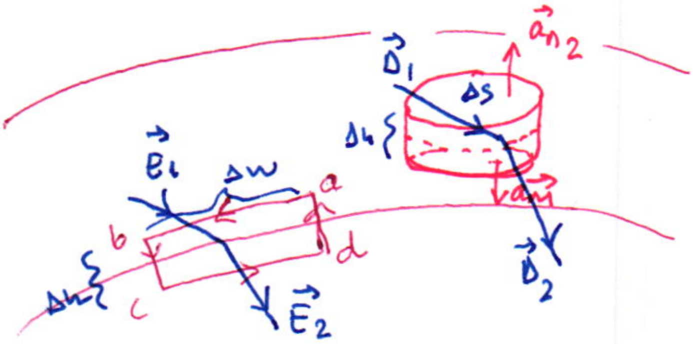
$$\varepsilon = \varepsilon' - j\varepsilon'' = \varepsilon_0 \varepsilon_r - j \frac{\sigma}{\omega}$$

$\tan\delta \ll 1$ → good insulator

$\tan\delta \sim 1$ → semi-conductor

$\tan\delta \gg 1$ → good conductor

Boundary Conditions for Electrostatic Fields



Consider the interface between two media. Recall that

$$\oint \vec{E}\, d\vec{l} = 0$$

$$\int \vec{D}\, d\vec{s} = \rho_v$$

Tangential Let's analyze $\int \vec{E}\, d\vec{l}$ on abcda path. As $\Delta h \to 0$ and ab & cd lines tend to a point like line we have;

$\int \vec{E}\, d\vec{l} = E_{1t} \cdot \Delta w - E_{2t}\, \Delta w = 0$, here bc & ad paths do not contribute to the integral

Hence, one may write;

$$\boxed{E_{1t} = E_{2t}}$$

Since the $\vec{E}$ field vanishes inside a conductor, no tangential component exist on the conductor surface even in the presence of charges. Also, this equality says that the tangential components of $\vec{E}$ fields are continuous.

Normal Let's analyze $\int \vec{D}\, d\vec{s} = \rho_v$ through the pillbox. As $\Delta h \to 0$, integral becomes

$(\vec{D}_1 \vec{a}_{n2} + \vec{D}_2 \vec{a}_{n1})\, \Delta s = \underbrace{\rho_v \cdot \Delta s}_{\text{total charge}}$ and $\vec{a}_{n2} = -\vec{a}_{n1}$ (normal vector is outward ~~the~~)

Then one may write

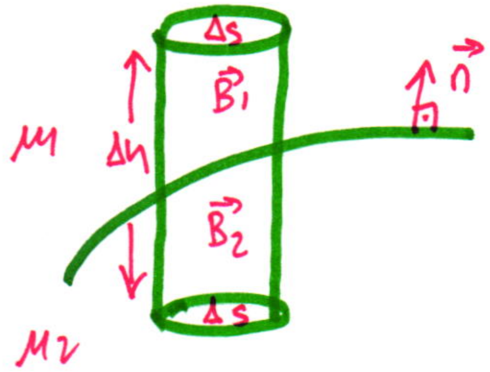
$\vec{a}_{n2}(\vec{D}_1 - \vec{D}_2) = \rho_v$ here ρ_v becomes ρ_s since tending $\Delta h \to 0$ forms volume to ~~a~~ surface

Finally,

$$\boxed{D_{1n} - D_{2n} = \rho_s}$$

Note that, under normal conditions there is no charge discontinuity between dielectric interfaces. Also, since $\vec{E}$ field is 0 inside a conductor, the normal component outward ~~from the~~ surface becomes $D_{1n} = \rho_s$ or $E_{1n} = \rho_s/\varepsilon$ as we obtained in conductors lecture.

Boundary Conditions for Magnetostatic Fields



$\int \vec{B} d\vec{s} = B_{1n} \cdot \Delta s - B_{2n} \cdot \Delta s$ as $\Delta h \to 0$

Recall $\nabla \cdot \vec{B} = 0$ and $\int \nabla \cdot \vec{B} dv = \int \vec{B} d\vec{s}$)

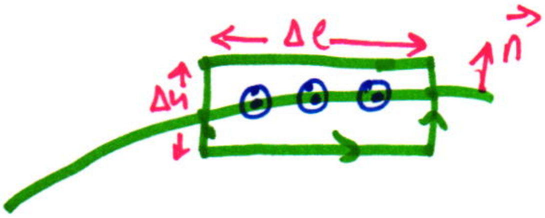
Hence

$B_{1n} \Delta s - B_{2n} \Delta s = 0 \rightarrow \boxed{B_{1n} = B_{2n}}$

In terms of magnetic field intensity $\vec{H}$,

$$\boxed{\mu_1 H_1 = \mu_2 H_2}$$

In vector form $\boxed{\vec{n} \cdot (\vec{B_1} - \vec{B_2}) = 0}$



$\oint_c \vec{H} d\vec{l} = H_{1t} \Delta l - H_{2t} \Delta l$ as $\Delta h \to 0$

$= \int \nabla \times \vec{H} d\vec{s} = \int \vec{J} d\vec{s} = I$

$\Rightarrow H_{1t} \Delta l - H_{2t} \Delta l = J \Delta l$

$\Rightarrow \boxed{H_{1t} - H_{2t} = J}$

In vector form, $\boxed{\vec{n} \times (\vec{H_1} - \vec{H_2}) = \vec{J}_{surface}}$

Derivations are not performed in detail because you are familiar with the followed approach due to the same concepts in electrostatics.

Continuity Relations

Recall that, for any given vector $\vec{A}$; $\nabla\cdot(\nabla\times\vec{A})\equiv 0$ (null identity)

$\nabla\cdot(\nabla\times\vec{H})\equiv 0$, $-\nabla\cdot(d\vec{D}/dt)\equiv\nabla\cdot\vec{J}$ $\quad(\nabla\times\vec{H}+d\vec{D}/dt=\vec{J})$

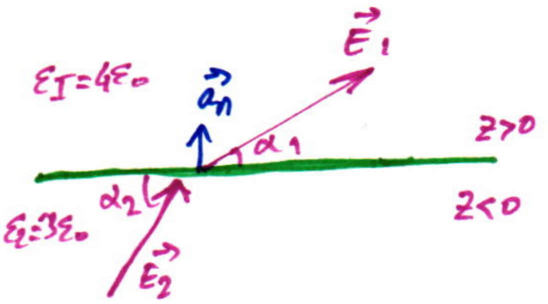
$\hookrightarrow \boxed{\nabla\cdot\vec{J}+d\rho_v/dt=0}$

$\hookrightarrow \int_v \nabla\cdot\vec{J}\,dv = -d/dt\int_v \rho_v\cdot dv$

$\hookrightarrow \int\vec{J}\,d\vec{s} = -d/dt\, Q \to$ charge

$\hookrightarrow \boxed{I_{surface} = \frac{-dQ}{dt}}$

\# medium I ($z>0$) and medium II ($z<0$) have dielectric constants $\varepsilon_I = 4\varepsilon_0$, $\varepsilon_{II}=3\varepsilon_0$, respectively. $\vec{E}=5\vec{a_x}-2\vec{a_y}+3\vec{a_z}$ [V/m] field exists in medium I. Determine $\vec{E}_{II}$ in medium II? Determine angles between $\vec{E}_I$, $\vec{E}_{II}$ and interfaces?



① $\vec{a_n}\times(\vec{E_1}-\vec{E_2})=0 \to \underbrace{\vec{E}_{1t}=\vec{E}_{2t}}_{\text{tangential components}}$

$\vec{E}_{1t}$ is composed of components on xy plane

Here $\vec{E}_{1t}=5\vec{a_x}-2\vec{a_y}$

$\vec{E}_{1t}=\vec{E}_{2t} \Rightarrow \underline{\vec{E}_{2t}=5\vec{a_x}-2\vec{a_y}}$

② $\vec{a_n}\cdot(\vec{D_1}-\vec{D_2})=\rho_s=0 \to \vec{D}_{1n}-\vec{D}_{2n}=0$

$\vec{D}_{1n}$ is the normal component (in $+\hat{z}$ direction) of $\vec{D}_1=\varepsilon_{Ir}\vec{E}_1$

(ε_{Ir} : relative dielectric constant 4; ε_{IIr} : " " 3)

Here $\vec{D}_{1n}=4\cdot 3\vec{a_z}=\vec{D}_{2n}\to\vec{D}_{2n}=12\vec{a_z}$

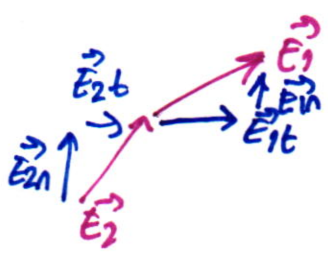
$\vec{D}_{2n}$ is the normal component of $\vec{D_2}=\varepsilon_{IIr}\vec{E_2}$

Hence $\vec{E}_{2n}=\vec{D}_{2n}/\varepsilon_{IIr}=12\vec{a_z}/3=\underline{4\vec{a_z}}$

Note that $\vec{E}_2=\vec{E}_{2t}+\vec{E}_{1n}\to \boxed{\vec{E_2}=5\vec{a_x}-2\vec{a_y}+4\vec{a_z}}$

$\tan\alpha_1=\frac{E_{1n}}{E_{1t}}=\frac{3}{\sqrt{5^2+2^2}}\Rightarrow\alpha_1=29.1°$

$\tan\alpha_2=\frac{E_{2n}}{E_{2t}}=\frac{4}{\sqrt{5^2+2^2}}\Rightarrow\alpha_2=36.6°$



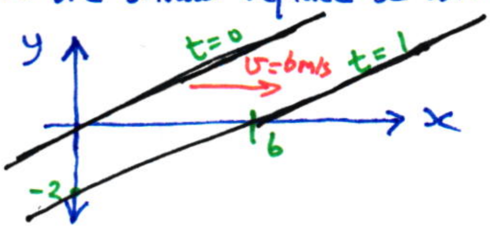
A Brief Introduction to Travelling waves and Standing waves

TRAVELLING WAVE

Consider $y = 1/3\,x$

If one desires to move this line in $+x$ direction with velocity 6 m/sec ...

... one should replace x with $x-6t$. So; $y = 1/3\,(x-6t)$

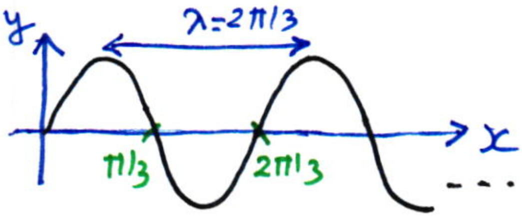


If we want to move with velocity v in $+x$ direction, we may simply replace x with $x - vt$

Similarly, If we want to move with velocity v in $-x$ direction, we may simply replace x with $x + vt$

Now consider $y = 2\sin 3x$.

3 → wave number k



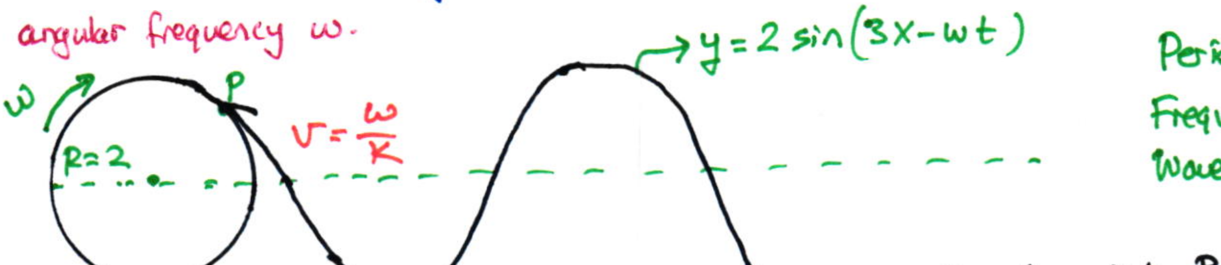
not moving!!!

$k = 2\pi/\lambda = 2\pi/(2\pi/3) = 3$

Again, if one desires to move this function with velocity 6 m/sec, one simply replaces x with $(x-6t)$. In this case, $y = 2\sin 3(x-6t)$

2 → Amplitude, 3 → wavenumber, 6 → velocity

Now consider a rotating circular disk moving in $+x$ direction with angular frequency ω.



Period T
Frequency f
Wavelength λ

$T = \dfrac{2\pi}{\omega}$; $\lambda = v \cdot T = \dfrac{v}{f}$; $f = v/\lambda$; $v = \omega/k$

If you choose $\omega = 18$; the path followed by point P will oscillate as a sinusoidal function $y = 2\sin 3(x - 6t) = 2\sin(3x - 18t)$

3 → k, 6 → v, 3 → k, 18 → ω

STANDING WAVES

Consider y_1 and y_2 waves travelling in $+x$ and $-x$ directions, respectively.

$\longrightarrow$ $y_1 = y_0 \sin(kx - wt)$ $\qquad$ $\longleftarrow$ $y_2 = y_0 \sin(kx + wt)$ $\qquad$ [$\rightarrow$ $+x$ direction, $\leftarrow$ $-x$ direction]

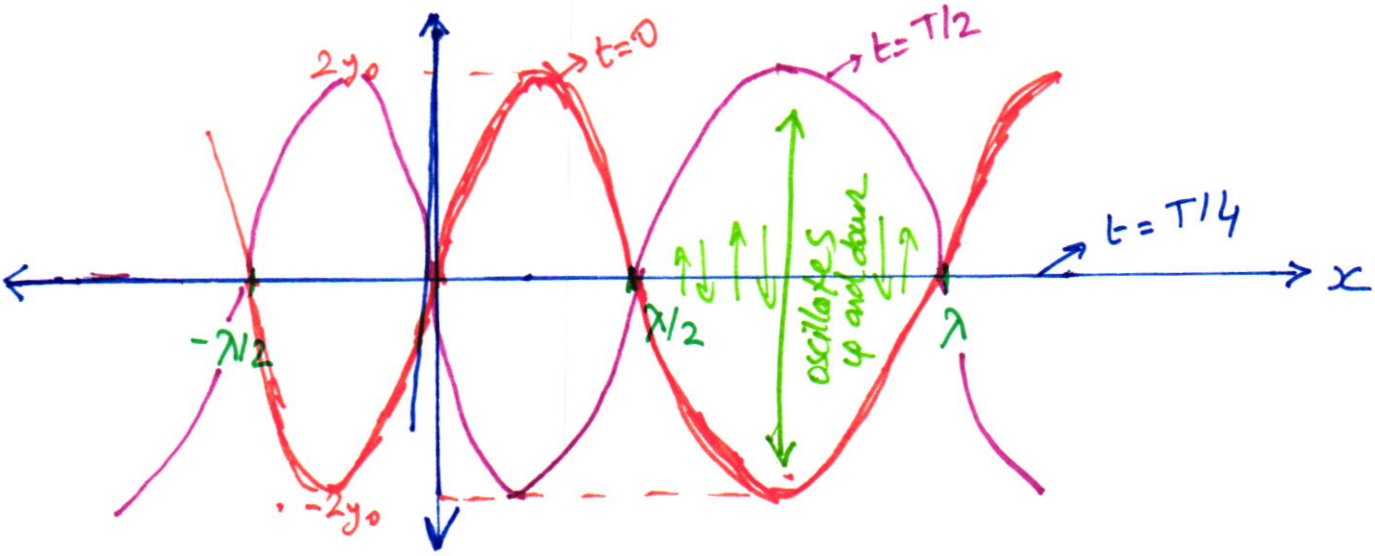
Total wave on the path of propagation is simply sum of y_1 and y_2. That is;

$$y = y_1 + y_2 = y_0 \left[\sin(kx - wt) + \sin(kx + wt)\right]$$

$$\Rightarrow \boxed{y = 2y_0 \underbrace{\sin(kx)}_{\text{Spatial information}} \underbrace{\cos(wt)}_{\text{timing information}}}$$

In this case, time information (wt) is separate from spatial information (kx)

Let's draw y for $t=0$, $t=T/4$ and $t=T/2$
$\downarrow$ $\cos(wt)=1$ $\quad$ $\downarrow$ $\cos(wt)=0$ $\quad$ $\downarrow$ $\cos(wt)=-1$



Total wave is fixed at $x = \ldots -\lambda, -\lambda/2, 0, \lambda/2, \lambda, \ldots$ due to $\sin(kx)$

Total wave oscillates between $2y_0$ and $-2y_0$ due to $\cos(wt)$

So the total wave is a standing wave.

