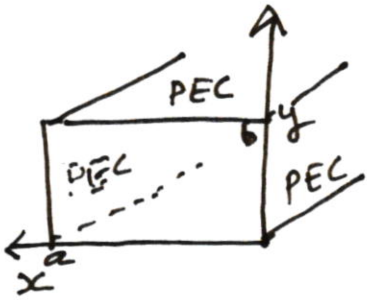
Rectangular Waveguides (TE, TM Modes are possible)



TE Modes ($E_z = 0$, $H_z \neq 0$)

$$(\nabla_t^2 + k_c^2) H_z = 0 = (\nabla_t^2 + k_c^2) h_z(x, y) e^{-i\beta z} = 0$$

Now let's seperate $h_z(x,y)$ in terms of multiplication as;

$$h_z(x,y) = X(x)\, Y(y)$$

and solve differential equation $(\nabla_t^2 + k_c^2) X(x) Y(y) = 0$

$$Y \frac{d^2X}{dx^2} + X \frac{d^2Y}{dy^2} + k_c^2 XY = 0 = \underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{f(x)} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{g(y)} + k_c^2$$

$f(x)$ and $g(y)$ are functions of x and y, respectively. They should be constant since k_c is constant. Hence, one may write;

$$\left.\begin{aligned} f(x) + g(y) &= -k_c^2 \\ \downarrow \quad\quad \downarrow \quad & \\ -k_x^2 - k_y^2 &= -k_c^2 \end{aligned}\right\} \quad k_x^2 + k_y^2 = k_c^2$$

$$f(x) = -k_x^2 \Rightarrow \frac{1}{X}\frac{d^2X}{dx^2} = -k_x^2 \rightarrow \frac{d^2X}{dx^2} + X k_x^2 = 0$$

$$g(y) = -k_y^2 = \frac{1}{Y}\frac{d^2Y}{dy^2} = -k_y^2 \rightarrow \frac{d^2Y}{dy^2} + Y k_y^2 = 0$$

Then,

$$X = A\cos(k_x x) + B\sin(k_x x)$$

$$Y = C\cos(k_y y) + D\sin(k_y y)$$

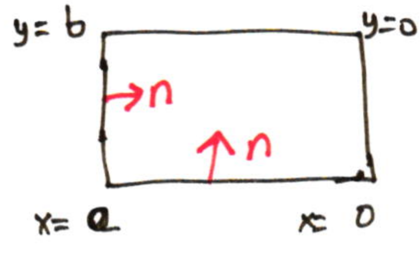
$h_z(x,y) = X(x)Y(y) = [A\cos(k_x x) + B\sin(k_x x)][C\cos(k_y y) + D\sin(k_y y)]$

$H_z(x,y,z) = h_z(x,y)\, e^{-i\beta z}$.

A, B, C, D constants can be determined (k_x and k_y as well) by Boundary conditions. Recall that $\frac{d\varphi}{dn} = 0$ on PEC where $\varphi = H_z$.

on PEC

$$\frac{d\varphi}{dn} = 0 \rightarrow \quad \left.\frac{dh_z}{dx}\right|_{x=0} = 0 \quad (1) \qquad \left.\frac{dh_z}{dx}\right|_{x=a} = 0 \quad (2)$$



$$\left.\frac{dh_z}{dy}\right|_{y=0} = 0 \quad (3) \qquad \left.\frac{dh_z}{dy}\right|_{y=b} = 0 \quad (4)$$

$$\frac{dh_z}{dx} = Y\,[-A k_x \sin(k_x x) + B k_x \cos(k_x x)] = 0$$

From (1), $\boxed{B=0}$; from (2) $\boxed{k_x = \frac{m\pi}{a}}$

$$\frac{dh_z}{dy} = X\,[-C k_y \sin(k_y y) + D k_y \cos(k_y y)] = 0$$

From (3), $\boxed{D=0}$; from (4) $\boxed{k_y = \frac{n\pi}{b}}$

So;

$$h_z(x,y) = \overbrace{A.C.}^{\text{arbitrary constant}} \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$$

$m = 0, 1, 2, \dots$ / $n = 1, 2, \dots$ or $m = 1, 2, \dots$ / $n = 0, 1, \dots$, $m = n = 0$ is not allowed

