MATH2056 Linear Algebra HW2 Solutions

2.1.4

(a) $$M_{32} = \begin{vmatrix} 2 & -1 & 1 \\ -3 & 0 & 3 \\ 3 & 1 & 4 \end{vmatrix} = 2\begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} - (-1)\begin{vmatrix} -3 & 3 \\ 3 & 4 \end{vmatrix} + 1\begin{vmatrix} -3 & 0 \\ 3 & 1 \end{vmatrix}$$

$$= 2(-3) + 1(-21) + 1(-3) = -30$$

$$C_{32} = (-1)^{3+2}M_{32} = -M_{32} = 30$$

(b) $$M_{44} = \begin{vmatrix} 2 & 3 & -1 \\ -3 & 2 & 0 \\ 3 & -2 & 1 \end{vmatrix} = 2\begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix} - 3\begin{vmatrix} -3 & 0 \\ 3 & 1 \end{vmatrix} + (-1)\begin{vmatrix} -3 & 2 \\ 3 & -2 \end{vmatrix}$$

$$= 2(2) - 3(-3) - 1(0) = 13$$

$$C_{44} = (-1)^{4+4}M_{44} = M_{44} = 13$$

(c) $$M_{41} = \begin{vmatrix} 3 & -1 & 1 \\ 2 & 0 & 3 \\ -2 & 1 & 0 \end{vmatrix} = 3\begin{vmatrix} 0 & 3 \\ 1 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} + 1\begin{vmatrix} 2 & 0 \\ -2 & 1 \end{vmatrix}$$

$$= 3(-3) + 1(6) + 1(2) = -1$$

$$C_{41} = (-1)^{4+1}M_{41} = -M_{41} = 1$$

(d) $$M_{24} = \begin{vmatrix} 2 & 3 & -1 \\ 3 & -2 & 1 \\ 3 & -2 & 1 \end{vmatrix} = 2\begin{vmatrix} -2 & 1 \\ -2 & 1 \end{vmatrix} - 3\begin{vmatrix} 3 & 1 \\ 3 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 3 & -2 \\ 3 & -2 \end{vmatrix}$$

$$= 2(0) - 3(0) - 1(0) = 0$$

$$C_{24} = (-1)^{2+4}M_{24} = M_{24} = 0$$

2.1.14

$$\begin{vmatrix} c & -4 & 3 \\ 2 & 1 & c^2 \\ 4 & c-1 & 2 \end{vmatrix} = \left|\begin{matrix} c & -4 & 3 \\ 2 & 1 & c^2 \\ 4 & c-1 & 2 \end{matrix}\right|\begin{matrix} c & -4 \\ 2 & 1 \\ 4 & c-1 \end{matrix}$$

$$= [2c - 16c^2 + 6(c-1)] - [12 + (c-1)c^3 - 16]$$

$$= 2c - 16c^2 + 6c - 6 - 12 - c^4 + c^3 + 16 = -c^4 + c^3 - 16c^2 + 8c - 2$$

2.1.18

Calculate the determinant by a cofactor expansion along the third row:

$$\det(A) = \begin{vmatrix} \lambda - 4 & 4 & 0 \\ -1 & \lambda & 0 \\ 0 & 0 & \lambda - 5 \end{vmatrix} = 0 - 0 + (\lambda - 5)\begin{vmatrix} \lambda - 4 & 4 \\ -1 & \lambda \end{vmatrix}$$

$$= (\lambda - 5)[(\lambda - 4)\lambda + 4] = (\lambda - 5)[\lambda^2 - 4\lambda + 4]$$

$$= (\lambda - 5)(\lambda - 2)^2$$

The determinant is zero if $\lambda = 2$ or $\lambda = 5$.