Wave Equation in Source-free Simple Media ($\rho_v = 0$, $\vec{J} = 0$ → no sources exist)

Maxwell's Equations become;

$$\nabla\times\vec{E} = -\mu\frac{d\vec{H}}{dt} \quad ; \quad \nabla\times\vec{H} = \varepsilon\frac{d\vec{E}}{dt} \quad ; \quad \nabla\cdot\vec{E} = 0 ; \quad \nabla\cdot\vec{H} = 0$$

// Recall vector identity $\nabla\times\nabla\times\vec{A} = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$

Hence;

$$\nabla\times\nabla\times\vec{E} = \nabla(\underbrace{\nabla\cdot\vec{E}}_{\text{sourcefree } 0}) - \nabla^2\vec{E} = -\mu\frac{d}{dt}(\nabla\times\vec{H}) = -\mu\frac{d}{dt}\varepsilon\frac{d\vec{E}}{dt}$$

$$\boxed{\nabla^2\vec{E} - \mu\varepsilon\frac{d^2\vec{E}}{dt^2} = 0} \quad (\mathrm{I})$$

(I) & (II) are homogeneous vector wave equations

$$\nabla\times\nabla\times\vec{H} = \nabla(\underbrace{\nabla\cdot\vec{H}}_{\substack{\text{no magnetic } \equiv 0 \\ \text{monopole exists}}}) - \nabla^2\vec{H} = \varepsilon\frac{d}{dt}(\nabla\times\vec{E}) = -\mu\varepsilon\frac{d^2\vec{H}}{dt^2}$$

$$\boxed{\nabla^2\vec{H} - \mu\varepsilon\frac{d^2\vec{H}}{dt^2} = 0} \quad (\mathrm{II})$$

(I) & (II) are in wave equation form $\left(\nabla^2 F - \frac{1}{v^2}\frac{d^2F}{dt^2} = 0\right)$. Therefore we may say v (velocity) of wave is $\left(\frac{1}{v^2} = \mu\varepsilon\right)$ → equal to $\frac{1}{\sqrt{\varepsilon\mu}^2}$

For free space $\varepsilon \cong \varepsilon_0 \cong \frac{1}{36\pi}10^{-9}$, $\mu \cong \mu_0 \cong 4\pi 10^{-7}$

Then $v = \frac{1}{\sqrt{\frac{1}{36\pi}10^{-9}\, 4\pi 10^{-7}}^2} \cong 3\,10^8\ \mathrm{m/sec}$ which is approximately speed of light

Maxwell's Equations also read that EM wave propagates with speed of light in free-space.

Now, let's assume that $\vec{E}$ and $\vec{H}$ are functions of (z,t) (You may replace z with x or y) wave equations in (I) & (II) become (let f represents $\vec{E}$ & $\vec{H}$)

$$\frac{d^2\psi}{dz^2} - \frac{1}{v^2}\frac{d^2\psi}{dt^2} = 0 \quad (\mathrm{III})$$

Solution of (III);

$$\psi = f(\overset{\rightarrow +z}{z - vt}) + g(\overset{\leftarrow -z}{z + vt})$$

where f and g are any functions, travelling in opposite directions; f travels in +z direction; while g travels in -z direction.

// One may readily replace z with x or y. In each case, analysed problem is one-dimensional wave equation and solution is identical. The one-dimensional wave equation can be solved by different approaches such as; d'Alembert's solution, Fourier Transform, Sepperation of variables.

An electromagnetic wave should satisfy wave equation.

$$\nabla^2\varphi = \frac{1}{\underbrace{h_1h_2h_3}_{\text{metric coefficients}}}\left[\frac{d}{du_1}\left(\frac{h_2h_3}{h_1}\frac{df}{du_1}\right) + \frac{d}{du_2}\left(\frac{h_1h_3}{h_2}\frac{df}{du_2}\right) + \frac{d}{du_3}\left(\frac{h_1h_2}{h_3}\frac{df}{du_3}\right)\right]$$

Check if given expressions are electromagnetic waves? ($c \hat{=}$ speed of light)

a) $\psi = \sin(2x - ct)$ b) $\varphi = \sin x\, e^{-t}$

a) ψ should satisfy $\frac{d^2\varphi}{dx^2} - \frac{1}{v^2}\frac{d^2\varphi}{dt^2} = 0$

$$\frac{d^2\varphi}{dx^2} = \frac{d}{dx}(2\cos(2x - ct)) = -4\sin(2x - ct)$$

$$\frac{d^2\psi}{dt^2} = \frac{d}{dt}(-c\cos(2x - ct)) = -c^2\sin(2x - ct)$$

$$-4\sin(2x - ct) - \frac{1}{v^2}(-c^2(\sin\alpha)) \overset{?}{=} 0 \qquad \alpha = 2x - ct$$

$$-4\sin(2x - ct) + \frac{c^2}{v^2}\sin(2x - ct) = 0 \rightarrow \boxed{v = \frac{c}{2}}$$

→ under this condition φ satisfies wave equation

b) $$\frac{d^2\varphi}{dx^2} = \frac{d}{dx}(\cos x\, e^{-t}) = -\sin x\, e^{-t}$$

$$\frac{d^2\varphi}{dt^2} = \frac{d}{dt}(-\sin x\, e^{-t}) = \sin x\, e^{-t}$$

$$-\sin x\, e^{-t} - \frac{1}{v^2}\sin x\, e^{-t} = 0 \Rightarrow \boxed{v^2 = -1}$$

→ this condition is not physical

here φ does not satisfy wave equation in any case and can not represent an EM wave.

Time-Harmonic Fields

So far, arbitrary time dependence of Maxwell's Equations ~~is~~ analyzed. The actual field quantities depend on the sources ρ_v and $\vec{J}$ in real applications.
Sinusoidal functions are easy to generate, to analyze. Since Maxwell's Equations are linear differential equations; produced $\vec{E}$ and $\vec{H}$ from sinusoidally time varying source functions $\rho_v, \vec{J}$ are in sinusoidal form as well (with the same frequency in the steady state). Time harmonic fields are sinusoidally ~~or~~ periodically time varying fields.

Review of Phasors

Phasors are time-independent quantities, involving amplitude and phase data.

Consider $i(t) = I\cos(\omega t+\phi)$ where $I \hat{=}$ amplitude, $\omega \hat{=}$ angular frequency
$\phi \hat{=}$ phase

We may rewrite

$$i(t) = Re[\underbrace{Ie^{i\phi}}_{\text{scalar phasor}}\underbrace{e^{i\omega t}}_{\text{time dependency}}]$$

Re means "the real part of"

$Ie^{i\phi}$ scalar phasor contains amplitude (I) and phase (ϕ) information but ~~is~~ independent of time (t). After scalar phasor has been determined, $i(t)$ can be found by multiplying $Ie^{i\phi}$ by $e^{i\omega t}$ and taking the real part of it. If one desires to obtain $i(t)$ as a sine wave, then simply take imaginary part of $Ie^{i\phi}.e^{i\omega t}$ product instead of real part.

Express $3\cos\omega t - 4\sin\omega t$ as a) $A_1\cos(\omega t+\theta_1)$ b) $A_2\sin(\omega t+\theta_2)$
Determine $A_1, \theta_1, A_2, \theta_2$.

a) This is $\cos(\omega t)$ reference. Phasor I_{s_1} $3\cos(\omega t) = Re[3e^{i\omega t}] = Re[I_s e^{i\omega t}]$

$$\Rightarrow I_{s_1} = 3$$

Phasor I_{s_2} $-4\sin(\omega t) = -4\cos(\omega t - \pi/2)$

$$= Re[-4e^{-i\pi/2}.e^{i\omega t}]$$
$$= Re[4e^{i\pi/2}.e^{i\omega t}]$$
$-1 = e^{i\pi}$

$$\Rightarrow I_{s_2} = 4e^{i\pi/2} = 4i$$

Hence $A_1\cos(\omega t+\theta_1) = Re[(I_{s_1}+I_{s_2})e^{i\omega t}] = Re[\underbrace{(3+4i)}_{5e^{i53.1^\circ}}e^{i\omega t}]$

$$\boxed{A_1\cos(\theta_1+\omega t) = 5\cos(53.1^\circ+\omega t)}$$

b) $3\cos(\omega t) = 3\sin(\omega t + \pi/2) = \mathrm{Im}[\underbrace{3e^{i\pi/2}}_{I_{s1}} e^{i\omega t}]$

Now we are to use sinus reference

$-4\sin(\omega t) = \mathrm{Im}[\underbrace{\underbrace{e^{i\pi}}_{-1} \cdot 4}_{I_{s2}} e^{i\omega t}]$

$A_2 \sin(\omega t + \theta_2) = \mathrm{Im}[(I_{s1} + I_{s2})e^{i\omega t}]$

$= \mathrm{Im}[\underbrace{(3i - 4)}_{5e^{i143.1^\circ}} e^{i\omega t}]$

$A_2 \sin(\omega t + \theta_2) = 5\sin(\omega t + 143.1^\circ)$

As you can see, phasor representation removes time dependency and make it simpler to calculate time-harmonic field relations.

Phasor Representation of Maxwell's Equations

a. $\nabla \times \vec{E} + \dfrac{d\vec{B}}{dt} = 0 \rightarrow \nabla \times \vec{E} = -j\omega\mu\vec{H}$

b. $\nabla \times \vec{H} - \dfrac{d\vec{D}}{dt} = 0 \rightarrow \nabla \times \vec{H} = j\omega\varepsilon\vec{E} + \vec{J}$

c. $\nabla \cdot \vec{D} = \rho_v \rightarrow \nabla \cdot \vec{E} = \rho_v/\varepsilon$

d. $\nabla \cdot \vec{B} = 0 \rightarrow \nabla \cdot \vec{H} = 0$

Note that $\vec{E}(\vec{r},t) = \mathrm{Re}[\vec{E}(\vec{r})e^{i\omega t}]$

So $\dfrac{d\vec{E}(\vec{r},t)}{dt}$ and $\int \vec{E}(\vec{r},t)dt$

would be represented by, respectively vector phasors $j\omega\vec{E}(\vec{r})$ and $\dfrac{\vec{E}(\vec{r})}{j\omega}$

If the simple medium (linear, isotropic, homogeneous medium) is conducting ($\sigma \neq 0$), a current $\vec{J} = \sigma\vec{E}$ will flow. Equation b can be expressed as;

$$\nabla \times \vec{H} = j\omega\varepsilon\vec{E} + \sigma\vec{E} = j\omega\vec{E}\underbrace{\left(\varepsilon + \frac{\sigma}{j\omega}\right)}_{\text{complex permittivity}} = j\omega\underbrace{\varepsilon_c}_{\text{complex permittivity}}\vec{E}$$

where ε_c can be expressed as $\varepsilon_c = \varepsilon' - j\varepsilon'' = \varepsilon - j\dfrac{\sigma}{\omega}$

// If you take time dependency as $e^{-i\omega t}$; ε_c becomes $\varepsilon_c = \varepsilon + j\dfrac{\sigma}{\omega}$

Recall that $\tan\delta = \varepsilon''/\varepsilon' = \dfrac{\sigma}{\omega\varepsilon}$ is a measure of the ratio between magnitude of the conduction current and displacement current

If one uses phasor representations of Maxwell's Equations and manipulates as we did earlier, wave equation becomes as;

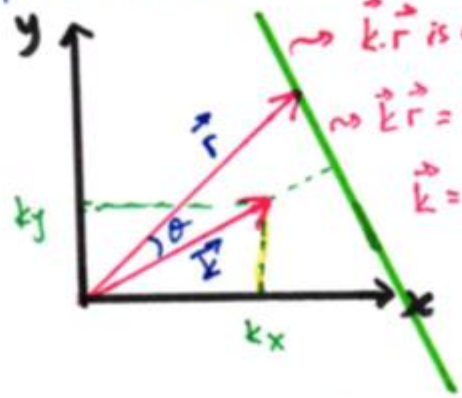
$$\nabla^2 \vec{E} + k^2 \vec{E} = 0$$

$$\nabla^2 \vec{H} + k^2 \vec{H} = 0$$

which is known as "Helmholtz Equation". Helmholtz Equation represents a time-independent form of wave equation, so the complexity of analysis is reduced.

Plane Electromagnetic Waves

Waves with wavefronts (surfaces of constant phase) which are infinite parallel planes are called "Plane Waves"



$\rightsquigarrow$ $\vec{k}.\vec{r}$ is constan (surfaces of constant phase)

$\rightsquigarrow$ $\vec{k}\vec{r} = |k||r|\cos\theta$ on entire infinite plane

$\vec{k} = k_x \vec{a_x} + k_y \vec{a_y} + k_z \vec{a_z}$ wave vector

$\vec{r} = x\vec{a_x} + y\vec{a_y} + z\vec{a_z}$ position vector

Plane Waves in Lossless Media

Consider a uniform plane wave characterized by a uniform E_x over plane surfaces perpendicular to z. Then;

$$\frac{d^2 E_x}{dy^2} = \frac{d^2 E_x}{dx^2} = 0 \ , \quad \nabla^2 \vec{E} = \frac{d^2 E_x}{dz^2} \rightarrow \frac{d^2 E_x}{dz^2} + k^2 E_x = 0$$

Solution of this ordinary differential equation is;

$$E_x(z) = E_x^+(z) + E_x^-(z) = \underbrace{E_0^+ e^{-jkz}}_{\rightarrow +z} + \underbrace{E_0^- e^{jkz}}_{\leftarrow -z}$$

E_0^+, E_0^- are amplitudes

Time domain (for the wave propagating in +z direction)

$$E_x^+(z,t) = Re\left[E_x^+(z) e^{jwt}\right] = Re\left[E_0^+ e^{j(wt-kz)}\right]$$

$$= E_0^+ \cos(\underbrace{wt - kz}_{\text{constant phase}}) \ [V/m]$$

The associated magnetic field $\vec{H}$? Recall that $\nabla \times \vec{E} = -j\omega\mu \vec{H}$ Hence;

$$\nabla \times \vec{E} = \begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x^+(z) & 0 & 0 \end{vmatrix} = \vec{a_y}\frac{d E_x^+(z)}{dz} = -j\omega\mu\left(H_x^+\vec{a_x} + H_y^+\vec{a_y} + H_z^+\vec{a_z}\right)$$

So, $H_x^+ = H_z^+ = 0$ and $H_y^+ = \frac{1}{-j\omega\mu}\frac{dE_x^+(z)}{dz} = \frac{1}{-j\omega\mu}\frac{d(E_0^+ e^{-jk_0 z})}{dz}$

$$\boxed{H_y^+(z) = \frac{k}{\omega\mu} E_x^+(z)} \; [A/m]$$

As it can be readily seen $\frac{|E_x^+(z)|}{|H_y^+(z)|} = \frac{\omega\mu}{k}$ which is called the intrinsic impedance

You may consider is as ratio of V/A. The value of $\frac{\omega\mu}{k} = v\mu = \frac{\mu}{\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}}$

$v = c$ for free space

$\doteq 120\pi \approx 377\,\Omega$

Time domain expression of H_y^+;

$H_y^+(z,t) = \mathrm{Re}\left[H_y^+(z)e^{j\omega t}\right] = \frac{E_0^+}{\eta}\cos(\omega t - kz)\;[A/m]$. Note that direction is in $\vec{a_y}$

where wave impedance $\eta \triangleq \frac{\omega\mu}{k}$

✓ For a uniform plane wave, the ratio of the magnitudes of $\vec{E}$ and $\vec{H}$ is the intrinsic impedance of the medium.

✓ $\vec{H}$ is perpendicular to $\vec{E}$ and both are normal to the direction propagation.