$\beta = \sqrt{k^2 - k_c^2}$ where $k_c^2 = k_x^2 + k_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$

$$\boxed{\beta_{mn} = \sqrt{k^2 - \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]}}$$

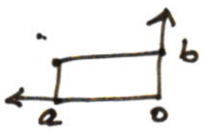
→ propagation constant of TE_{mn} mode

Each mode has its own cut-off frequency ($\beta_{mn}=0$)

$\beta_{mn}=0 \rightarrow k^2 = k_c^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$ where $k = \frac{2\pi f}{v_p}$

Hence $\boxed{f_{c_{mn}} = \frac{c}{2\pi\sqrt{\varepsilon_r\mu_r}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}}$ → cut-off frequency of the TE_{mn} mode.

At a given frequency f
- ↗ $f > f_c$ → propagating mode
- → $f = f_c$ → cut-off condition
- ↘ $f < f_c$ → evanescent mode

Generally $a>b$ is chosen for waveguide applications 

Then,

$$f_{c_{10}}^{TE} = \frac{c}{2a\sqrt{\varepsilon_r\mu_r}}$$

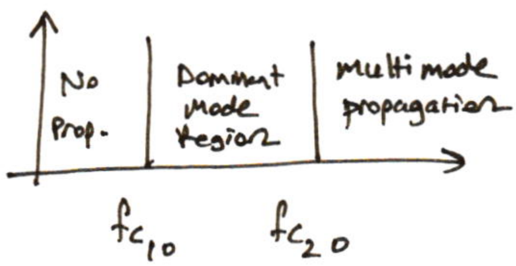
$$f_{c_{20}}^{TE} = \frac{c}{a\sqrt{\varepsilon_r\mu_r}}$$

$$f_{c_{01}}^{TE} = \frac{c}{2b\sqrt{\varepsilon_r\mu_r}}$$

$$f_{c_{11}}^{TE} = \frac{c}{2}\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}\,\frac{1}{\sqrt{\varepsilon_r\mu_r}}$$

If $a > 2b$

$f_{c_{10}} < f_{c_{20}} < f_{c_{01}} < f_{c_{11}}$



The mode with the lowest cut-off frequency is the "dominant mode".

For a given geometry $a>b$, TE_{10} is the dominant mode.

TM Modes $(E_z \neq 0, H_z = 0)$

$$(\nabla_t^2 + k_c^2) E_z = 0 = (\nabla_t^2 + k_c^2)\, e_z(x,y)\, e^{-i\beta z}$$

Letting $e_z(x,y) = X(x)\,Y(y)$ we may write

$$Y \frac{d^2 X}{dx^2} + X \frac{d^2 Y}{dy^2} + XYk_c^2 = 0$$

$$\rightarrow \underbrace{\frac{1}{X}\frac{d^2X}{dx^2}}_{-k_x^2} + \underbrace{\frac{1}{Y}\frac{d^2Y}{dy^2}}_{-k_y^2} + k_c^2 = 0 \rightarrow k_c^2 = k_x^2 + k_y^2$$

which gives;

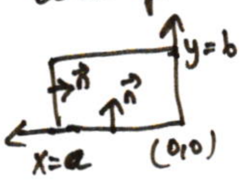
$$\frac{1}{X}\frac{d^2X}{dx^2} + k_x^2 = 0 \quad ; \quad \frac{1}{Y}\frac{d^2Y}{dy^2} + k_y^2 = 0$$

Solution of differential equations are;

$$X = A\cos(k_x x) + B\sin(k_x x)$$

$$Y = C\cos(k_y y) + D\sin(k_y y)$$

Boundary condition is that $\varphi = 0$ on PEC where $\varphi = E_z$



At $x=0$, $e_z = 0 \rightarrow \boxed{A = 0}$

At $x=a$, $e_z = 0 \rightarrow \boxed{k_x = \frac{m\pi}{a}}$

At $y=0$, $e_z = 0 \rightarrow \boxed{C = 0}$

At $y=b$, $e_z = 0 \rightarrow \boxed{k_y = \frac{n\pi}{b}}$

So, for TM modes;

$$E_z(x,y,z) = B.D.\ \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) e^{-i\beta_{mn} z}$$

where

$$\beta_{mn} = \sqrt{k^2 + k_c^2} = \sqrt{k^2 - k_x^2 - k_y^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