2.1.26

Calculate the determinant by a cofactor expansion along the first row:

$$\det(A) = 4\begin{vmatrix} 3 & 3 & -1 & 0 \\ 2 & 4 & 2 & 3 \\ 4 & 6 & 2 & 3 \\ 2 & 4 & 2 & 3 \end{vmatrix} - 0 + 0 - 1\begin{vmatrix} 3 & 3 & 3 & 0 \\ 1 & 2 & 4 & 3 \\ 9 & 4 & 6 & 3 \\ 2 & 2 & 4 & 3 \end{vmatrix} + 0$$

Calculate each of the two determinants by a cofactor expansion along its first row:

$$\begin{vmatrix} 3 & 3 & -1 & 0 \\ 2 & 4 & 2 & 3 \\ 4 & 6 & 2 & 3 \\ 2 & 4 & 2 & 3 \end{vmatrix} = 3\begin{vmatrix} 4 & 2 & 3 \\ 6 & 2 & 3 \\ 4 & 2 & 3 \end{vmatrix} - 3\begin{vmatrix} 2 & 2 & 3 \\ 4 & 2 & 3 \\ 2 & 2 & 3 \end{vmatrix} + (-1)\begin{vmatrix} 2 & 4 & 3 \\ 4 & 6 & 3 \\ 2 & 4 & 3 \end{vmatrix} - 0$$

$$= 3(0) - 3(0) - 1(0) - 0 = 0$$

$$\begin{vmatrix} 3 & 3 & 3 & 0 \\ 1 & 2 & 4 & 3 \\ 9 & 4 & 6 & 3 \\ 2 & 2 & 4 & 3 \end{vmatrix} = 3\begin{vmatrix} 2 & 4 & 3 \\ 4 & 6 & 3 \\ 2 & 4 & 3 \end{vmatrix} - 3\begin{vmatrix} 1 & 4 & 3 \\ 9 & 6 & 3 \\ 2 & 4 & 3 \end{vmatrix} + 3\begin{vmatrix} 1 & 2 & 3 \\ 9 & 4 & 3 \\ 2 & 2 & 3 \end{vmatrix} - 0$$

$$= 3(0) - 3(-6) + 3(-6) - 0 = 0$$

Therefore $\det(A) = 4(0) - 0 + 0 - 1(0) = 0$.

2.1.41

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1),$$

where (x_1, y_1) and (x_2, y_2) are two points on the line with $x_2 \neq x_1$. This is equivalent to the point-slope form above, where the slope is explicitly given as $(y_2 - y_1)/(x_2 - x_1)$.

Multiplying both sides of this equation by $(x_2 - x_1)$ yields a form of the line generally referred to as the **symmetric form**:

$$(x_2 - x_1)(y - y_1) = (y_2 - y_1)(x - x_1).$$

Expanding the products and regrouping the terms leads to the general form:

$$x\,(y_2 - y_1) - y\,(x_2 - x_1) = x_1 y_2 - x_2 y_1$$

$$\det(A) = x\begin{vmatrix} b_1 & 1 \\ b_2 & 1 \end{vmatrix} - y\begin{vmatrix} a_1 & 1 \\ a_2 & 1 \end{vmatrix} + 1\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$$

$x(b_1-b_2)+y(a_1-a_2)+(a_1b_2-a_2b_1)=0$

If we replace a_1, a_2, b_1, and b_2 with x_1, x_2, y_1, and y_2 respectively, two equations are same.

3.2.20

(a) $-\frac{1}{\|\mathbf{u}\|}\mathbf{u} = -\frac{1}{\sqrt{(-12)^2+(-5)^2}}(-12,-5) = -\frac{1}{13}(-12,-5) = (\frac{12}{13},\frac{5}{13}).$

(b) $-\frac{1}{\|\mathbf{u}\|}\mathbf{u} = -\frac{1}{\sqrt{3^2+(-3)^2+(-3)^2}}(3,-3,-3) = -\frac{1}{3\sqrt{3}}(3,-3,-3) = (-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})$

(c) $-\frac{1}{\|\mathbf{u}\|}\mathbf{u} = -\frac{1}{\sqrt{(-6)^2+8^2}}(-6,8) = -\frac{1}{10}(-6,8) = (\frac{3}{5},-\frac{4}{5})$

(d) $-\frac{1}{\|\mathbf{u}\|}\mathbf{u} = -\frac{1}{\sqrt{(-3)^2+1^2+(\sqrt{6})^2+3^2}}(-3,1,\sqrt{6},3) = -\frac{1}{5}(-3,1,\sqrt{6},3) = (\frac{3}{5},-\frac{1}{5},-\frac{\sqrt{6}}{5},-\frac{3}{5})$

2.2.14

$$\begin{vmatrix} 1 & -2 & 3 & 1 \\ 5 & -9 & 6 & 3 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ -1 & 2 & -6 & -2 \\ 2 & 8 & 6 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 2 & 8 & 6 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 12 & 0 & -1 \end{vmatrix}$$

–5 times the first row was added to the second row.