Transverse Electromagnetic Waves

We now consider the propagation of a uniform planewave in 3 dimensional space

Recall that $\vec{k} = k_x\vec{a_x} + k_y\vec{a_y} + k_z\vec{a_z}$ (wave vector)

$\vec{r} = x\vec{a_x} + y\vec{a_y} + z\vec{a_z}$ (position vector ~~with~~ with tail at origin)

So, we may write;

$$\vec{E}(\vec{r}) = E_0 \cdot e^{-i\vec{k}\vec{r}} = E_0 e^{-ik\vec{a_n}\cdot\vec{r}}$$

where $\vec{a_n}$ is the unit vector in the direction of propagation

$\vec{k} = k\vec{a_n}$ and $k^2 = k_x^2 + k_y^2 + k_z^2 = \omega^2 \varepsilon\mu = \frac{2\pi}{\lambda}$

Using phaser representation of M.E., we may write;

$$\boxed{\vec{H}(\vec{r}) = \frac{1}{\eta}\vec{a_n} \times \vec{E}(\vec{r}) \ [A/m]}$$

for which $\vec{E}(\vec{r}) = E_0 e^{-i\vec{k}\vec{r}} = E_0 e^{-ik\vec{a_n}\vec{r}}$ is used. Also, note that $\vec{E} \perp \vec{H} \perp \vec{k}$

Similarly;

$$\boxed{\vec{E}(\vec{r}) = \eta \vec{H}(\vec{r}) \times \vec{a_n} \ [V/m]}$$

Plane Waves in Conducting Medium

The Helmholtz equation to be solved becomes;

$\nabla^2\vec{E} + k_c^2\vec{E} = 0$

where the wavenumber $k_c = \omega\sqrt{\mu\varepsilon_c}$ is a complex number since $\varepsilon_c = \varepsilon' - j\varepsilon''$ is a complex quantity. Let's define a propagation constant γ as;

$\gamma = ik_c = i\omega\sqrt{\mu\varepsilon_c}$ (1/m) Recall that $\varepsilon_c = \varepsilon + \frac{\sigma}{i\omega}$

$$\gamma = \alpha + i\beta = i\omega\sqrt{\mu\left(\varepsilon + \frac{\sigma}{i\omega}\right)}$$

$$\gamma = i\omega\sqrt{\mu\varepsilon}\left(1 + \frac{\sigma}{i\omega\varepsilon}\right)^{1/2}$$

// If medium is lossless (that is $\sigma = 0$), $\alpha = 0$ and $\beta = \omega\sqrt{\varepsilon\mu} = k$

The Helmholtz equation becomes;

$$\nabla^2 \vec{E} - \gamma^2 \vec{E} = 0$$

which has a uniform wave solution, propagating in +z direction and -z direction;

$$\vec{E} = \vec{a_x} E_0^+ e^{-\gamma z} + \vec{a_x} E_0^- e^{\gamma z}$$

$$\underbrace{\vec{a_x} E_0^+ e^{-\gamma z}}_{+z\ \text{direction}} \quad \underbrace{\vec{a_x} E_0^- e^{\gamma z}}_{-z\ \text{direction}}$$

$$\vec{E} = \vec{a_x} E_0^+ e^{-\gamma z} = \vec{a_x} E_0^+ \underbrace{e^{-\alpha z}}_{\text{attenuates}} \cdot \underbrace{e^{-j\beta z}}_{\text{propagates}}$$

The wave attenuates proportional to $e^{-\alpha z}$ and propagates with a phase shift $e^{-j\beta z}$ (that occurs with amount of β per meter)

α: attenuation constant [Np/m]

β: propagation constant [rad/m]

Low-loss Dielectric

$\varepsilon_c = \varepsilon' - j\varepsilon''$ and $\varepsilon'' << \varepsilon'$ ($\sigma/\omega\varepsilon << 1$) (Property of a low loss dielectric)

Recall that $\gamma = \alpha + i\beta = i\omega\sqrt{\mu\varepsilon}\left(1 + \frac{\sigma}{j\omega\varepsilon}\right)^{1/2}$. $(1+x)^{1/2} \cong 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \text{H.O.T}$

is the binomial series. If one replaces x with $\sigma/j\omega\varepsilon$;

for a good dielectric

$$\gamma = \alpha + i\beta \cong i\omega\sqrt{\mu\varepsilon}\left(1 + \frac{\sigma}{j2\omega\varepsilon} + \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right)$$

which yields;

$$\alpha \cong \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} \ [\text{Np/m}]$$

$$\beta \cong \omega\sqrt{\mu\varepsilon}\left[1 + \frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right] \ [\text{rad/m}]$$

The intrinsic impedance of a low-loss dielectric is approximately

$$\eta_c = \frac{\omega\mu}{k_c} \cong \frac{\omega\mu}{\omega\sqrt{\varepsilon\mu}\left(1 + \frac{\sigma}{j2\omega\varepsilon}\right)} \cong \sqrt{\frac{\mu}{\varepsilon}}\left(1 + i\frac{\sigma}{2\omega\varepsilon}\right) \ [\Omega]$$

Note that, complex behaviour of η_c is also related to out of phase relation of $\vec{E}$ & $\vec{H}$ in lossy medium.

phase velocity $v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\varepsilon}}\left[1+\frac{1}{8}\left(\frac{\sigma}{\omega\varepsilon}\right)^2\right]$ [m/s]

Good Conductor

$\varepsilon'' \gg \varepsilon'$ or $\sigma/\omega\varepsilon \gg 1 \rightarrow$ the condition for a good conductor

$$\gamma \cong i\omega\sqrt{\mu\varepsilon}\sqrt{\frac{\sigma}{i\omega\varepsilon}} = \sqrt{i\omega\mu\sigma} = \frac{1+i}{\sqrt{2}}\sqrt{\omega\mu\sigma} \rightarrow \sqrt{i} = e^{i\pi/4} = \frac{1+i}{\sqrt{2}}$$

$\left(1+\frac{\sigma}{\omega\varepsilon i} \cong \frac{\sigma}{i\omega\varepsilon}\right)$ Hence;

$$\gamma \cong \alpha + i\beta \cong (1+i)\sqrt{\pi f\mu\sigma} \qquad \boxed{\beta = \alpha \cong \sqrt{\pi f \mu\sigma}}$$

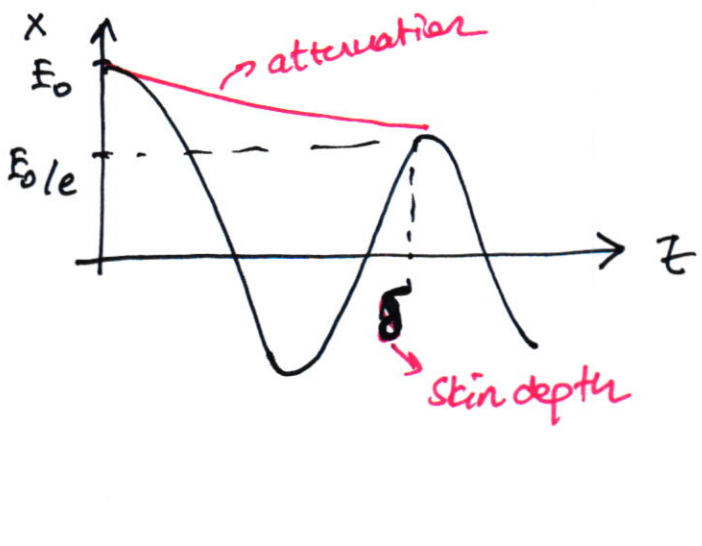
$$\eta_c = \frac{\omega\mu}{k} \cong \frac{\omega\mu}{(\alpha+i\beta)/i} = \frac{i\,\omega\mu}{(1+i)\sqrt{\pi f\mu\sigma}} = (1+i)\sqrt{\frac{\pi f\mu}{\sigma}} \quad [\Omega]$$

$$v_p = \frac{\omega}{\beta} \cong \sqrt{\frac{2\omega}{\mu\sigma}} \quad [\text{m/sec}]$$

Skin Depth(δ)

The distance δ is the skindepth for which the amplitude of a travelling wave decreases by a factor $1/e$

$$\boxed{\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f\mu\sigma}} \quad [\text{m}]}$$



$$\vec{E} = E_0 e^{-\alpha z}\cos(\omega t - \beta z)\vec{a_x}$$

Group Velocity

Recall that phase velocity $v_p = \frac{\omega}{\beta}$ and for plane waves in a lossless medium, $\beta = \omega\sqrt{\varepsilon\mu}$. As a result, $v_p = 1/\sqrt{\mu\varepsilon}$ which is independent of frequency.

For the cases that the phase velocity varies by frequency (e.g., lossy dielectrics, waveguides, etc.), each component with different frequency propagates with different velocity. Consequently the total signal dispers, and such mediums are called "dispersive".

Consider two waves y_1 and y_2 with same amplitude but different phase velocities.

$$y_1 = \cos([\omega_0 + \Delta\omega]t - [\beta_0 + \Delta\beta]z)$$

$$y_2 = \cos([\omega_0 - \Delta\omega]t - [\beta_0 - \Delta\beta]z)$$

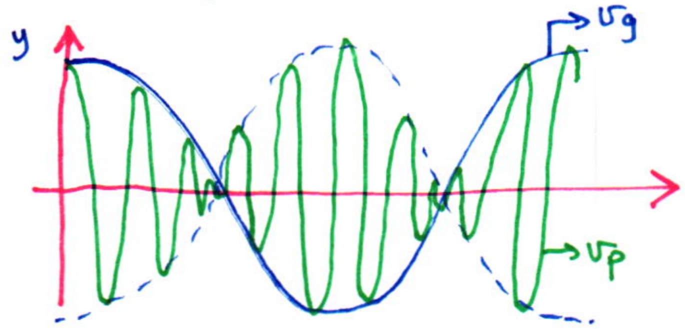
If both travels, total wave $y = y_1 + y_2$

$$= 2\underbrace{\cos(\Delta\beta z - \Delta\omega t)}_{\text{I}}\ \underbrace{\cos(\beta_0 z - \omega_0 t)}_{\text{II}}$$

$\text{II} \to v_p = \omega_0/\beta_0$

$\text{I} \to v_g = \frac{\Delta\omega}{\Delta\beta} = \lim_{\Delta\beta\to 0} \frac{1}{d\beta/d\omega}$

If $\Delta\omega \ll \omega_0$; the total wave looks like



A mathematical connect may be seen by;

$$\frac{d\beta}{d\omega} = \frac{d}{d\omega}\left(\frac{\omega}{v_p}\right) = \frac{v_p - \omega\, dv_p/d\omega}{v_p^2} \to v_g = \frac{1}{d\beta/d\omega} = \frac{v_p}{1 - \frac{\omega}{v_p}\frac{dv_p}{d\omega}}$$

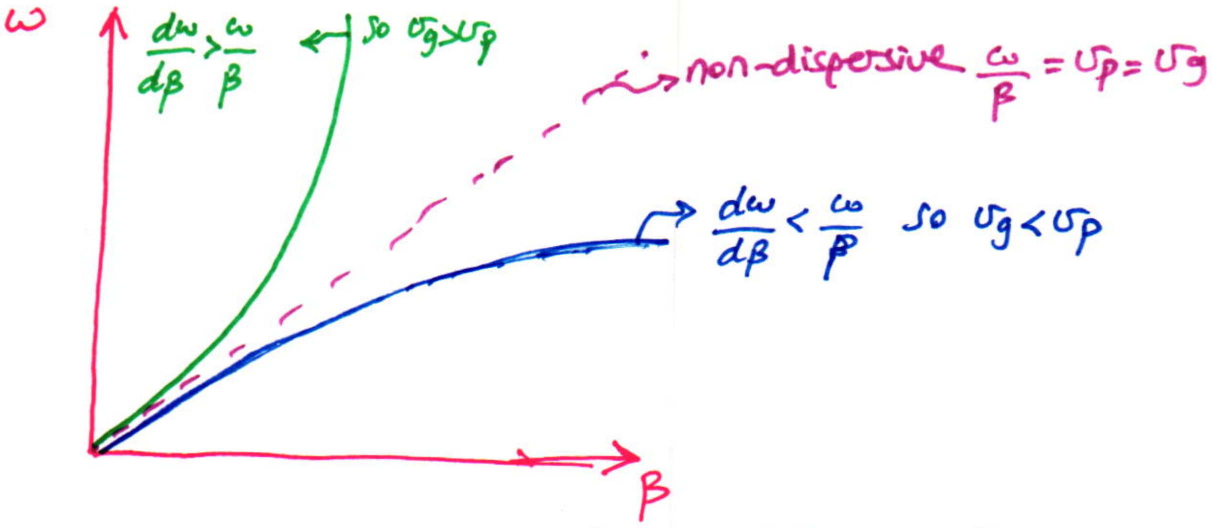
If $\frac{dv_p}{d\omega} = 0$ (phase velocity doesn't vary by frequency)

then $v_g = v_p$ (non-dispersive medium)

a) No dispersion $dv_p/d\omega = 0$, $v_p = v_g$

b) Normal dispersion $\frac{dv_p}{d\omega} < 0$, $v_g < v_p$

c) Anomalous dispersion $\frac{dv_p}{d\omega} > 0$, $v_g > v_p$



Flow of the Electromagnetic Power and Poynting Vector

$\nabla \times \vec{E} = -\dfrac{d\vec{B}}{dt}$ and $\nabla \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot (\nabla \times \vec{E}) - \vec{E}(\nabla \times \vec{H})$

$\nabla \times \vec{H} = \vec{J} + \dfrac{d\vec{D}}{dt}$ $\qquad \nabla \cdot (\vec{E} \times \vec{H}) = -\vec{H}\dfrac{d\vec{B}}{dt} - \vec{E}\dfrac{d\vec{D}}{dt} - \vec{E}\vec{J}$

In a simple medium $\vec{B} = \mu \vec{H}$, $\vec{D} = \varepsilon \vec{E}$. So;

$$\vec{H}\frac{d\vec{B}}{dt} = \vec{H}\frac{d(\mu\vec{H})}{dt} = \frac{1}{2}\frac{d(\mu\vec{H}\cdot\vec{H})}{dt} = \frac{d}{dt}\left(\frac{\mu H^2}{2}\right)$$

$$\vec{E}\frac{d\vec{D}}{dt} = \vec{E}\frac{d(\varepsilon\vec{E})}{dt} = \frac{1}{2}\frac{d(\varepsilon\vec{E}\cdot\vec{E})}{dt} = \frac{d}{dt}\left(\frac{\varepsilon}{2}E^2\right)$$

and

$\vec{E}\cdot\vec{J} = \vec{E}(\sigma\vec{E}) = \sigma E^2$. let's rewrite

$$\nabla \cdot (\vec{E} \times \vec{H}) = -\frac{d}{dt}\left(\mu H^2/2 + \varepsilon E^2/2\right) - \sigma E^2$$

integrating through a volume

$$\int_v \nabla \cdot (\vec{E} \times \vec{H})\,dv = \oint_S \vec{E} \times \vec{H} \cdot d\vec{s} = \underbrace{-\frac{d}{dt}}_{\text{rate of decrease}} \int_v \Big(\underbrace{\frac{1}{2}\varepsilon E^2}_{\text{stored electric energy}} + \underbrace{\frac{1}{2}\mu H^2}_{\text{stored magnetic energy}}\Big)dv - \underbrace{\int_v \sigma E^2 dv}_{\text{dissipated ohmic power}}$$

divergence theorem

Due to the law of conservation of energy, right hand side terms in total, must be equal to the power, leaving the volume through surface S. Thus $\vec{E} \times \vec{H}$ represents the power flow per unit area.

$$\vec{P} = \vec{E} \times \vec{H} \; [W/m^2]$$

Poynting vector

$$-\oint_S \vec{P}\cdot d\vec{s} = \frac{d}{dt}\int_v (w_e + w_M)\,dv + \int_v P_\sigma\,dv$$

where

$w_e = \frac{1}{2}\varepsilon E^2$: electric energy density

$w_M = \frac{1}{2}\mu H^2$: magnetic energy density

$P_\sigma = \sigma E^2$: ohmic power density

$\vec{E}\perp\vec{H}\perp\vec{P}$

Instantaneous and Average Power Densities

// Some mathematical relations:

// $Re[A] = \frac{1}{2}[A + A^*]$, where $*$ denotes complex conjugate

// $Re[A]\times Re[B] = \frac{1}{2}[A+A^*]\times\frac{1}{2}[B+B^*]$ and $(A\times B^*)^* = A^*\times B$

// $= \frac{1}{4}\left[(A\times B^* + A^*\times B) + (A\times B + A^*\times B^*)\right]$

// $\boxed{Re[A]\times Re[B] = \frac{1}{2} Re\left[A\times B^* + A\times B\right]}$

let $A = E(z)e^{i\omega t}$, $B = H(z)e^{i\omega t}$ where $\vec{E}(z) = \vec{a}_x E_x(z) = a_x E_0\, e^{-(\alpha+i\beta)z}$

Hence, $E(z,t) = Re[E(z)e^{i\omega t}] = \vec{a}_x E_0 e^{-\alpha z}\cos(\omega t - \beta z)$

Besides, $H(z) = \vec{a}_y H_y(z) = \vec{a}_y \frac{E_0}{|\eta|} e^{-\alpha z} e^{-i(\beta z + \theta_\eta)}$; Note that $\eta = |\eta| e^{i\theta_\eta}$

So, $H(z,t) = Re[H(z)e^{i\omega t}] = \vec{a}_y\, E_0/|\eta|\; e^{-\alpha z}\cos(\omega t - \beta z - \theta_\eta)$