For TM modes m or n can not be 0. TM_{10} or TM_{01} does not exist.

$$f_{cmn}\Big|_{\beta_{mn}=0} = \frac{c}{2\pi\sqrt{\varepsilon_r\mu_r}}\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}$$

Here, TE_{10} is the dominant mode for RWG (for $a>b$)

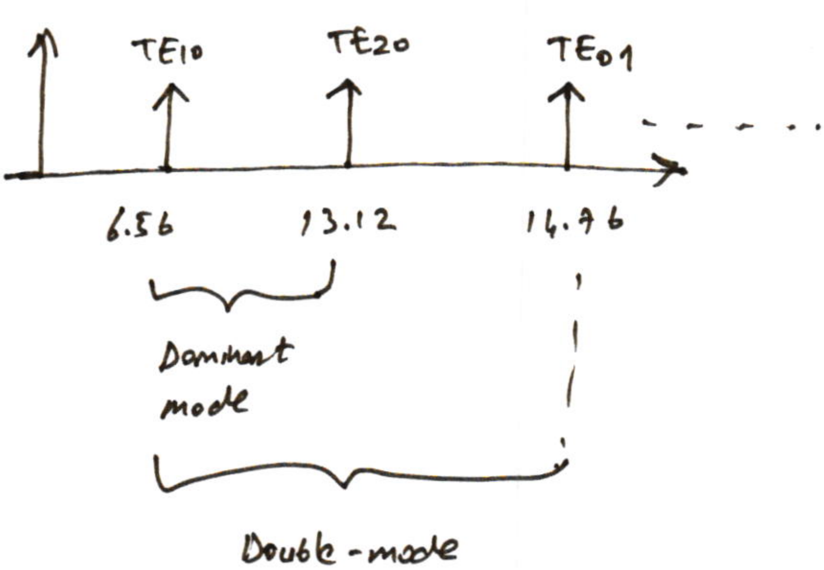
For $a = 2.286$ cm, $b = 1.016$ cm

i. Find first cut-off frequencies

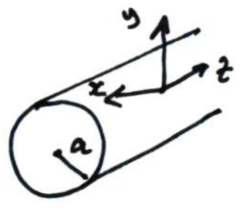
ii. Find frequency bandwith of dominant mode

iii. Find frequency bandwith of double-mode region

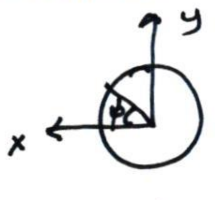
Mode	m	n	f_{cmn} [GHz]
TE	1	0	6.562
TE	2	0	13.12
TE	0	1	14.76
TE, TM	1	1	16.156
TE, TM	1	2	30.248
TE, TM	2	1	19.753



CIRCULAR WAVEGUIDES



Geometry



Cross-Section

TE MODES $(E_z = 0,\ H_z \neq 0)$

$$(\nabla_t^2 + k_c^2)\varphi = 0 \text{ where } \varphi = H_z \text{ and } \frac{d\varphi}{dn} = 0 \text{ on PEC}$$

$$\nabla_t^2 \varphi = \nabla\cdot(\nabla\varphi) = \nabla\cdot\left(\frac{d\varphi}{d\rho}\vec{a_\rho} + \frac{1}{\rho}\frac{d\varphi}{d\phi}\vec{a_\phi} + \frac{d\varphi}{dz}\vec{a_z}\right)$$

$$\nabla_t^2\varphi = \frac{1}{\rho}\left[\frac{d}{d\rho}\left(\rho\frac{d\varphi}{d\rho}\right) + \frac{d}{d\phi}\left(\frac{1}{\rho}\frac{d\varphi}{d\phi}\right) + \frac{d^2\varphi}{dz^2}\right]$$

$\varphi = H_z(\rho,\phi,z) = h_z(\rho,\phi)e^{-i\beta z}$ and let's seperate $h_z = R(\rho)P(\phi)$

$$\nabla_t^2 h_z = \frac{d^2h_z}{d\rho^2} + \frac{1}{\rho}\frac{dh_z}{d\rho} + \frac{1}{\rho^2}\frac{d^2h_z}{d\phi^2}$$

$$= P\frac{d^2R}{d\rho^2} + P\frac{dR}{d\rho}\cdot\frac{1}{\rho} + R\frac{1}{\rho^2}\frac{d^2P}{d\phi^2} + k_c^2RP = 0$$

