3.3.14. The plane $x - 4y - 3z - 2 = 0$ has a normal vector $(1, -4, -3)$.

The plane $3x - 12y - 9z - 7 = 0$ has a normal vector $(3, -12, -9)$.

The two normal vectors are parallel: $(3, -12, -9) = 3(1, -4, -3)$ therefore the planes are parallel as well.

3.3.16. The normal vectors of the two planes are parallel: $(8, -2, -4) = -2(-4, 1, 2)$ therefore the planes are parallel as well.

3.3.18. The normal vectors of the two planes are orthogonal:

$$(1, -2, 3) \cdot (-2, 5, 4) = (1)(-2) + (-2)(5) + (3)(4) = 0$$

therefore the given planes are perpendicular.

3.3.20. **(a)** From Formula (12) on p.148,

$$\|\text{proj}_{\mathbf{a}}\mathbf{u}\| = \frac{|\mathbf{u} \cdot \mathbf{a}|}{\|\mathbf{a}\|} = \frac{|(5)(2) + (6)(-1)|}{\sqrt{2^2 + (-1)^2}} = \frac{4}{\sqrt{5}}$$

(b) From Formula (12) on p.148,

$$\|\text{proj}_{\mathbf{a}}\mathbf{u}\| = \frac{|\mathbf{u} \cdot \mathbf{a}|}{\|\mathbf{a}\|} = \frac{|(3)(1) + (-2)(2) + (6)(-7)|}{\sqrt{1^2 + 2^2 + (-7)^2}} = \frac{43}{\sqrt{54}} = \frac{43}{3\sqrt{6}}$$

3.3.22. $\mathbf{u} \cdot \mathbf{a} = (-1)(-2) + (-2)(3) = -4$, $\|\mathbf{a}\|^2 = (-2)^2 + 3^2 = 13$,

the vector component of $\mathbf{u}$ along $\mathbf{a}$ is $\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = -\frac{4}{13}(-2, 3) = \left(\frac{8}{13}, -\frac{12}{13}\right)$,

the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$ is

$$\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = (-1, -2) - \left(\frac{8}{13}, -\frac{12}{13}\right) = \left(-\frac{21}{13}, -\frac{14}{13}\right)$$

3.3.28. $\mathbf{u} \cdot \mathbf{a} = (5)(2) + (0)(1) + (-3)(-1) + (7)(-1) = 6$, $\|\mathbf{a}\|^2 = 2^2 + 1^2 + (-1)^2 + (-1)^2 = 7$,

the vector component of $\mathbf{u}$ along $\mathbf{a}$ is $\text{proj}_{\mathbf{a}}\mathbf{u} = \frac{\mathbf{u} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}\mathbf{a} = \frac{6}{7}(2, 1, -1, -1) = \left(\frac{12}{7}, \frac{6}{7}, -\frac{6}{7}, -\frac{6}{7}\right)$,

the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$ is

$$\mathbf{u} - \text{proj}_{\mathbf{a}}\mathbf{u} = (5, 0, -3, 7) - \left(\frac{12}{7}, \frac{6}{7}, -\frac{6}{7}, -\frac{6}{7}\right) = \left(\frac{23}{7}, -\frac{6}{7}, -\frac{15}{7}, \frac{55}{7}\right)$$

3.3.30. From Theorem 3.3.4(a) the distance between the point and the line is

$$D = \frac{|(1)(-1) + (-3)(4) + 2|}{\sqrt{1^2 + (-3)^2}} = \frac{11}{\sqrt{10}}$$

3.4.6. A point on the line: $(0, 7, 4)$; a vector parallel to the line: $(4, 0, 3)$.

3.4.8. A point on the line: $(0, -5, 1)$; a vector parallel to the line: $(0, 5, -1)$.

3.4.12. The vector equation in Formula (6) on p.154 can be expressed as

$$(x, y, z) = (0, 5, -4) + t_1(0, 0, -5) + t_2(1, -3, -2).$$

This yields the parametric equations $x = t_2$, $y = 5 - 3t_2$, $z = -4 - 5t_1 - 2t_2$.

3.4.26. (a) The augmented matrix of the homogeneous system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & \frac{11}{5} & 0 \\ 0 & 1 & -\frac{2}{5} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. A general solution of the system is $x_1 = -\frac{11}{5}t,\ x_2 = \frac{2}{5}t,\ x_3 = t$.

(b) Multiplying $\begin{bmatrix} 1 & -2 & 3 \\ 2 & 1 & 4 \\ 1 & -7 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ yields $\begin{bmatrix} 2 \\ 7 \\ -1 \end{bmatrix}$ therefore $x_1 = x_2 = x_3 = 1$ is a solution of the nonhomogeneous system.