The instantaneous expression for the Poynting vector becomes;

$$\vec{P}(z,t) = \vec{E}(z,t)\times\vec{H}(z,t) = Re[E(z)e^{i\omega t}]\times Re[H(z)e^{i\omega t}]$$

$$= \frac{1}{2} Re\Big[\underbrace{E(z)e^{i\omega t}\times H^*(z)e^{-i\omega t}}_{\frac{E_0^2}{|\eta|}\underbrace{e^{i\theta_\eta}}_{\text{Real Part}}\cdot\vec{a}_z} + \underbrace{E(z)e^{i\omega t}\times H(z)e^{i\omega t}}_{\frac{E_0^2}{|\eta|}e^{-2\alpha z}\underbrace{e^{-2j\beta z}e^{-i\theta_\eta}e^{2i\omega t}}_{\text{Real part}}\cdot\vec{a}_z}\Big]$$

$$\vec{P}(z,t) = \frac{1}{2}\frac{E_0^2}{|\eta|}e^{-2\alpha z}\left[\cos(\theta_\eta) + \cos(2\omega t - 2\beta z - \theta_\eta)\right]\cdot\vec{a}_z$$

Average value is more meaningful than instantaneous value. Time-average Poynting vector $P_{av}(z)$ becomes;

$$\vec{P}_{av}(z) = \frac{1}{T}\int_0^T \vec{P}(z,t)\,dt = \vec{a}_z \frac{E_0^2}{2|\eta|} e^{-2\alpha z} \cos(\theta_\eta) \quad [W/m^2]$$

where T is the period of the wave. If you examine ~~on~~ the derivations, it's clear to say that

$$\vec{P}_{av}(z) = \frac{1}{2} \mathrm{Re}\left[E(z) \times H^*(z)\right]$$

or in general;

$$\boxed{\vec{P}_{av} = \frac{1}{2} \mathrm{Re}\left[E(z) \times H^*(z)\right]} \quad [W/m^2]$$

Polarization of Plane Waves

Consider electric field components of a z-directed propagating plane wave;

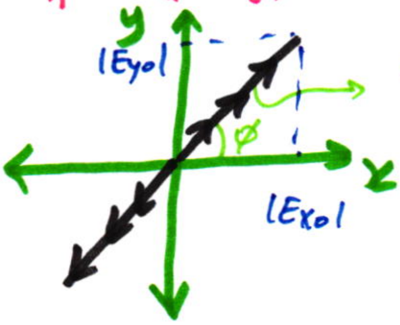
$\vec{E}(z) = (E_{x0}\vec{a_x} + E_{y0}\cdot\vec{a_y})e^{-j\beta z}$

$= (|E_{x0}|e^{j\theta_x}\vec{a_x} + |E_{y0}|e^{j\theta_y}\vec{a_y})e^{-j\beta z}$

Time domain

$\vec{E}(z,t) = \vec{a_x}|E_{x0}|\cos(\omega t - \beta z + \theta_x) + \vec{a_y}|E_{y0}|\cos(\omega t - \beta z + \theta_y)$

If $\theta_x = \theta_y$, then this is a "linearly polarized" wave. See figure below;



wave depicts on this constant line

$\tan\phi = \frac{|E_{y0}|}{|E_{x0}|} \Rightarrow$ constant

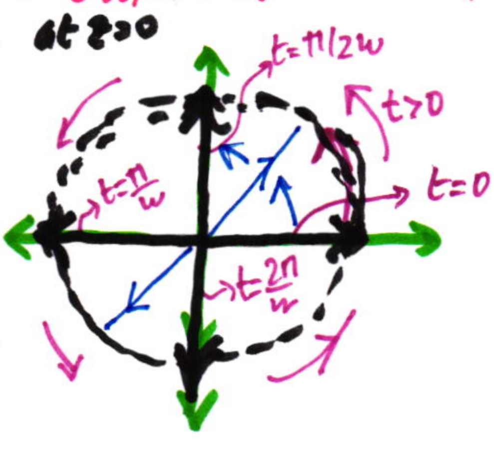
Now, let's give $\pi/2$ phase difference between θ_x and θ_y. Consequently one reaches minimum when the other one peaks maximum. Meanwhile, they plot a circle on a perpendicular plane to the direction of propagation if $|E_{x0}| = |E_{y0}| = E_0$

Let $\theta_x = 0$ and $\theta_y = \pi/2$.

$\vec{E}(z) = E_0(\vec{a_x} + e^{j\pi/2}\vec{a_y})e^{-j\beta z}$

$\rightarrow \vec{E}(z,t) = E_0(\cos(\omega t - \beta z)\vec{a_x} + \sin(\omega t - \beta z)\vec{a_y})$

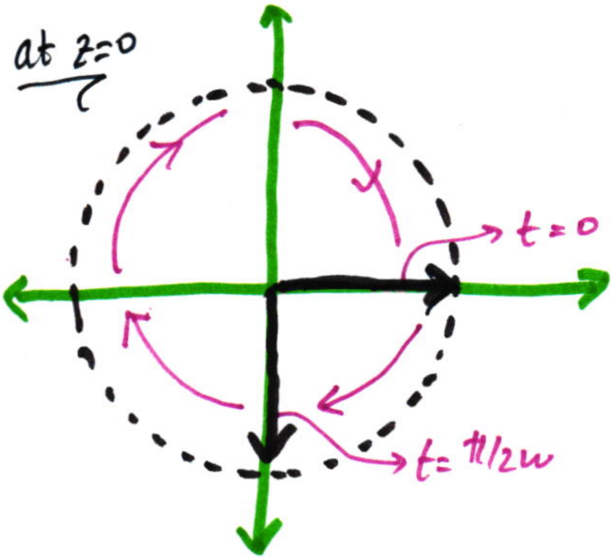
at $z=0$



This is called as "right hand circular polarization)

a.k.a rhcp

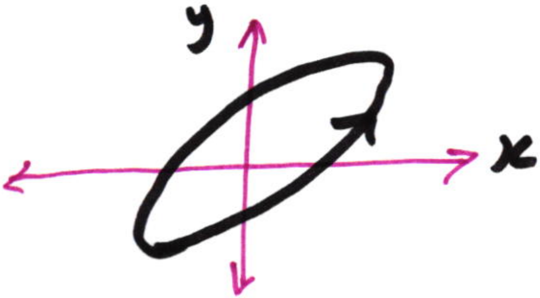
What if $\theta_x = 0$ and $\theta_y = -\pi/2$?



$$\vec{E}(z,t) = E_0 \left(\cos(\omega t - \beta z)\vec{a_x} - \sin(\omega t - \beta t)\vec{a_y}\right)$$

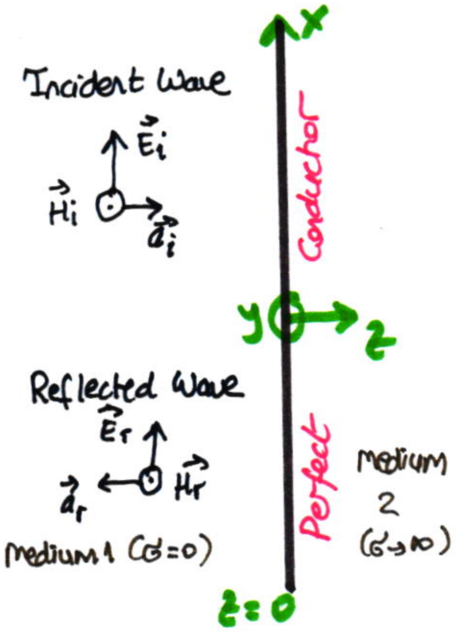
This time, field vector rotates in clockwise direction and that is known as "left hand circular polarization"!

If both amplitudes and phases differ from each other, the most general case, which is known as "elliptical polarization", occurs



Normal Incidence at a Conducting Plane

Let's assume the incident wave travels in a lossless medium ($\sigma = 0$) and impinges on a perfect conductor ($\sigma \to \infty$).



The incident wave travels in +z direction, and PEC plane is at $z=0$. $\vec{E}_i(z)$ and $\vec{H}_i(z)$ phasors are;

$$\vec{E}_i(z) = \vec{a}_x E_{i0} \cdot e^{-j\beta_1 z}$$

$$\vec{H}_i(z) = \vec{a}_y H_{i0} e^{-j\beta_1 z}$$

In medium 2, both $\vec{E}$ and $\vec{H}$ vanishes ($E_2 = H_2 = 0$). The reflected $\vec{E}$ can be written as;

$$\vec{E}_r(z) = \vec{a}_x E_{r0} e^{j\beta_1 z}$$

The total $\vec{E}$ in medium 1 is $\vec{E}_i + \vec{E}_r$, that is;

$$\vec{E}_1(z) = \vec{E}_i(z) + \vec{E}_r(z) = \vec{a}_x \left(E_{i0} e^{-j\beta_1 z} + E_{r0} e^{j\beta_1 z}\right)$$

At $z=0$, boundary condition yields that $E_{1t} = E_{2t} = 0$. That is;

$$\vec{E}_1(z=0) = \vec{E}_2(z=0) = 0 \rightarrow \vec{a}_x (E_{i0} + E_{r0}) = 0 \rightarrow \boxed{E_{r0} = -E_{i0}}$$

Hence;

$$\vec{E}_1(z) = \vec{a}_x E_{i0}\left(e^{-j\beta_1 z} - e^{j\beta_1 z}\right) = \underline{-\vec{a}_x 2j E_{i0} \sin \beta_1 z} \leftarrow \text{Total } \vec{E} \text{ in medium 1}$$

Using relation between $\vec{H}$ and $\vec{E}$, one may determine $\vec{H}$ reflected as;

$$\vec{H}_r(z) = \frac{1}{\eta_1} \vec{a}_r \times \vec{E}_r(z) = \frac{1}{\eta_1} -\vec{a}_z \times -\vec{a}_x E_{i0} e^{j\beta_1 z} = \underline{\vec{a}_y \frac{E_{i0}}{\eta_1} e^{j\beta_1 z}}$$

Similarly $\vec{H}_i(z) = \frac{1}{\eta_1} \vec{a}_z \times \vec{E}_i(z) = \frac{1}{\eta_1} \vec{a}_z \times \vec{a}_x E_{i0} e^{-j\beta_1 z} = \vec{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z}$. So, the total $\vec{H}$ field in medium 1 is;

$$\vec{H}_1(z) = \vec{H}_i(z) + \vec{H}_r(z) = \vec{a}_y \frac{E_{i0}}{\eta_1}\left(e^{j\beta_1 z} + e^{-j\beta_1 z}\right) = \underline{\vec{a}_y \frac{2E_{i0}}{\eta_1} \cos(\beta_1 z)} \leftarrow \text{Total } \vec{H} \text{ in medium 1}$$

Recall that $\vec{P}_{av} = \frac{1}{2} Re\left[\vec{E} \times \vec{H}^*\right]$. For the analysis above $Re\left[\vec{E} \times \vec{H}^*\right] = 0$, which indicates that no power in average exist.

Let's examine time-space behavior of the total field in medium 1.

$$\vec{E}_1(z,t) = Re\,[\vec{E}_1(z)e^{j\omega t}] = \vec{a_x}\, 2E_{i0} \sin\beta_1 z \sin\omega t;$$

$$\vec{H}_1(z,t) = Re\,[\vec{H}_1(z)e^{j\omega t}] = \vec{a_y}\, \frac{2}{\eta_1} E_{i0} \cos\beta_1 z \cos\omega t.$$

note that: Standing waves!

Note that $\vec{E}_1(z)$ and $\vec{H}_1(z)$ have their minima and maxima at the PEC boundary, respectively. Besides they are out of phase by $\pi/2$ in time ($\sin\omega t \Leftrightarrow \cos\omega t$) and shifted in space by $\lambda/4$ ($\beta.\lambda/4 = \frac{2\pi}{\lambda}.\frac{\lambda}{4} = \pi/2$).

A y-polarized uniform plane wave with a frequency 100 MHz propagates in air and impinges normally on a PEC plane at x=0. $|\vec{E_i}|_{max} = 6\,mV/m$. a) Phasors of $\vec{E_i}, \vec{H_i}$ b) $\vec{E_r}, \vec{H_r}$ c) $\vec{E}_{total}, \vec{H}_{total}$ d) First 0 of E_{total}?

$$\omega = 2\pi f = 2\pi 10^8 \text{ rad/sn}; \quad \beta_1 = \frac{\omega}{c} = 2\pi/3 \text{ rad/m}, \quad \eta_1 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi$$

a) $\vec{E_i}(x) = \vec{a_y}\, 6\times10^{-3} e^{-j2\pi x/3}$ (V/m); $\vec{H_i}(x) = \vec{a_z}\, \frac{10^{-4}}{2\pi} e^{-j2\pi x/3}$ (A/m)

$\vec{E_i}(x,t) = \vec{a_y}\, 6\times10^{-3}\cos(2\pi10^8 t - \frac{2\pi}{3}x)$; $\vec{H_i}(x,t) = \vec{a_z}\frac{10^{-4}}{2\pi}\cos(2\pi10^8 t - \frac{2\pi}{3}x)$

b) $\vec{E_r}(x) = -\vec{a_y}\, 6\times10^{-3} e^{j2\pi x/3}$; $\vec{H_r}(x) = \vec{a_z}\,\frac{10^{-4}}{2\pi} e^{j2\pi x/3}$

$\vec{E_r}(x,t) = -\vec{a_y}\, 6\times10^{-3}\cos(2\pi10^8 t + \frac{2\pi}{3}x)$, $\vec{H_r}(x,t) = \vec{a_z}\frac{10^{-4}}{2\pi}\cos(2\pi10^8 t + \frac{2\pi}{3}x)$

c) $E_1(x) = E_i(x) + E_r(x) = -\vec{a_y}\, j\, 12\times10^{-3}\sin(\frac{2\pi}{3}x)$

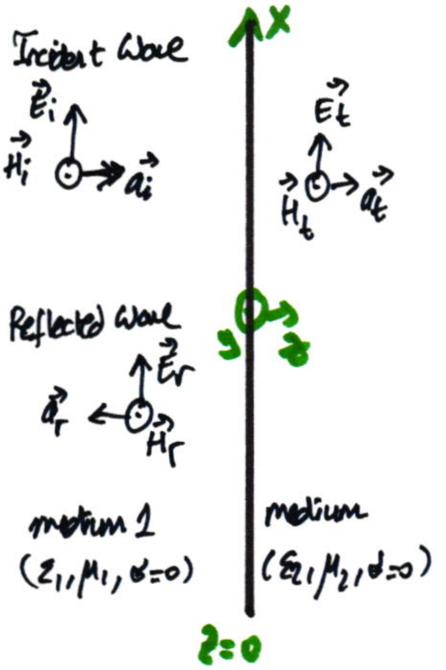
$H_1(x) = H_i(x) + H_r(x) = \vec{a_z}\,\frac{10^{-4}}{2\pi}\cos(\frac{2\pi}{3}x)$

$E_1(x,t) = \vec{a_y}\, 12\times10^{-3}\sin(\frac{2\pi}{3}x)\sin(2\pi\times10^8 t)$

$H_1(x,t) = \vec{a_z}\,\frac{10^{-4}}{\pi}\cos(\frac{2\pi}{3}x)\cos(2\pi\times10^8 t)$

d) $\frac{2\pi}{3}x = -\pi \Rightarrow x = -3/2$ meter

Normal Incidence at a Dielectric Boundary



$$\underbrace{\vec{E}_i(z) = \vec{a}_x E_{i0} e^{-j\beta_1 z} \quad ; \quad \vec{H}_i(z) = \vec{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z}}_{\text{incident fields}}$$

$$\underbrace{\vec{E}_r(z) = \vec{a}_x E_{r0} e^{j\beta_1 z} \quad ; \quad \vec{H}_r(z) = -\vec{a}_y \frac{E_{r0}}{\eta_1} e^{j\beta_1 z}}_{\text{reflected fields}}$$

$$\underbrace{\vec{E}_t(z) = \vec{a}_x E_{t0} e^{-j\beta_2 z} \; ; \; \vec{H}_t(z) = \vec{a}_y \frac{E_{t0}}{\eta_2} e^{-j\beta_2 z}}_{\text{transmitted fields}}$$

Utilizing boundary conditions, we may write;

$E_{1t} = E_{2t}$ and $H_{1t} = H_{2t}$. Hence;

$$E_i(0) + E_r(0) = E_t(0) \rightarrow E_{i0} + E_{r0} = E_{t0}$$

$$H_i(0) + H_r(0) = H_t(0) \rightarrow \frac{1}{\eta_1}(E_{i0} - E_{r0}) = \frac{E_{t0}}{\eta_2}$$

$$\rightarrow E_{r0} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} E_{i0} \; ; \; E_{t0} = \frac{2\eta_2}{\eta_2 + \eta_1} E_{i0}$$

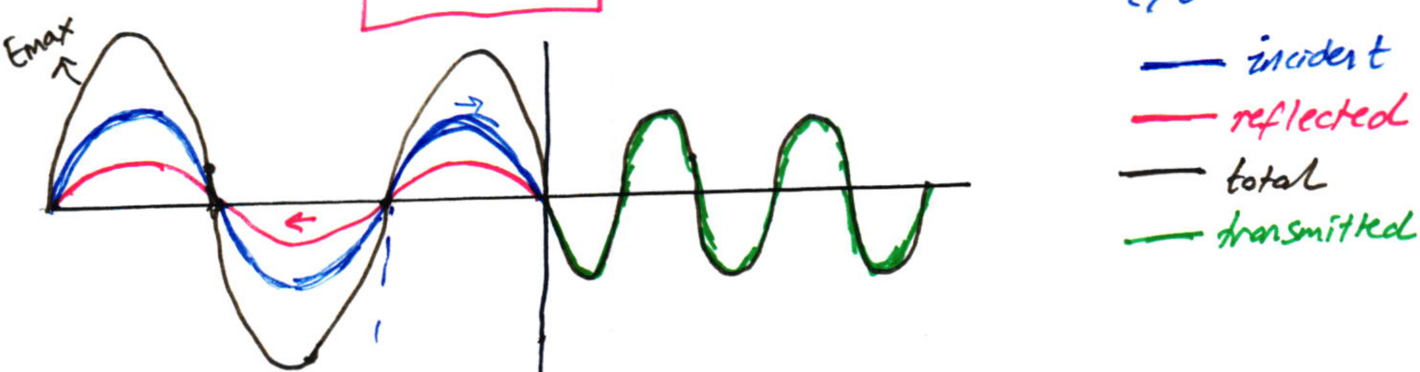
$\frac{E_{r0}}{E_{i0}}$ and $\frac{E_{t0}}{E_{i0}}$ are called reflection coefficient and transmission coefficient, respectively.