$\times\frac{1}{RP}$

$$= \frac{1}{R}\frac{d^2R}{d\rho^2} + \frac{1}{R}\frac{1}{\rho}\frac{dR}{d\rho} + \frac{1}{P}\frac{1}{\rho^2}\frac{d^2P}{d\phi^2} + k_c^2 = 0$$

$\times\rho^2$

$$= \frac{\rho^2}{R}\frac{d^2R}{d\rho^2} + \frac{\rho}{R}\frac{dR}{d\rho} + k_c^2\rho^2 = -\frac{1}{P}\frac{d^2P}{d\phi^2}$$

or $\times R$

$$= \boxed{\rho^2R'' + \rho R' + (\rho^2k_c^2 - k_\phi^2)R = 0} \text{ where } k_\phi = \frac{-P''}{P}$$

which is known as Bessel's Differential Equation

The solution is

$$R = C \cdot J_{k\phi}(k_c \rho) + D\, N_{k\phi}(k_c \rho)$$

where $J_{k\phi}(x)$ is the Bessel function of the first kind of order v and

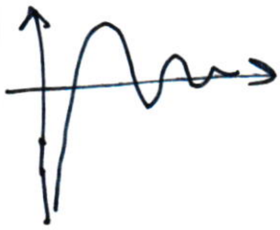
$N_v(x)$ is the Bessel function of the second kind of order v.

$$R \cdot P = H_z(\rho, \phi, z) = [C J_{k\phi}(k_c \rho) + D \cdot N_{k\phi}(k_c \rho)][A \sin(k_\phi \cdot \phi) + B \cos(k_\phi \cdot \phi)] e^{-i\beta z}$$

since $P'' + k_\phi^2 P = 0 \Rightarrow P = A \sin(k_\phi \phi) + B \cos(k_\phi \phi)$

★ $H_z(\rho, \phi, z) = H_z(\rho, \phi + 2\pi t, z)$ where t is an integer due to the circular symmetry. This can be true only if $k\phi$ becomes an integer. So, $k\phi = n$

★ $N_v(x)$ has the property →



as $x \to 0$, $N_v(x) \to -\infty$

Since $\rho = 0$ is in the waveguide domain, $N_v(x)$ can not be a physical solution. There $D \to 0$. Hence;

$$h_z(\rho, \phi) = C \cdot [A \sin(n\phi) + B \cos(n\phi)] J_n(k_c \rho)$$

$\vec{n} = \vec{a_\rho}$, $\left.\frac{dh_z}{dn}\right|_{\rho = a} = 0 \rightarrow \frac{d J_n(k_c \rho)}{d\rho} = 0 \rightarrow k_c^{nm} = \frac{\rho'_{nm}}{a}$ where ρ'_{nm} is the m^{th} root of derivative of n^{th} order bessel fuction

$$\beta_{nm} = \sqrt{k^2 - \left(\frac{\rho'_{nm}}{a}\right)^2}$$

ρ'_{nm} values for some TE^{mn} modes in circular waveguide

n \ m	1	2	3
0	3.83	7.016	10.174
1	$\boxed{1.841}$	5.331	8.536
2	3.054	6.706	9.970

minimum value
TE^{11} is the dominant mode in circular waveguide

TM Modes

Similar derivations are valid. Hence

$$E_z(\rho,\phi,z) = [A\sin(n\phi) + B\cos(n\phi)].C.J_n(k_c\rho)e^{-i\beta z}$$

$\psi = 0$ on PEC where $\psi = E_z \Rightarrow k_c = \frac{p_{nm}}{a}$ where p_{nm} is the

m^{th} root of the n^{th} degree bessel fuction J_n

Some p_{nm} values

n \ m	1	2	3
0	2.4	5.52	8.65
1	3.83	7.01	10.17
2	5.13	8.417	11.62

$$\beta_{nm} = \sqrt{k^2 - \left(\frac{p_{nm}}{a}\right)^2}$$

Still, there exists no lower mode in TM modes than TE^{11} which is the dominant mode for circular waveguide.