(c) The vector form of a general solution of the nonhomogeneous system is

$$(x_1, x_2, x_3) = \underbrace{(1,1,1)}_{\substack{\text{particular}\\ \text{solution}\\ \text{of the}\\ \text{nonhomogeneous}\\ \text{system}}} + \underbrace{\left(-\frac{11}{5}t,\ \frac{2}{5}t, t\right)}_{\substack{\text{general}\\ \text{solution}\\ \text{of the}\\ \text{homogeneous}\\ \text{system}}}$$

(d) The augmented matrix of the homogeneous system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & \frac{11}{5} & \frac{16}{5} \\ 0 & 1 & -\frac{2}{5} & \frac{3}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$. A general solution of the system is $x_1 = \frac{16}{5} - \frac{11}{5}s,\ x_2 = \frac{3}{5} + \frac{2}{5}s,\ x_3 = s$.

If we let $s = 1 + t$ then this agrees with the solution we obtained in part (c).

3.5.18. From Theorem 3.5.4(b), the volume of the parallelepiped is equal to $\left|\det\begin{bmatrix} 3 & 1 & 2 \\ 4 & 5 & 1 \\ 1 & 2 & 4 \end{bmatrix}\right| = 45.$

3.5.20. $\begin{vmatrix} 5 & -2 & 1 \\ 4 & -1 & 1 \\ 1 & -1 & 0 \end{vmatrix} = 0$ therefore by Theorem 3.5.5 these vectors lie in the same plane when they have the same initial point.

3.5.37. (a) $\mathbf{a} = \overrightarrow{PQ} = (3, -1, -3)$

$\mathbf{b} = \overrightarrow{PR} = (2, -1, 1)$

$\mathbf{c} = \overrightarrow{PS} = (4, -4, 3)$

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 3 & -1 & -3 \\ 2 & -1 & 1 \\ 4 & -4 & 3 \end{vmatrix}$$
$$= 3\begin{vmatrix} -1 & 1 \\ -4 & 3 \end{vmatrix} + 1\begin{vmatrix} 2 & 1 \\ 4 & 3 \end{vmatrix} - 3\begin{vmatrix} 2 & -1 \\ 4 & -4 \end{vmatrix}$$
$$= 3(1) + 1(2) - 3(-4)$$
$$= 17$$

The volume is $\frac{1}{6}|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})| = \frac{17}{6}$.

(b) $\mathbf{a} = \overrightarrow{PQ} = (1, 2, -1)$

$\mathbf{b} = \overrightarrow{PR} = (3, 4, 0)$

$\mathbf{c} = \overrightarrow{PS} = (-1, -3, 4)$

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \\ -1 & -3 & 4 \end{vmatrix}$$
$$= -1\begin{vmatrix} 3 & 4 \\ -1 & -3 \end{vmatrix} + 4\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$
$$= -(-5) + 4(-2)$$
$$= -3$$

The volume is $\frac{1}{6}|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})| = \frac{3}{6} = \frac{1}{2}$.

4.1.6. Let V denote the set of all n-tuples of real numbers of the form $(x, x, \dots, x)$.

Axiom 1: $(x, x, \dots, x) + (y, y, \dots, y) = (x + y, x + y, \dots, x + y)$ is in V for all real x and y;

Axiom 2: $(x, x, \dots, x) + (y, y, \dots, y) = (x + y, x + y, \dots, x + y) = (y + x, y + x, \dots, y + x)$
$= (y, y, \dots, y) + (x, x, \dots, x)$ for all real x and y;

Axiom 3: $(x, x, \dots, x) + \big((y, y, \dots, y) + (z, z, \dots, z)\big) = (x, x, \dots, x) + (y + z, y + z, \dots, y + z)$
$= (x + y + z, x + y + z, \dots, x + y + z) = (x + y, x + y, \dots, x + y) + (z, z, \dots, z)$
$= \big((x, x, \dots, x) + (y, y, \dots, y)\big) + (z, z, \dots, z)$ for all real x, y, and z;

Axiom 4: taking $\mathbf{0} = (0, 0, \dots, 0)$, we have $(0, 0, \dots, 0) + (x, x, \dots, x) = (x, x, \dots, x)$ and
$(x, x, \dots, x) + (0, 0, \dots, 0) = (x, x, \dots, x)$ for all real x;

Axiom 5: for each $\mathbf{u} = (x, x, \dots, x)$, let $-\mathbf{u} = (-x, -x, \dots, -x)$;
then $(x, x, \dots, x) + (-x, -x, \dots, -x) = (0, 0, \dots, 0)$ and
$(-x, -x, \dots, -x) + (x, x, \dots, x) = (0, 0, \dots, 0)$;

Axiom 6: $k(x, x, \dots, x) = (kx, kx, \dots, kx)$ is in V for all real k and x;

Axiom 7: $k\big((x, x, \dots, x) + (y, y, \dots, y)\big) = k(x + y, x + y, \dots, x + y)$
$= (kx + ky, kx + ky, \dots, kx + ky)$
$= k(x, x, \dots, x) + k(y, y, \dots, y)$ for all real k, x, and y;

Axiom 8: $(k + m)(x, x, \dots, x) = \big((k + m)x, (k + m)x, \dots, (k + m)x\big)$
$= (kx + mx, kx + mx, \dots, kx + mx) = k(x, x, \dots, x) + m(x, x, \dots, x)$ for all real k, m, and x;

Axiom 9: $k\big(m(x, x, \dots, x)\big) = k(mx, mx, \dots, mx) = (kmx, kmx, \dots, kmx) = (km)(x, x, \dots, x)$
for all real k, m, and x;

Axiom 10: $1(x, x, \dots, x) = (x, x, \dots, x)$ for all real x.