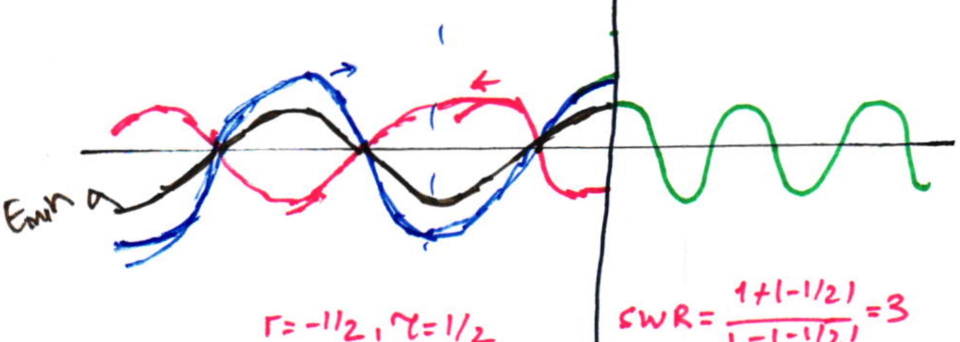
$$\underbrace{\Gamma = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}}_{\text{reflection coefficient}} \quad ; \quad \underbrace{\tau = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2}{\eta_2 + \eta_1}}_{\text{transmission coefficient}}$$

$$\boxed{1 + \Gamma = \tau}$$

Γ may be less or greater than 0

$\tau > 0$





$$SWR = \frac{|E_{max}|}{|E_{min}|} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

$\Gamma = -1/2$, $\tau = 1/2$

$(\varepsilon_r, \mu_r) = (1, 1)$

$$SWR = \frac{1 + |-1/2|}{1 - |-1/2|} = 3$$

$(\varepsilon_r, \mu_r) = (9, 1)$

Phasor

$$\vec{B}_p = \frac{20}{i}\vec{a_x} + 10e^{i\frac{2\pi}{3}x}\vec{a_y} \rightarrow \vec{B}(x,t)?$$

$$\vec{B}(x,t) = Re\left[\vec{B}_p \cdot e^{i\omega t}\right]$$

$$\frac{1}{i} = -i = e^{-i\pi/2}$$

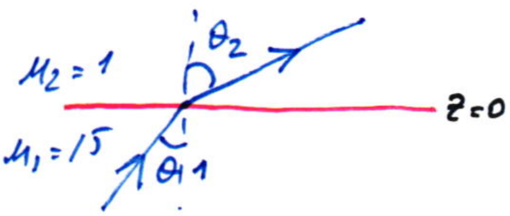
$$= Re\left[\vec{a_x}\left(20e^{i(\omega t - \pi/2)}\right) + \vec{a_y}\left(10e^{i(\omega t + \frac{2\pi}{3}x)}\right)\right]$$

$$= 20\cos(\omega t - \pi/2)\vec{a_x} + 10\cos(\omega t + \frac{2\pi}{3}x)\vec{a_y}$$

Boundary Condition

At $z=0$, $\vec{B_1} = 1.2\vec{a_x} + 0.8\vec{a_y} + 0.4\vec{a_z} \Rightarrow \vec{H_1}, \vec{H_2}, \vec{B_2}$? (Source free media)

θ_1, θ_2?



$$\vec{B_1} = \mu_1\vec{H_1} \qquad B_{1n} = B_{2n}$$

$$\vec{B_2} = \mu_2\vec{H_2} \qquad H_{1t} = H_{2t}$$

$B_{1n} = 0.4 \Rightarrow B_{2n} = 0.4$ (in $\vec{a_z}$ direction)

$$\vec{H_1} = \frac{1.2\vec{a_x} + 0.8\vec{a_y} + 0.4\vec{a_z}}{15} \Rightarrow H_{1t} = (1.2\vec{a_x} + 0.8\vec{a_y})/15$$

$$H_{2t} = \frac{B_{2t}}{\mu_2} = H_{1t} \Rightarrow B_{2t} = \frac{1.2}{15}\vec{a_x} + \frac{0.8}{15}\vec{a_y}$$

$$\vec{B_2} = \vec{B_{2n}} + \vec{B_{2t}} = \dots$$

$$\vec{H_1} = \vec{H_{1n}} + \vec{H_{1t}} = \dots$$

$$\tan\theta_1 = \frac{|B_{1t}|}{|B_{1n}|} = \dots$$

$$\tan\theta_2 = \frac{|B_{2t}|}{|B_{2n}|} = \dots$$

In a sourceless medium ($\vec{J}=0, \rho_v=0$), $\vec{E}=E_0 \sin(\omega t-\beta z)\vec{a_x}$ is given.

Determine $\vec{H}$? Determine β?

$$\nabla\times\vec{E}+\frac{d\vec{B}}{dt}=0 \; ; \quad \nabla\times\vec{E}=\begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ d/dx & d/dy & d/dz \\ E_x & 0 & 0 \end{vmatrix} = \vec{a_y}\,\frac{dE_x}{dz}$$

$$= -\vec{a_y}\,\beta\cos(\omega t-\beta z)E_0 = \frac{-d\vec{B}}{dt}$$

$$\Rightarrow \vec{B} = \int \beta\cos(\omega t-\beta z)E_0\vec{a_y}\,.dt = \frac{\beta}{\omega}\sin(\omega t-\beta z)E_0\vec{a_y} + \cancel{C}\,.\vec{a_y}$$

integral constant we may neglect it

$$\boxed{\Rightarrow \vec{H} = \frac{\beta E_0}{\omega\mu_0}\sin(\omega t-\beta z)\vec{a_y}}$$

$$\nabla\times\vec{H}=\frac{d\vec{D}}{dt} \Rightarrow \nabla\times\vec{H}=\begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ 0 & 0 & d/dz \\ 0 & H_y & 0 \end{vmatrix} = \frac{-dH_y}{dz}\vec{a_x} = \frac{\beta^2 E_0}{\omega\mu_0}\cos(\omega t-\beta z)\vec{a_x}$$

$$= \frac{d\vec{D}}{dt} \rightarrow \vec{D} = \frac{\beta^2 E_0}{\omega\mu_0}\int \cos(\omega t-\beta z)\vec{a_x}\,dt$$

$$\vec{D} = \frac{\beta^2 E_0}{\omega^2\mu_0}\sin(\omega t-\beta z)\vec{a_x}$$

$$\vec{E}.\varepsilon_0=\vec{D} \Rightarrow \frac{\beta^2 E_0}{\omega^2\varepsilon_0\mu_0} = E_0 \rightarrow \boxed{\beta=\omega\sqrt{\varepsilon_0\mu_0}}$$

Maxwell

Boundary Conditions

$\vec{B}_1 = 2.4\vec{a_x} + 10\vec{a_z}$

$\vec{B}_2 = 25.75\vec{a_x} - 17.7\vec{a_y} + 10\vec{a_z}$

$\mu_2 = 5$ $\quad \vec{a_n} = \vec{a_z}$ $\quad z>0$

$z=0$

$\mu_1 = 1.5$ $\quad z<0$

Determine the current density $\vec{J}_s$ at origin

recall that $\vec{a_n} \times (\vec{H}_2 - \vec{H}_1) = 0$

$$\vec{H}_1 = \frac{\vec{B}_1}{\mu_0\mu_1} = \frac{1}{\mu_0}\left(1.6\vec{a_x} + 6.66\vec{a_z}\right) \text{ A/m}$$

$$\vec{H}_2 = \frac{\vec{B}_2}{\mu_0\mu_2} = \frac{1}{\mu_0}\left(5.15\vec{a_x} - 3.54\vec{a_y} + 2\vec{a_z}\right) \text{ A/m}$$

$\vec{J}_s = \vec{a_z} \times (\vec{H}_2 - \vec{H}_1)$ which only contributes via tangential components that is $\vec{a_x}$ & $\vec{a_y}$ components

$$\equiv \vec{a_z} \times \left([5.15 - 1.6]\vec{a_x} - 3.54\vec{a_y}\right) \cdot \frac{1}{\mu_0}$$

$$\boxed{\vec{J}_s = (3.55\vec{a_y} + 3.54\vec{a_x})/\mu_0 \text{ A/m}}$$

Phasor

$\vec{A} = 10\cos(10^8 t - 10x + \pi/6)\vec{a_z}$. In phasor form?

$$\vec{A}(x,t) = Re[\underbrace{\vec{A}(x)}_{\text{phasor}} e^{i\omega t}] = Re[10 e^{i(10^8 t - 10x + \pi/6)} \cdot \vec{a_z}]$$

$$\Rightarrow \boxed{\vec{A}(x) = 10 e^{-i(10x - \pi/6)} \cdot \vec{a_z}}$$

Propagation Matrices

Consider an electric field, linearly polarized in $\vec{a_x}$ and propagating in $\vec{a_z}$, in a lossless, homogeneous and isotropic medium (dielectric).

$$E(z) = E_{0+}\, e^{-i\beta z} + E_{0-}\, e^{i\beta z} = E_+(z) + E_-(z)$$

$$H(z) = \frac{1}{\eta}\left[E_{0+} e^{-i\beta z} - E_{0-} e^{i\beta z}\right] = \frac{1}{\eta}\left[E_+(z) - E_-(z)\right]$$

forward $\rightarrow$ $E_+(z) = E_{0+} e^{-i\beta z} = \frac{1}{2}\left[E(z) + \eta H(z)\right]$

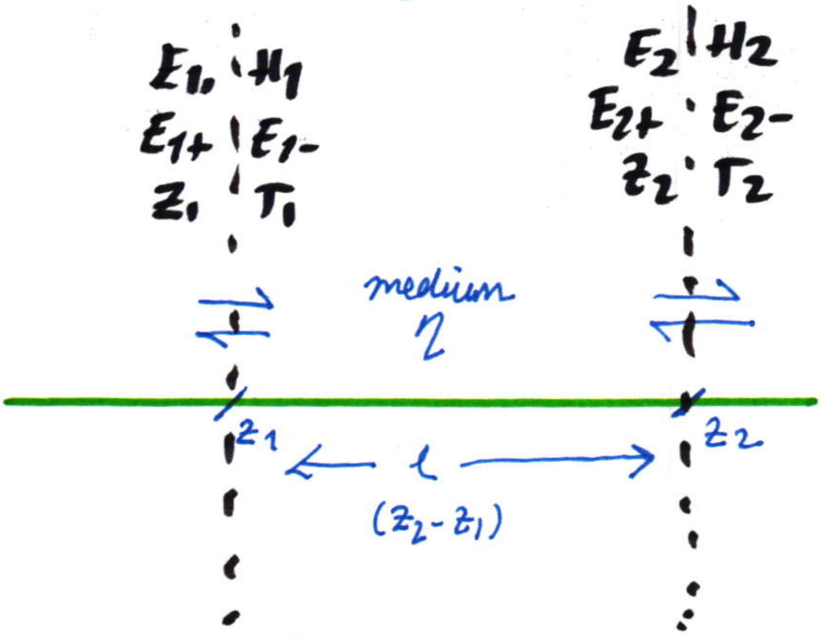
backward $\rightarrow$ $E_-(z) = E_{0-} e^{+i\beta z} = \frac{1}{2}\left[E(z) - \eta H(z)\right]$

In matrix form

$$\begin{bmatrix} E \\ H \end{bmatrix} = \underset{a)}{\begin{bmatrix} 1 & 1 \\ 1/\eta & -1/\eta \end{bmatrix}} \begin{bmatrix} E_+ \\ E_- \end{bmatrix}, \quad \begin{bmatrix} E_+ \\ E_- \end{bmatrix} = \underset{b)}{\frac{1}{2}\begin{bmatrix} 1 & \eta \\ 1 & -\eta \end{bmatrix}} \begin{bmatrix} E \\ H \end{bmatrix}$$

Wave Impedance at z, $Z(z) = \dfrac{E(z)}{H(z)}$

Reflection Coefficient at z, $\Gamma(z) = \dfrac{E_-(z)}{E_+(z)}$



$$E_{2+} = E_{0+} e^{-i\beta z_2}$$

$$E_{1+} = E_{0+} e^{-i\beta z_1} = E_{0+} e^{-i\beta(z_2 - l)}$$

$$= e^{i\beta l} E_{2+}$$

$$E_{1-} = e^{-i\beta l} E_{2-}$$

$$E_{1+} = e^{i\beta l} E_{2+}$$

$\leftarrow$ similarly one may obtain

in matrix form;

$$\begin{bmatrix} E_{1+} \\ E_{1-} \end{bmatrix} = \underbrace{\begin{bmatrix} e^{i\beta l} & 0 \\ 0 & e^{-i\beta l} \end{bmatrix}}_{\text{propagation matrix for the forward and backword waves}} \begin{bmatrix} E_{2+} \\ E_{2-} \end{bmatrix}$$

Now, let's relate (E_1, H_1) and (E_2, H_2). Insterting in Eq a), one obtains;

$$\begin{bmatrix} E_1 \\ H_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1/\eta_1 & -1/\eta_1 \end{bmatrix} \begin{bmatrix} E_{1+} \\ E_{1-} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1/\eta_1 & -1/\eta_1 \end{bmatrix} \underbrace{\begin{bmatrix} e^{i\beta l} & 0 \\ 0 & e^{-i\beta l} \end{bmatrix} \begin{bmatrix} E_{2+} \\ E_{2-} \end{bmatrix}}_{\begin{bmatrix} E_{1+} \\ E_{1-} \end{bmatrix}}$$

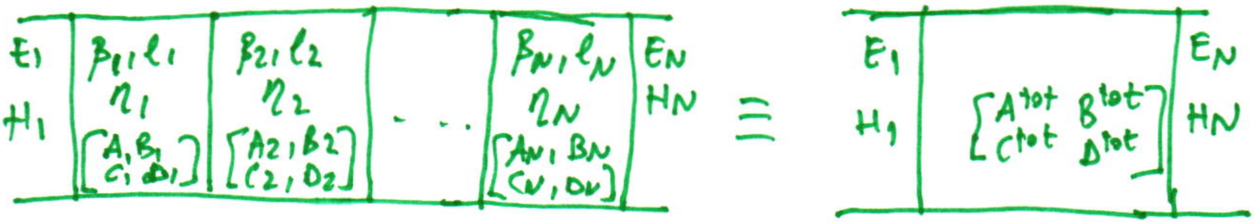
and using Eq b);

$$\begin{bmatrix} E_1 \\ H_1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1/\eta_1 & -1/\eta_1 \end{bmatrix} \begin{bmatrix} e^{i\beta l} & 0 \\ 0 & e^{-i\beta l} \end{bmatrix} \underbrace{\begin{bmatrix} 1 & \eta_1 \\ 1 & -\eta_1 \end{bmatrix} \begin{bmatrix} E_2 \\ H_2 \end{bmatrix}}_{2\begin{bmatrix} E_{2+} \\ E_{2-} \end{bmatrix}}$$