The first row was added to the third row.

–2 times the first row was added to the fourth row.

$$= \begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & -3 & -1 \\ 0 & 0 & 108 & 23 \end{vmatrix} = -3\begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 108 & 23 \end{vmatrix} = -3\begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & -13 \end{vmatrix}$$

–12 times the second row was added to the fourth row.

A common factor of –3 from the third row was taken through the determinant sign.

–108 times the third row was added to the fourth row.

$$= (-3)(-13)\begin{vmatrix} 1 & -2 & 3 & 1 \\ 0 & 1 & -9 & -2 \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{vmatrix} = (-3)(-13)(1) = 39$$

A common factor of –13 from the third row was taken through the determinant sign.

2.2.31.

On the matrix at the right side, if we add the first column to the last and then add the second column to the last, we will get the matrix on the left side. These column operations don't affect the values of the determinant, so determinant of the matrices are equal.

2.3.6

$$\det(AB) = \begin{vmatrix} 6 & 15 & 26 \\ 2 & -4 & -3 \\ -2 & 10 & 12 \end{vmatrix} = -66; \quad \det(BA) = \begin{vmatrix} 5 & 8 & -3 \\ -6 & 14 & 7 \\ 5 & -2 & -5 \end{vmatrix} = -66$$

$$\det(A+B) = \begin{vmatrix} 1 & 7 & -2 \\ 2 & 1 & 2 \\ -2 & 5 & 1 \end{vmatrix} = -75; \quad \det(A) = 2; \quad \det(B) = -33;$$

$\det(A+B) \neq \det(A) + \det(B)$.

2.3.18

Expanding along the first row,

$$\det\begin{pmatrix} 2 & 1 & 0 \\ k & 2 & k \\ 2 & 4 & 2 \end{pmatrix} = 2(4-4k) - 1(2k-2k) = 8-8k = 0 \text{ when } k = 1.$$

Thus, the matrix is invertible for all $k \neq 1$.

2.3.20

$\det(A) = -6$ is nonzero, therefore by Theorem 2.3.3, A is invertible.

The cofactors of A are:

$$\begin{array}{lll} C_{11} = -12 & C_{12} = -4 & C_{13} = 6 \\ C_{21} = 0 & C_{22} = -2 & C_{23} = 0 \\ C_{31} = -9 & C_{32} = -4 & C_{33} = 6 \end{array}$$

The matrix of cofactors is $\begin{bmatrix} -12 & -4 & 6 \\ 0 & -2 & 0 \\ -9 & -4 & 6 \end{bmatrix}$

and the adjoint matrix is $\operatorname{adj}(A) = \begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix}.$

From Theorem 2.3.6, we have

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{-6}\begin{bmatrix} -12 & 0 & -9 \\ -4 & -2 & -4 \\ 6 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 0 & \frac{3}{2} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -1 & 0 & -1 \end{bmatrix}.$$

2.3.32

(a) $A = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 3 & 7 & -1 & 1 \\ 7 & 3 & -5 & 8 \\ 1 & 1 & 1 & 2 \end{bmatrix}$, $A_1 = \begin{bmatrix} 6 & 1 & 1 & 1 \\ 1 & 7 & -1 & 1 \\ -3 & 3 & -5 & 8 \\ 3 & 1 & 1 & 2 \end{bmatrix}$, $A_2 = \begin{bmatrix} 4 & 6 & 1 & 1 \\ 3 & 1 & -1 & 1 \\ 7 & -3 & -5 & 8 \\ 1 & 3 & 1 & 2 \end{bmatrix}$, $A_3 = \begin{bmatrix} 4 & 1 & 6 & 1 \\ 3 & 7 & 1 & 1 \\ 7 & 3 & -3 & 8 \\ 1 & 1 & 3 & 2 \end{bmatrix}$, $A_4 = \begin{bmatrix} 4 & 1 & 1 & 6 \\ 3 & 7 & -1 & 1 \\ 7 & 3 & -5 & -3 \\ 1 & 1 & 1 & 3 \end{bmatrix}$;

$$x = \frac{\det(A_1)}{\det(A)} = \frac{-424}{-424} = 1, \quad y = \frac{\det(A_2)}{\det(A)} = \frac{0}{-424} = 0,$$

$$z = \frac{\det(A_3)}{\det(A)} = \frac{-848}{-424} = 2, \quad w = \frac{\det(A_4)}{\det(A)} = \frac{0}{-424} = 0$$