After some algebra

$$\begin{bmatrix} E_1 \\ H_1 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\beta l) & i\eta \sin(\beta l) \\ i\sin(\beta l)/\eta & \cos(\beta l) \end{bmatrix}}_{\text{This matrix some-times is called as ABCD matrix}} \begin{bmatrix} E_2 \\ H_2 \end{bmatrix}$$

Note that ABCD matrix is constituted of medium parameters (β, η, l). If a multilayer structure is composed of N layers, one may easily obtain the total ABCD matrix simply by multiplying ABCD matrices of each layer in given sequence. For instance



where $\begin{bmatrix} A^{tot} & B^{tot} \\ C^{tot} & D^{tot} \end{bmatrix} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \cdots \begin{bmatrix} A_N & B_N \\ C_N & D_N \end{bmatrix}$

Now let's analyze the system in terms of impedances. Recall that $Z = E/H$

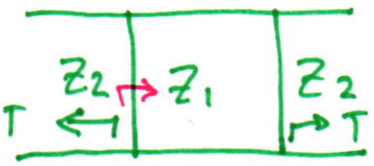
Impedance of medium $Z_1 = \dfrac{E_1}{H_1} = \dfrac{E_2 \cos(\beta_1 l_1) + i\eta \sin(\beta l) \cdot H_2}{i\, E_2/\eta \, \sin(\beta_1 l_1) + \cos(\beta_1 l_1) H_2}$

For simplicity, let's name $\begin{bmatrix} \cos(\beta l) & i\eta \sin(\beta l) \\ i/\eta \, \sin(\beta l) & \cos(\beta l) \end{bmatrix} \equiv \begin{bmatrix} A & B \\ C & D \end{bmatrix}$

Manipulating, one derives;

$$Z_1 = \frac{E_2/H_2 \cdot A + B}{\eta D + \eta \frac{E_2}{H_2} C} \cdot \eta = \frac{Z_2 A + B}{A + C \cdot Z_2} \qquad (A \equiv D)$$

Now consider the system below



Recall that reflection $\Gamma = \dfrac{Z_1 - Z_2}{Z_1 + Z_2}$ can be defined in terms of impedances.

So, $\Gamma = \dfrac{B - C Z_2^2}{2 Z_2 A + C Z_2 + B}$

Now let's analyze the system in terms of impedances. Recall that $Z = E/H$

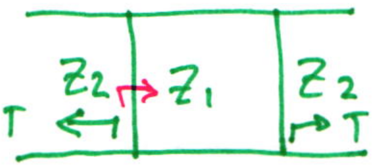
Impedance of medium $Z_1 = \dfrac{E_1}{H_1} = \dfrac{E_2 \cos(\beta_1 l_1) + i\eta \sin(\beta l)\cdot H_2}{i E_2/\eta \,\sin(\beta_1 l_1) + \cos(\beta_1 l_1) H_2}$

For simplicity, let's name $\begin{bmatrix} \cos(\beta l) & i\eta \sin(\beta l) \\ i/\eta \sin(\beta l) & \cos(\beta l) \end{bmatrix} \equiv \begin{bmatrix} A & B \\ C & D \end{bmatrix}$

Manipulating, one derives;

$$Z_1 = \frac{E_2/H_2 \cdot A + B}{\eta D + \eta \frac{E_2}{H_2} C} \cdot \eta = \frac{Z_2 A + B}{A + C \cdot Z_2} \qquad (A \equiv D)$$

Now consider the system below



Recall that reflection $\Gamma = \dfrac{Z_1 - Z_2}{Z_1 + Z_2}$ can be defined in terms of impedances.

So, $\boxed{\Gamma = \dfrac{B - C Z_2^2}{2 Z_2 A + C Z_2 + B}}$ → reflection coefficient in terms of ABCD parameters

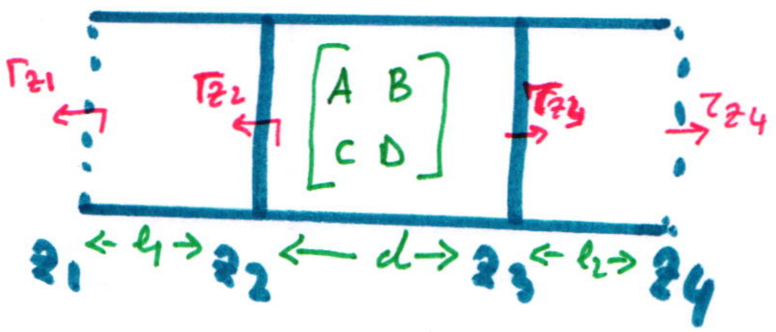
transmitted

$$T = \frac{E_{2+}}{E_{1+}} = \frac{E_2 + Z_2 \cdot H_2}{E_1 + Z_2 \cdot H_1} = \frac{H_2 (Z_2 + Z_2)}{\underbrace{E_2 A + H_2 B}_{E_1} + Z_2 (\underbrace{E_2 C + A H_2}_{H_1})}$$

incident

$\boxed{T = \dfrac{2 Z_2}{2 Z_2 A + B + Z_2^2 C}}$ → transmission coefficient in terms of ABCD parameters

Now, let's assume that we desire to observe reflection and transmission not at the discontinuity boundary, but at a different plane. That is;



We derived $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ which relates field quantities at z_2 and z_3 planes as well as reflection coefficient Γ and transmission coefficient τ.

If one aims to determine Γ and τ at z_1 and z_4, respectively; one simply relates field expressions. Namely;

$$\Gamma_{z_1} = \frac{E_{0-} \, e^{i\beta z_1}}{E_{0+} \cdot e^{-i\beta z_1}} = |\Gamma| \cdot e^{2i\beta z_1} \quad ; \quad \Gamma_{z_2} = \frac{E_{0-} \cdot e^{i\beta z_2}}{E_{0+} \, e^{-i\beta z_2}} = |\Gamma| \, e^{2i\beta z_2}$$

Then;

$$\Gamma_{z_1} = \Gamma_{z_2} \cdot e^{-2i\beta(z_2 - z_1)} = \Gamma_{z_2} \cdot e^{-2i\beta l_1}$$

where Γ_{z_2} is derived via propagation matrix.

Similarly; τ_{z_3} and τ_{z_4} are related through it

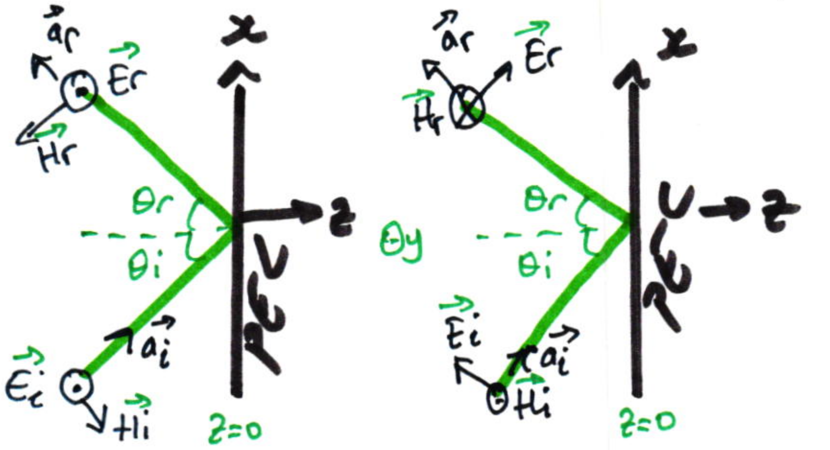
$$\tau_{z_3} = \frac{E^2_{0+} \cdot e^{-i\beta z_3}}{E_{0+} \, e^{-i\beta z_2}} = |\tau| \, e^{-i\beta(z_3 - z_2)} \quad ; \quad \tau_{z_4} = \frac{E^2_{0+} \cdot e^{-i\beta z_4}}{E_{0+} \cdot e^{-i\beta z_1}}$$

So ; $\tau_{z_4} = |\tau| \cdot e^{-i\beta(z_3 - z_2)} \cdot e^{-i\beta l_2} \cdot e^{i\beta l_1}$

$$\tau_{z_4} = \tau_{z_3} \cdot e^{-i\beta(l_2 + l_1)}$$

where τ_{z_3} is derived through propagation matrix.

Oblique Incidence at a Plane Conducting Boundary



Perpendicular Polarization   Parallel Polarization

The behavior of the reflected wave depends on the polarization of the incident field.

The most general case is the superposition of Perpendicular and parallel polarizations. So, we may generalize by summing two solutions. let's analyze two cases.

Perpendicular Polarization (a.k.a. Horizontal polarization or $\vec{E}$ polarization)

$\vec{a}_i = \vec{a}_x \sin\theta_i + \vec{a}_z \cos\theta_i$ (normal unit vector to the wavefront of incident wave)

$$\vec{E}_i(x,z) = \vec{a}_y E_{i0} \cdot e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$; incident $\vec{E}$ field

$$\vec{H}_i = \frac{1}{\eta_1} \vec{a}_i \times \vec{E}_i = \frac{E_{i0}}{\eta_1}(-\vec{a}_x\cos\theta_i + \vec{a}_z \sin\theta_i) e^{-j\beta_1(x \cdot \sin\theta_i + z\cos\theta_i)}$$; incident $\vec{H}$ field

$\vec{a}_r = \vec{a}_x \sin\theta_r - \vec{a}_z \cos\theta_r$ (unit vector indicating direction of prop. for reflected wave)

$$\vec{E}_r = \vec{a}_y E_{r0} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}_r = \frac{1}{\eta_1} \vec{a}_r \times \vec{E}_r = \frac{E_{i0}}{\eta_1}(-\vec{a}_x \cos\theta_i - \vec{a}_z \sin\theta_i) e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)}$$

At PEC boundary ($z=0$, xy plane), total $\vec{E}$ ($=\vec{E}_i + \vec{E}_r$) vanishes due to the boundary condition (i.e. $E_{tangential} = 0$)

Here;

$$\vec{E}_{total} = (\vec{E}_i + \vec{E}_r)\Big|_{z=0} = \vec{a}_y\left(E_{i0} \cdot e^{-j\beta_1 x\sin\theta_i} + E_{r0} e^{-j\beta_1 x \sin\theta_r}\right) = 0$$

In order for this equation to hold for $\forall\{\theta_i, \theta_r, x\}$, $\boxed{E_i = -E_r \text{ and } \theta_i = \theta_r}$

So, $$\boxed{\vec{E}_r = -\vec{a}_y E_{r0} e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)}}$$

Similarly, associated magnetic field $\vec{H}_r$ becomes;

$$\vec{H}_r = \frac{E_{io}}{\eta_1}(-\vec{a}_x \cos\theta_i - \vec{a}_z \sin\theta_i)e^{-i\beta_1(x\sin\theta_i - z\cos\theta_i)}$$

Total fields are vectorral sum of incident and reflected waves. For $\vec{E}$ field

$$\vec{E}_{total} = \vec{E}_i + \vec{E}_r = \vec{a}_y E_{io}(e^{-i\beta_1 z\cos\theta_i} - e^{i\beta_1 z\cos\theta_i})e^{-i\beta_1 x\sin\theta_i}$$

$$\boxed{\vec{E}_{total} = -\vec{a}_y . i2E_{io}\sin(\beta_1 z\cos\theta_i)e^{-i\beta_1 x\sin\theta_i}}$$

For $\vec{H}$ field;

$$\vec{H}_{total} = \vec{H}_i + \vec{H}_r = -2\frac{E_{io}}{\eta_1}(\vec{a}_x \cos\theta_i \cos(\beta_1 z\cos\theta_i)e^{-i\beta_1 x\sin\theta_i}$$
$$+ \vec{a}_z\, i \sin\theta_i \sin(\beta_1 z\cos\theta_i)e^{-i\beta_1 x\sin\theta_i}$$

A uniform planewave with angular frequency ω, is incident on a PEC boundary. Wave is perpendicularly polarized. (free-space)

a) Induced current $\vec{J}$ on the PEC wall? b) the time-average $\vec{P}$?

$$\vec{H}_{total}\Big|_{z=0} = \frac{-2}{\eta_1}E_{io}\,\vec{a}_x\cos\theta_i \cdot e^{-i\beta_1 x\sin\theta_i}$$

$$\vec{J} = \vec{a}_n \times \vec{H}_{total}\Big|_{z=0}, \quad \vec{a}_n = -\vec{a}_z$$

$$\boxed{\vec{J} = \vec{a}_y \frac{2}{\eta_1}E_{io}\cos\theta_i\, e^{-i\beta_1 x\sin\theta_i}}$$

where $\beta_1 = \frac{\omega}{c}$.

$$P_{av} = \frac{1}{2}\mathrm{Re}[\vec{E}_{total} \times \vec{H}_{total}^*]$$

$$= \vec{a}_x\, 2\frac{E_{io}^2}{\eta_1}\sin\theta_i \sin^2(\beta_1 z\cos\theta_i)$$

Parallel Polarization (aka Vertical Polarization or $\vec{H}$ polarization)

Now assume that $\vec{E_i}$ is parallel to the plane of incidence. E_i and E_r have (x,z) components while $H_{i,r}$ has only y component.

For incident waves;

$$\vec{E}_i(x,z) = E_{i0}\,(\vec{a_x}\cos\theta_i - \vec{a_z}\sin\theta_i)\,e^{-i\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}_i(x,z) = \vec{a_y}\,\frac{E_{i0}}{\eta_1}\,e^{-i\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

For reflected waves;

$$\vec{E}_r(x,z) = E_{r0}\,(\vec{a_x}\cos\theta_r + \vec{a_z}\sin\theta_r)\,e^{-i\beta_1(x\sin\theta_r - z\cos\theta_i)}$$

$$\vec{H}_r(x,z) = -\vec{a_y}\,\frac{E_{r0}}{\eta_1}\,e^{-i\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

can be written in phasor form.

Recall that, on the PEC boundary $\vec{E}^{tang}_{total}$ vanishes which is vectorial sum of $\vec{E}^{tang}_{incident} + \vec{E}^{tang}_{reflected}$. That is;

$$\vec{E}^{tang}_{total}(x,0) = \vec{E}^{tang}_i(x,0) + \vec{E}^{tang}_r(x,0) = 0$$

Hence;

$$\vec{E}^{tang}_{total}(x,0) = \vec{a_x}\,(E_{i0}\cos\theta_i\,e^{-i\beta_1 x\sin\theta_i} + E_{r0}\cos\theta_r\,e^{-i\beta_1 x\sin\theta_r}) = 0$$

↳ if it holds for $\forall x$, $\boxed{E_{i0} = -E_{r0} \text{ and } \theta_i = \theta_r}$. Substituting these relations, one obtains total field in free-space as;

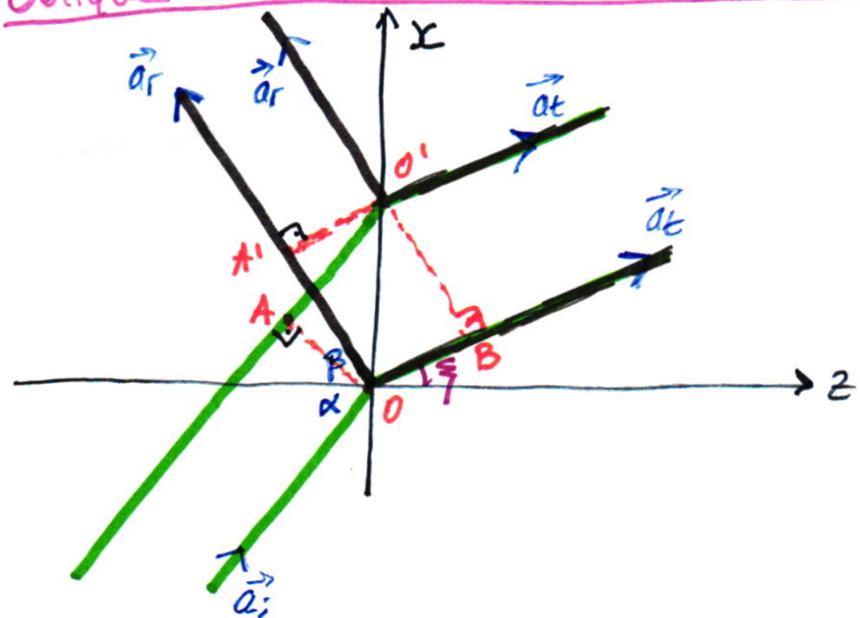
$$\vec{E}_{total}(x,z) = \vec{a_x}\,E_{i0}\cos\theta_i\,e^{-i\beta_1 x\cos\theta_i}\,(e^{-i\beta_1 z\cos\theta_i} - e^{i\beta_1 z\cos\theta_i})$$
$$-\vec{a_z}\,E_{i0}\sin\theta_i\,e^{-i\beta_1 x\cos\theta_i}\,(e^{-i\beta_1 z\cos\theta_i} + e^{i\beta_1 z\cos\theta_i})$$

$$= (-2i\,\vec{a_x}\,E_{i0}\cos\theta_i\sin(\beta_1 z\cos\theta_i) - 2E_{i0}\,\vec{a_z}\sin\theta_i\cos(\beta_1 z\cos\theta_i)) \times e^{-i\beta_1 x\sin\theta_i}$$

Similarly $\vec{H}_{total}(x,z) = \vec{H}_i(x,z) + \vec{H}_r(x,z)$

$$\boxed{\vec{H}_{total} = 2\vec{a_y}\,\frac{E_{i0}}{\eta_1}\cos(\beta_1 z\cos\theta_i)\,e^{-i\beta_1 x\sin\theta_i}}$$

Oblique Incidence at a Plane Dielectric Boundary

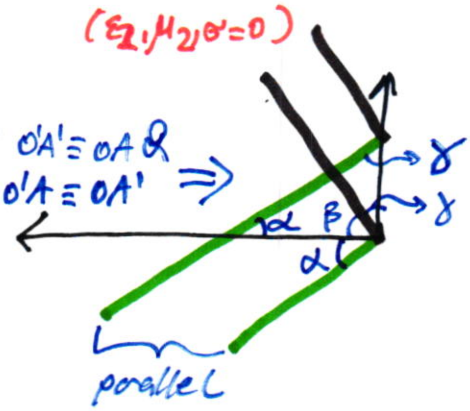


Medium 1
$(\varepsilon_1, \mu_1, \sigma = 0)$

Medium 2
$(\varepsilon_2, \mu_2, \sigma = 0)$

In medium, wave propagates with constant velocity.
While distance between two propagating waves is constant (i.e., $OA \equiv O'A'$), the distance that travelled by these two waves is equal (i.e., $OA' \equiv AO'$).
Now, let's examine figure geometrically.

If $O'A' \equiv OA$ &
$O'A \equiv OA'$ $\Rightarrow$



$\alpha + \gamma \equiv 90° \equiv \beta + \gamma \rightarrow \boxed{\alpha = \beta}$ So the angle of incidence is equal to the angle of refraction.

While transmitted wave reaches point B, propagating wave travels from point A to point O' (with different velocities but with same duration).

Hence duration $= \dfrac{OB}{v_2} = \dfrac{O'A}{v_1}$ where $OB = \sin\xi \cdot OO'$, $O'A = \sin\beta \cdot OO'$

v_2, v_1 $\rightarrow$ velocities in medium 2 and medium 1, respectively.

So, one may derive $\dfrac{\sin\xi}{v_2} = \dfrac{\sin\beta}{v_1}$

For clarity let's name $\alpha = \theta_i$ (angle of incidence); $\beta = \theta_r$ (angle of refraction)
$\xi = \theta_t$ (angle of transmission)

$\theta_i = \theta_r$ and $\dfrac{\sin\theta_t}{\sin\theta_i} = \dfrac{v_2}{v_1} = \dfrac{n_1}{n_2}$ **(Snell's law)**

n_1 and n_2 are refractive indices of medium 1 & 2, respectively.

$n \triangleq \dfrac{c}{v_p}$ where c is speed of light and v_p is the speed of wave of the medium.

Total Reflection

With respect to Snell's law, if a wave propagates from a medium to a less dense medium, $\theta_t > \theta_i$ occurs. Let's investigate if $\theta_t = \pi/2$?

Let's name $\theta_t = \pi/2 \Rightarrow \theta_c = \theta_i$ critical angle w.r.t. Snell's Law;

$$\sin\theta_c = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \quad \left(\text{since } v_1/v_2 = \sqrt{\varepsilon_2/\varepsilon_1} \text{ and } \sin\theta_t\big|_{\pi/2} = 1\right)$$

Hence

$$\theta_c = \sin^{-1}\left(\sqrt{\varepsilon_2/\varepsilon_1}\right) = \sin^{-1}(n_2/n_1) \quad (\text{for } \mu_1 = \mu_2, \text{ otherwise } \theta_c = \sin^{-1}\left(\sqrt{\varepsilon_2\mu_2/\varepsilon_1\mu_1}\right))$$

Now let's analyze the situation if θ_i becomes larger than θ_c. Then;

$$\sin\theta_t = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\sin\theta_i > 1 \rightarrow \cos\theta_t = \sqrt{1-\sin^2\theta_t} = \pm i\sqrt{\frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_i - 1}$$

The phase term of wave in medium 2 is (as used to be);

$$e^{-i\beta_2 \vec{a}_t \cdot \vec{r}} \quad \text{where } \vec{a}_t = \vec{a}_x \cdot \sin\theta_t + \vec{a}_z \cdot \cos\theta_t, \ \vec{r} = x\vec{a}_x + y\vec{a}_y + z\vec{a}_z$$

So;

$$\underbrace{e^{-i\beta_2 \cdot z \cdot \cos\theta_t}} \cdot e^{-i\beta_2 \cdot x \cdot \sin\theta_t} \quad \text{becomes the phase term}$$

If $\cos\theta_t = i\sqrt{\frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_i - 1} \rightarrow$ exponential term increases non-physically.

So, one should choose $\cos\theta_t = -i\sqrt{\frac{\varepsilon_1}{\varepsilon_2}\sin^2\theta_i - 1}$ which gives a decaying component in z direction

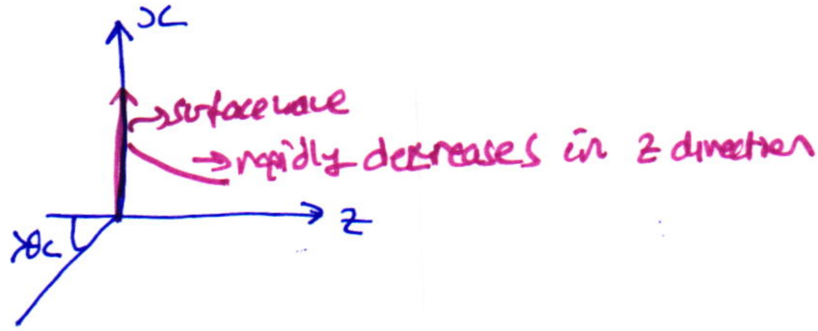
That is

$\rightarrow$ z component decreases with an exponential factor $e^{-z\beta_2\sqrt{\varepsilon_1/\varepsilon_2 \cdot \sin^2\theta_i - 1}}$

The component in x direction (on the interface) propagates with a phase term

$$e^{-i\beta_2\sqrt{\varepsilon_1/\varepsilon_2}\cdot\sin\theta_i\cdot x}$$

So, if $\theta_i > \theta_c$, the wave is bond to the interface and called as surface wave.



Perpendicular Polarization

Incident waves:

$$\vec{E}_i(x,z) = \vec{a_y}\, E_{i0}\cdot e^{-i\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}_i(x,z) = (-\vec{a_x}\cos\theta_i + \vec{a_z}\sin\theta_i)\,\frac{E_{i0}}{\eta_1}\, e^{-i\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

Reflected waves;

$$\vec{E}_r(x,z) = \vec{a_y}\, E_{r0}\cdot e^{-i\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}_r(x,z) = (\vec{a_x}\cos\theta_r + \vec{a_z}\sin\theta_r)\,\frac{E_{r0}}{\eta_1}\, e^{-i\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

Transmitted waves;

$$\vec{E}_t(x,z) = \vec{a_y}\, E_{t0}\, e^{-i\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$\vec{H}_t(x,z) = (-\vec{a_x}\cos\theta_t + \vec{a_z}\sin\theta_t)\, e^{-i\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

On the boundary $(z=0)$

$\vec{a_y}\left(\vec{E}_i(x,0) + \vec{E}_r(x,0)\right) = \vec{E}_t(x,0)\vec{a_y}$ and $\vec{a_x}\vec{H}_i(x,0) + \vec{H}_r(x,0)\cdot\vec{a_x} = \vec{a_x}\vec{H}_t(x,0)$

due to the boundary condition related to the tangential components ($E_{1t} = E_{2t}$, $\underbrace{H_{1t} = H_{2t}}_{\text{no free current } \vec{J} \text{ on the interface}}$)

So, one may write; $(z=0)$

$$E_{i0}\cdot e^{-i\beta_1 x\sin\theta_i} + E_{r0}\cdot e^{-i\beta_1 x\sin\theta_r} = E_{t0}\cdot e^{-i\beta_2 \sin\theta_t x}$$

$$\frac{1}{\eta_1}\left(-E_{i0}\cdot\cos\theta_i\, e^{-i\beta_1 x\sin\theta_i} + E_{r0}\cos\theta_r\cdot e^{-i\beta_1 x\sin\theta_r}\right) = -\frac{E_{t0}}{\eta_2}\cos\theta_t\cdot e^{-i\beta_2 x\sin\theta_t}$$

For $\forall x$, if these equations hold, then;

$\beta_1 x\sin\theta_i = \beta_1 x\sin\theta_r = \beta_2 x\sin\theta_t$ which is consistent with Snell's law.

So,

$$E_{i0} + E_{r0} = E_{t0} \quad \text{and} \quad \frac{1}{\eta_1}(E_{i0} - E_{r0})\cos\theta_i = \frac{E_{t0}}{\eta_2}\cos\theta_t$$

$$\Gamma_\perp = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2\cdot\cos\theta_i - \eta_1\cdot\cos\theta_t}{\eta_2\cdot\cos\theta_i + \eta_1\cdot\cos\theta_t}$$ and

reflection for perpendicular polarization coefficient

$$\tau_\perp = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2\cos\theta_i}{\cos\theta_i\,\eta_2 + \cos\theta_t\,\eta_1}$$

transmission coefficient for perpendicular polarization

which gives

$$\boxed{1 + \Gamma_\perp = \tau_\perp}$$

Now let's examine if no reflection occurs, that is $\Gamma_\perp = 0$. That requires;

$$\eta_2\cdot\cos\theta_i = \eta_1\cdot\cos\theta_t$$

Then

$$\sin^2\theta_i = 1 - \left(\frac{\eta_1}{\eta_2}\right)^2\cos^2\theta_t \quad \text{where } \cos^2\theta_t = 1 - \sin^2\theta_t$$

$$= 1 - \frac{n_1^2}{n_2^2}\sin^2\theta_i$$

$$\Rightarrow \sin^2\theta_i = 1 - \left(\frac{\eta_1}{\eta_2}\right)^2\left(1 - \frac{n_1^2}{n_2^2}\sin^2\theta_i\right)$$

$$\Rightarrow \sin\theta_i = \sqrt{\frac{\varepsilon_2/\varepsilon_1 - \mu_2/\mu_1}{\mu_1/\mu_2 - \mu_2/\mu_1}}$$

$\sin\theta_i \leq 1 \Rightarrow$ $\varepsilon_2/\varepsilon_1 \leq \mu_1/\mu_2$ should be satisfied for such an angle to be

If $\mu_1 = \mu_2$, $\sin\theta_i \to \infty$ which is not lelong to a real angle θ_i.

So, for $\mu_1 = \mu_2$ (most of dielectric materials have μ_0 magnetic permeability) there exist no a real incidence for which the total reflection is zero. This angle θ_i is called as "Brewster angle".

Parallel Polarization

Incident waves:

$$\vec{E}_i(x,z) = E_{i0}(\vec{a}_x \cos\theta_i - \vec{a}_z \sin\theta_i) e^{-i\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\vec{H}_i(x,z) = \frac{E_{i0}}{\eta_1} \vec{a}_y e^{-i\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

Reflected waves:

$$\vec{E}_r(x,z) = E_{r0}(\vec{a}_x \cos\theta_r + \vec{a}_z \sin\theta_r) e^{-i\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\vec{H}_r(x,z) = -\vec{a}_y \frac{E_{r0}}{\eta_1} e^{-i\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

Transmitted waves

$$\vec{E}_t(x,z) = E_{t0}(\vec{a}_x \cos\theta_t - \vec{a}_z \sin\theta_t) e^{-i\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

$$\vec{H}_t(x,t) = \vec{a}_y \frac{E_{t0}}{\eta_2} e^{-i\beta_2(x\sin\theta_t + z\cos\theta_t)}$$

On the boundary $(z=0)$

$$(E_{i0} + E_{r0})\cos\theta_i = E_{t0}\cos\theta_t$$

$$\frac{1}{\eta_1}(E_{i0} - E_{r0}) = \frac{1}{\eta_2} E_{t0}$$

Here $\Gamma_{\parallel} = \dfrac{E_{r0}}{E_{i0}} = \dfrac{\eta_2\cos\theta_t - \eta_1\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$ (reflection coefficient)

$\tau_{\parallel} = \dfrac{E_{t0}}{E_{i0}} = \dfrac{2\eta_2\cos\theta_i}{\eta_2\cos\theta_t + \eta_1\cos\theta_i}$ (transmission coefficient)

If one lets $T_{\parallel}=0$, $\eta_2 \cos\theta_t = \eta_1 \cos\theta_i$ should be satisfied where $\theta_i = \theta_{B\parallel}$ is the Brewster angle

$$\Rightarrow \cos\theta_i = \frac{\eta_2}{\eta_1}\cos\theta_t$$

using Snell's Law

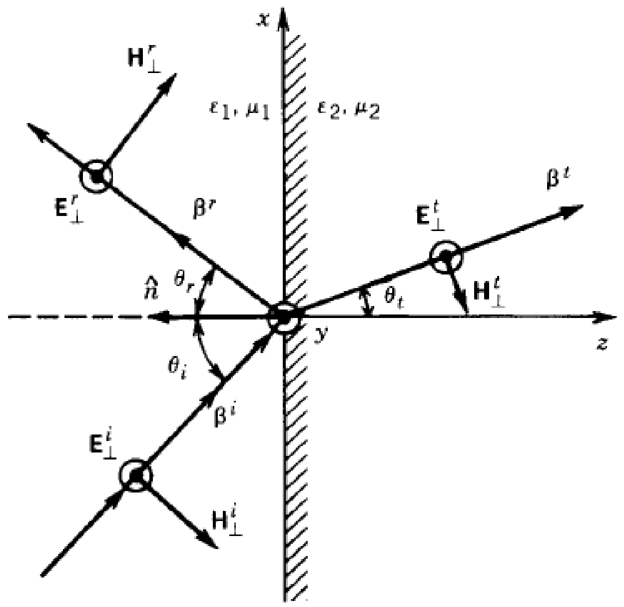
$$\Rightarrow (1-\sin^2\theta_i) = \frac{\eta_2^2}{\eta_1^2}\left(1-\frac{n_1^2}{n_2^2}\sin^2\theta_i\right)$$

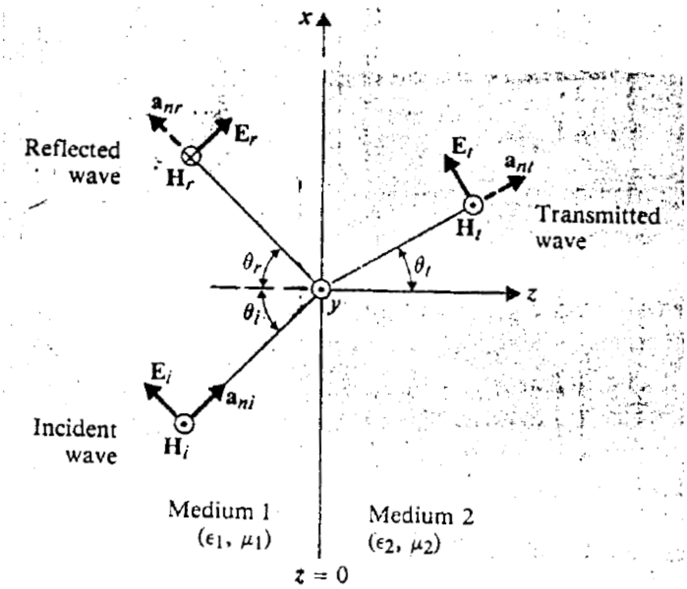
$$\boxed{\sin\theta_i = \sqrt{\frac{\varepsilon_2/\varepsilon_1 - \mu_2/\mu_1}{\varepsilon_2/\varepsilon_1 - \varepsilon_1/\varepsilon_2}}}$$

where θ_i is the Brewster angle for parallel polarization

* Total reflection is independent of polarization. Only limitation is that the wave to propagate into a less dense medium (i.e., $\mu_2\varepsilon_2 < \mu_1\varepsilon_1$)

* For non-magnetic materials (or $\mu_1 = \mu_2$) there exist no real incidence angle so that $\Gamma = 0$ (total transmission) for perpendicular polarization.