(b) The augmented matrix of the system $\begin{bmatrix} 4 & 1 & 1 & 1 & 6 \\ 3 & 7 & -1 & 1 & 1 \\ 7 & 3 & -5 & 8 & -3 \\ 1 & 1 & 1 & 2 & 3 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ therefore the system has only one solution: $x = 1,\ y = 0,\ z = 2,$ and $w = 0$.

(c) The method in part (b) requires fewer computations.

2.3.36

(a) $\det(-A) = \det((-1)A) = (-1)^4 \det(A) = \det(A) = -2$ (using Formula (1) on p. 106)

(b) $\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{-2} = -\frac{1}{2}$ (using Theorem 2.3.5)

(c) $\det(2A^T) = 2^4 \det(A^T) = 16 \det(A) = -32$ (using Formula (1) on p. 106 and Theorem 2.2.2)

(d) $\det(A^3) = \det(AAA) = \det(A)\det(A)\det(A) = (-2)^3 = -8$ (using Theorem 2.3.4)

2.3.39.

According to Theorem 2.3.4 $\det(AB) = \det(A)\det(B)$. So;

det(A^T A)=det(A^T).det(A) on the other side det(A.A^T)=det(A).det(A^T)

According to Theorem 2.2.2 det(A)=det(A^T), So;

$\det(A^T A) = \det(AA^T)$.

3.1.12

(a) For any positive real number k, the vector $\mathbf{u} = k\mathbf{v}$ has the same direction as $\mathbf{v}$. For example, letting $k = 1$, we have $\mathbf{u} = (6, 7, -3)$. If the initial point is $P(-1, 3, -5)$ then the terminal point has coordinates $(-1 + 6, 3 + 7, -5 - 3)$, i.e., $(5, 10, -8)$.

(b) For any negative real number k, the vector $\mathbf{u} = k\mathbf{v}$ is oppositely directed to $\mathbf{v}$. For example, letting $k = -1$, we have $\mathbf{u} = (-6, -7, 3)$. If the initial point is $P(-1, 3, -5)$ then the terminal point has coordinates $(-1 - 6, 3 - 7, -5 + 3)$, i.e., $(-7, -4, -2)$.

3.1.14

(a) $\mathbf{v} - \mathbf{w} = (4 - 6, 0 - (-1), -8 - (-4)) = (-2, 1, -4)$

(b) $6\mathbf{u} + 2\mathbf{v} = (-18, 6, 12) + (8, 0, -16) = (-10, 6, -4)$

(c) $-\mathbf{v} + \mathbf{u} = (-4, 0, 8) + (-3, 1, 2) = (-7, 1, 10)$

(d) $5(\mathbf{v} - 4\mathbf{u}) = 5[(4, 0, -8) - (-12, 4, 8)] = 5(16, -4, -16) = (80, -20, -80)$

(e) $-3(\mathbf{v} - 8\mathbf{w}) = -3[(4, 0, -8) - (48, -8, -32)] = -3(-44, 8, 24) = (132, -24, -72)$

(f) $(2\mathbf{u} - 7\mathbf{w}) - (8\mathbf{v} + \mathbf{u}) = [(-6, 2, 4) - (42, -7, -28)] - [(32, 0, -64) + (-3, 1, 2)]$
$= (-48,\ 9, 32) - (29, 1, -62) = (-77, 8, 94)$

3.1.34

The midpoint of the line segment connecting the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

Therefore we have $\left(\frac{1 + x_2}{2}, \frac{3 + y_2}{2}, \frac{7 + z_2}{2}\right) = (4, 0, -6)$.

This vector equation is equivalent to a system of three linear equations in three unknowns that

$\frac{1 + x_2}{2} = 4 \quad \Leftrightarrow \quad x_2 = 7 \quad , \quad \frac{3 + y_2}{2} = 0 \quad \Leftrightarrow \quad y_2 = -3 \quad , \quad \frac{7 + z_2}{2} = -6 \quad \Leftrightarrow \quad z_2 = -19$

We conclude that the point Q is $(7, -3, -19)$.