Perpendicular Polarization

Parallel Polarization

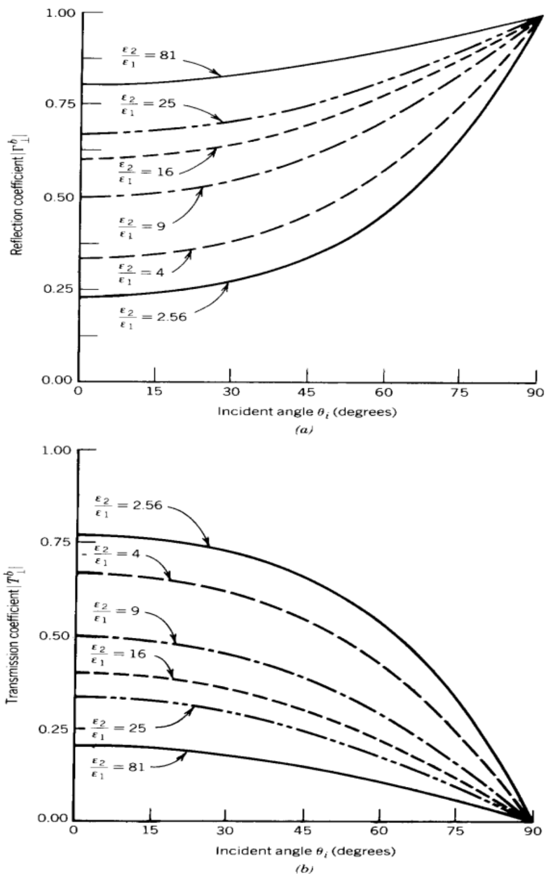


FIGURE 5-3 Magnitude of (*a*) reflection and (*b*) transmission coefficients for perpendicular polarization as a function of incident angle.

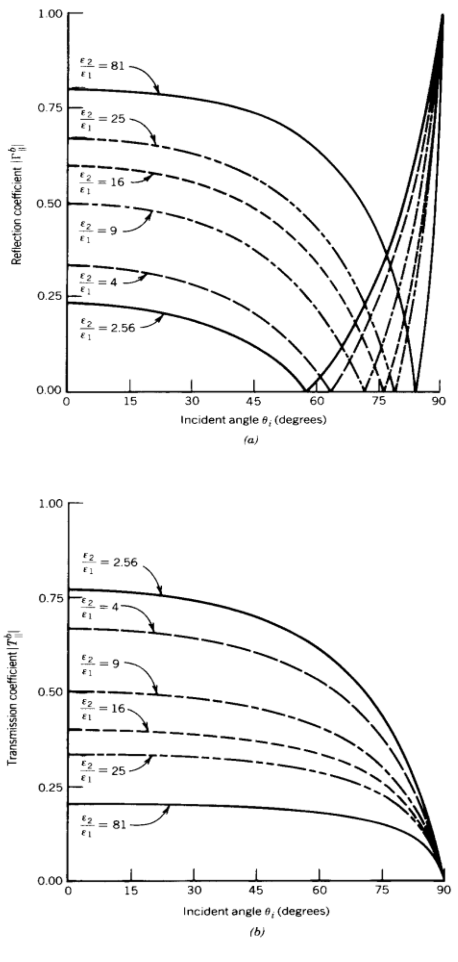
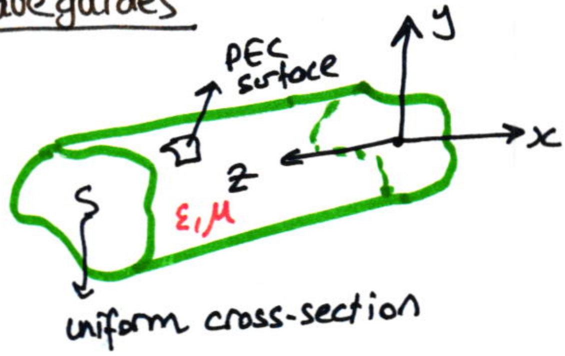


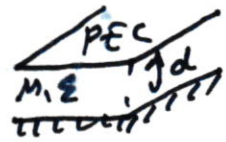
FIGURE 5-5 Magnitude of (*a*) reflection and (*b*) transmission coefficients for parallel polarization as a function of incident angle.

Waveguides



If ε, μ and S are invariant in z direction, waveguide is uniform. In this course, uniform waveguides will be analyzed.

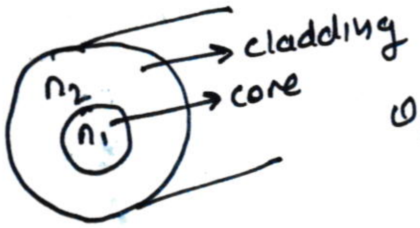
Waveguide Types



Parallel Plate waveguides (TEM, TE, TM)

Rectangular waveguides (TE, TM)

Circular waveguides (TE, TM)



Optical Fiber Waveguide (Hybrid Modes)

Many waves are possible in a waveguide. Each possible wave is called a <u>mode</u>.

We will assume time harmonic EM waves with an $e^{i\omega t}$ time dependence and wave propagation along z axis.

Classification of Modes

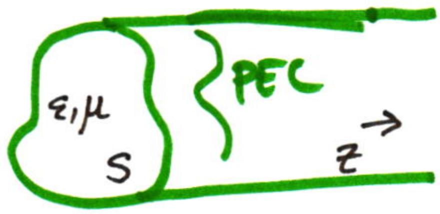
1 TEM Modes: Transverse Electric and magnetic, $E_z = H_z = 0$

2 TE Modes: Transverse Electric, $E_z = 0$, $H_z \neq 0$

3 TM Modes: Transverse Magnetic, $E_z \neq 0$, $H_z = 0$

4 Hybrid Modes: $E_z \neq 0$, $H_z \neq 0$

Uniform Waveguides with Perfectly Conductor Boundaries



uniform, lossless waveguide

EM wave propagates in +z direction

$$\vec{E}(x,y,z) = [\underbrace{\vec{e}_t(x,y)}_{\text{transverse component}} + \underbrace{\vec{a}_z\, e_z(x,y)}_{\text{longitudinal component}}]\, e^{-i\beta z}$$

$$\vec{H}(x,y,z) = [\overbrace{\vec{h}_t(x,y)}^{\text{transverse component}} + \overbrace{\vec{a}_z\, h_z(x,y)}^{\text{longitudinal component}}]\, e^{-i\beta z}$$

Recall that we had derived $\nabla^2\vec{E} + k^2\vec{E} = 0$, $\nabla^2\vec{H} + k^2\vec{H} = 0$ for source-free-medium.

$$\vec{E} = E_x\vec{a_x} + E_y\vec{a_y} + E_z\vec{a_z}$$

$\rightarrow$ $\nabla^2 E_x + k^2 E_x = 0$

$\nabla^2 E_y + k^2 E_y = 0$

$\boxed{\nabla^2 E_z + k^2 E_z = 0}$ similarly $\boxed{\nabla^2 H_z + k^2 H_z = 0}$

$$\nabla^2 E_z = \underbrace{\boxed{\frac{d^2E_z}{dx^2} + \frac{d^2E_z}{dy^2}}}_{\text{transverse part } \nabla_t^2 E_z} + \frac{d^2E_z}{dz^2}$$

$E_z = e_z(x,y)\,e^{-i\beta z}$, $H_z = h_z(x,y)\,e^{-i\beta z}$

$\rightarrow$ $\dfrac{d^2E_z}{dz^2} = (-i\beta)^2 E_z = -\beta^2 E_z$

Hence;

$$\nabla^2 E_z + k^2 E_z = \nabla_t^2 E_z + \underbrace{(k^2 - \beta^2)}_{k_c^2} E_z = 0$$

One may write;

$\boxed{\nabla_t^2 E_z + k_c^2 E_z = 0}$ or $\boxed{(\nabla_t^2 + k_c^2)\, e_z(x,y)\, e^{-i\beta z} = 0}$

For further analysis, let's write

$$(\nabla_t^2 + k_c^2)\, e_z(x,y) = 0$$

Similarly

$$(\nabla_t^2 + k_c^2)\, h_z(x,y) = 0$$

Once the e_z and h_z components are determined, other components can be found via Maxwell's Equations.

$$\nabla \times \vec{E} = -i\omega\mu \vec{H}$$

$$\begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ d/dx & d/dy & d/dz \\ E_x & E_y & E_z \end{vmatrix} = -i\omega\mu \left[H_x \vec{a_x} + H_y \vec{a_y} + H_z \vec{a_z} \right]$$

$$\Rightarrow -i\omega\mu H_x = \frac{d}{dy} E_z - \frac{d}{dz} E_y = \frac{dE_z}{dy} + i\beta E_y$$

$$-i\omega\mu H_y = \frac{d}{dz} E_x - \frac{d}{dx} E_z = -i\beta E_x - \frac{d}{dx} E_z$$

$$-i\omega\mu H_z = \frac{d}{dx} E_y - \frac{d}{dy} E_x$$

Similarly $\nabla \times \vec{H} = i\omega\varepsilon \vec{E}$

$$\begin{vmatrix} \vec{a_x} & \vec{a_y} & \vec{a_z} \\ d/dx & d/dy & d/dz \\ H_x & H_y & H_z \end{vmatrix} = i\omega\varepsilon \left[E_x \vec{a_x} + E_y \vec{a_y} + E_z \vec{a_z} \right]$$

$$\Rightarrow i\omega\varepsilon E_x = \frac{d}{dy} H_z - \frac{d}{dz} H_y = \frac{d}{dy} H_z + i\beta H_y$$

$$i\omega\varepsilon E_y = \frac{d}{dz} H_x - \frac{d}{dx} H_z = -i\beta H_x - \frac{d}{dx} H_z$$

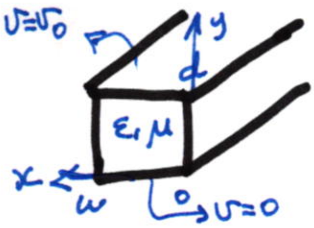
$$i\omega\varepsilon E_z = \frac{d}{dx} H_y - \frac{d}{dy} H_x$$

So, 4 components are a function of H_z & E_z

$H_x = f(E_z, H_z)$; $H_y = g(E_z, H_z)$; $E_x = u(E_z, H_z)$; $E_y = v(E_z, H_z)$

	TEM Modes	TE Modes ($E_z=0$)	TM Modes ($H_z=0$)
β	$\omega\sqrt{\varepsilon\mu}$	$\sqrt{k^2-k_c^2}$	$\sqrt{k^2-k_c^2}$
φ	V	H_z	E_z
Boundary Conditions on PEC	Constant on PEC	$\frac{d\varphi}{dn}=0$ Neumann B.C.	$\varphi=0$ Dirichlet B.C.
φ equation	$\nabla_t^2\varphi=0$	$(\nabla_t^2+k_c^2)\varphi=0$	$(\nabla_t^2+k_c^2)\varphi=0$
Transverse Component	$\vec{E}_t=-\nabla_t\varphi\cdot e^{-i\beta z}$	$\vec{H}_t=\frac{-i\beta}{k_c^2}\nabla_t\varphi$	$\vec{E}_t=\frac{-i\beta}{k_c^2}\nabla_t\varphi$
Wave Impedance	$Z^{TEM}=\sqrt{\frac{\mu}{\varepsilon}}$	$Z^{TE}=\omega\mu/\beta$	$Z^{TM}=\beta/\omega\varepsilon$
Transverse Component	$\vec{H}_t=\frac{1}{Z^{TEM}}\vec{a}_z\times\vec{E}_t$	$\vec{E}_t=Z^{TE}\vec{H}_t\times\vec{a}_z$	$\vec{H}_t=\frac{1}{Z^{TM}}\vec{a}_z\times\vec{E}_t$

Parallel Plate Waveguide (Supports TEM, TE, TM modes)



xz plane at z=0 and z=d are PEC

wave propagates in z direction

TEM Mode Solutions

$$\nabla_t^2\varphi=0 \Rightarrow \nabla_t^2 V=0 \Rightarrow \frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}=0$$

$$w \gg d \Rightarrow \frac{dV}{dx}=0 \Rightarrow \frac{d^2V}{dy^2}=0 \Rightarrow V(y)=Ay+B$$

$$V(0)=0 \Rightarrow B=0$$

$$V(d)=V_0 \Rightarrow A=V_0/d$$

$$\Rightarrow V(y)=y\frac{V_0}{d}$$

$$\vec{e}_t = -\nabla_t v = -\cancel{\frac{dv}{dx}}^{\,0}\,\vec{a}_x - \frac{dv}{dy}\vec{a}_y = -\frac{V_0}{d}\vec{a}_y$$

$$\boxed{\vec{E}_t = -\frac{V_0}{d} e^{-i\beta z}\vec{a}_y}\,,\quad E_z = 0,\ E_x = 0$$

$$\vec{H}_t = \frac{1}{Z_{TEM}}\vec{a}_z \times \vec{E}_t = \sqrt{\frac{\varepsilon}{\mu}}\,\vec{a}_x \frac{V_0}{d} e^{-i\beta z}\,,\quad H_y = H_z = 0$$

$\beta = \omega\sqrt{\varepsilon\mu}$. If no propagation occurs, $\beta = 0$

$\beta = 0$ for $f = 0 \Rightarrow$ this is cut-off frequency.

So, one may say a parallel plate waveguide is an all-pass filter.

TM Mode Solutions

$H_z = 0,\ E_z \neq 0,\ \varphi = E_z$

$$(\nabla_t^2 + k_c^2)\, e_z = 0 \rightarrow \left(\cancel{\frac{d^2}{dx^2}}^{\,0} + \frac{d^2}{dy^2} + k_c^2\right) e_z = \left(\frac{d^2}{dy^2} + k_c^2\right) e_z = 0$$

$e_z = A\cos(k_c y) + B\sin(k_c y)$ $\leftarrow$ solution

Bandary Conditions

at $y = 0$ and $y = d$ $\ e_z = 0 \rightarrow A = 0$ and $k_c = \dfrac{n\pi}{d}$

$$\Rightarrow e_z = B\sin\left(\frac{n\pi}{d}y\right)$$

$$\boxed{E_z = B\sin\left(\frac{n\pi}{d}y\right)e^{-i\beta_n z}}\ ;\ \boxed{\vec{E}_t = \frac{-i\beta}{k_c}\nabla_t E_z}$$

at cut-off frequency, $\beta = 0$ (no propagation)

$$\beta = \sqrt{k^2 - \left(\frac{n\pi}{d}\right)^2} = 0 \rightarrow \boxed{f_{cn} = \frac{n}{2d}\frac{1}{\sqrt{\varepsilon\mu}}}$$ cut-off frequency for n^{th} mode

$k \downarrow$

$\dfrac{2\pi f}{v} \rightarrow \dfrac{1}{\sqrt{\varepsilon\mu}}$

$n = 1, 2, 3, \ldots$

TE Mode Solutions

$E_z = 0, \; H_z \neq 0, \; \varphi = H_z$

$(\nabla_t^2 + k_c^2) h_z = 0 \rightarrow h_z = A\cos(k_c y) + B\sin(k_c y)$

Boundary Conditions $y=0, y=d$: $\frac{d\varphi}{dy} = 0 \rightarrow B = 0$ and $\boxed{h_z = A_n \cos\left(\frac{n\pi}{d} \cdot y\right)}$

$H_z = A_n \cos\left(\frac{n\pi}{d} y\right) e^{-i\beta z}$

$\vec{H}_t = \frac{-i\beta}{k_c} \nabla_t H_z$

TE_n modes $n = 1, 2, 3, \ldots$

Dominant mode is the first possible mode in waveguide. For Parallel plate waveguide, Dominant mode is TEM mode.