3.2.2

(a) $\|\mathbf{v}\| = \sqrt{(-5)^2 + 12^2} = \sqrt{169} = 13;$

$\frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{13}(-5, 12) = (-\frac{5}{13}, \frac{12}{13});\quad -\frac{1}{\|\mathbf{v}\|}\mathbf{v} = -\frac{1}{13}(-5, 12) = (\frac{5}{13}, -\frac{12}{13})$

(b) $\|\mathbf{v}\| = \sqrt{1^2 + (-1)^2 + 2^2} = \sqrt{6}\,;$

$\frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{6}}(1, -1, 2) = (\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}});\quad -\frac{1}{\|\mathbf{v}\|}\mathbf{v} = -\frac{1}{\sqrt{6}}(1, -1, 2) = (-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}})$

(c) $\|\mathbf{v}\| = \sqrt{(-2)^2 + 3^2 + 3^2 + (-1)^2} = \sqrt{23}\,;$

$\frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{23}}(-2, 3, 3, -1) = (-\frac{2}{\sqrt{23}}, \frac{3}{\sqrt{23}}, \frac{3}{\sqrt{23}}, -\frac{1}{\sqrt{23}});$

$-\frac{1}{\|\mathbf{v}\|}\mathbf{v} = -\frac{1}{\sqrt{23}}(-2, 3, 3, -1) = (\frac{2}{\sqrt{23}}, -\frac{3}{\sqrt{23}}, -\frac{3}{\sqrt{23}}, \frac{1}{\sqrt{23}})$

3.2.6

(a) $\|\mathbf{u}\| - 2\|\mathbf{v}\| - 3\|\mathbf{w}\|$

$= \sqrt{(-2)^2 + (-1)^2 + 4^2 + 5^2} - 2\sqrt{3^2 + 1^2 + (-5)^2 + 7^2} - 3\sqrt{(-6)^2 + 2^2 + 1^2 + 1^2}$

$= \sqrt{46} - 2\sqrt{84} - 3\sqrt{42} = \sqrt{46} - 4\sqrt{21} - 3\sqrt{42}$

(b) $-2\mathbf{v} = (-6, -2, 10, -14)\,,\quad -3\mathbf{w} = (18, -6, -3, -3)$

$\|\mathbf{u}\| + \|-2\mathbf{v}\| + \|-3\mathbf{w}\|$

$= \sqrt{(-2)^2 + (-1)^2 + 4^2 + 5^2} + \sqrt{(-6)^2 + (-2)^2 + 10^2 + (-14)^2}$

$\quad + \sqrt{18^2 + (-6)^2 + (-3)^2 + (-3)^2}$

$= \sqrt{46} + \sqrt{336} + \sqrt{378} = \sqrt{46} + 4\sqrt{21} + 3\sqrt{42}$

(c) $\mathbf{u} - \mathbf{v} = (-5, -2, 9, -2),\quad \|\mathbf{u} - \mathbf{v}\| = \sqrt{(-5)^2 + (-2)^2 + 9^2 + (-2)^2} = \sqrt{114}$

$\|\mathbf{u} - \mathbf{v}\|\mathbf{w} = (-6\sqrt{114}, 2\sqrt{114}, \sqrt{114}, \sqrt{114});\quad \big\|\|\mathbf{u} - \mathbf{v}\|\mathbf{w}\big\| = \sqrt{4788} = 6\sqrt{133}$

3.2.12

(a) $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(1-5)^2 + (2-1)^2 + (-3-2)^2 + \big(0-(-2)\big)^2} = \sqrt{46}$

(b) $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|$

$= \sqrt{\big(2-(-2)\big)^2 + \big(-1-(-1)\big)^2 + (-4-0)^2 + (1-3)^3 + (0-7)^2 + (6-2)^2 + \big(-3-(-5)\big)^2 + (1-1)^2} = \sqrt{105}$

(c) $d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(0-2)^2 + (1-1)^2 + (1-0)^2 + \big(1-(-1)\big)^2 + (2-3)^2} = \sqrt{10}$