This is a vector space – all axioms hold.

4.1.8. Axiom 1 fails since a sum of two 2×2 invertible matrices may or may not be invertible, e.g., both $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ are invertible, but $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not invertible.

Axiom 6 fails whenever $k = 0$.

4.2.10. (a) For $-9 - 7x - 15x^2$ to be a linear combination of the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, there must exist scalars a, b, and c such that

$$a(2 + x + 4x^2) + b(1 - x + 3x^2) + c(3 + 2x + 5x^2) = -9 - 7x - 15x^2$$

holds for all real x values. Grouping the terms according to the powers of x yields

$$(2a + b + 3c) + (a - b + 2c)x + (4a + 3b + 5c)x^2 = -9 - 7x - 15x^2$$

Since this equality must hold for every real value x, the coefficients associated with the like powers of x on both sides must match. This results in the linear system

$$\begin{array}{rcrcrcr} 2a & + & 1b & + & 3c & = & -9 \\ 1a & - & 1b & + & 2c & = & -7 \\ 4a & + & 3b & + & 5c & = & -15 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$. There is only one solution to this system, $a = -2$, $b = 1$, $c = -2$, therefore

$$-9 - 7x - 15x^2 = -2\mathbf{p}_1 + 1\mathbf{p}_2 - 2\mathbf{p}_3.$$

(b) For $6 + 11x + 6x^2$ to be a linear combination of the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, there must exist scalars a, b, and c such that

$$a(2 + x + 4x^2) + b(1 - x + 3x^2) + c(3 + 2x + 5x^2) = 6 + 11x + 6x^2$$

holds for all real x values. Grouping the terms according to the powers of x yields

$$(2a + b + 3c) + (a - b + 2c)x + (4a + 3b + 5c)x^2 = 6 + 11x + 6x^2$$

Since this equality must hold for every real value x, the coefficients associated with the like powers of x on both sides must match. This results in the linear system

$$\begin{array}{ccccccc} 2a & + & 1b & + & 3c & = & 6 \\ 1a & - & 1b & + & 2c & = & 11 \\ 4a & + & 3b & + & 5c & = & 6 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$. There is only one solution to this system, $a = 4,\ b = -5,\ c = 1,$ therefore

$$6 + 11x + 6x^2 = 4\mathbf{p}_1 - 5\mathbf{p}_2 + 1\mathbf{p}_3.$$

(c) For 0 to be a linear combination of the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, there must exist scalars a, b, and c such that

$$a(2 + x + 4x^2) + b(1 - x + 3x^2) + c(3 + 2x + 5x^2) = 0$$

holds for all real x values. Grouping the terms according to the powers of x yields

$$(2a + b + 3c) + (a - b + 2c)x + (4a + 3b + 5c)x^2 = 0 + 0x + 0x^2$$

Since this equality must hold for every real value x, the coefficients associated with the like powers of x on both sides must match. This results in the linear system

$$\begin{array}{ccccccc} 2a & + & 1b & + & 3c & = & 0 \\ 1a & - & 1b & + & 2c & = & 0 \\ 4a & + & 3b & + & 5c & = & 0 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. There is only one solution to this system, $a = 0,\ b = 0,\ c = 0,$ therefore $0 = 0\mathbf{p}_1 + 0\mathbf{p}_2 + 0\mathbf{p}_3$.

(d) For $7 + 8x + 9x^2$ to be a linear combination of the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, there must exist scalars a, b, and c such that

$$a(2 + x + 4x^2) + b(1 - x + 3x^2) + c(3 + 2x + 5x^2) = 7 + 8x + 9x^2$$

holds for all real x values. Grouping the terms according to the powers of x yields

$$(2a + b + 3c) + (a - b + 2c)x + (4a + 3b + 5c)x^2 = 7 + 8x + 9x^2$$

Since this equality must hold for every real value x, the coefficients associated with the like powers of x on both sides must match. This results in the linear system

$$\begin{array}{ccccccc} 2a & + & 1b & + & 3c & = & 7 \\ 1a & - & 1b & + & 2c & = & 8 \\ 4a & + & 3b & + & 5c & = & 9 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$. There is only one solution to this system, $a = 0,\ b = -2,\ c = 3,$ therefore

$$7 + 8x + 9x^2 = 0\mathbf{p}_1 - 2\mathbf{p}_2 + 3\mathbf{p}_3.$$

4.2.12. (a) In order for the vector $(2,3,-7,3)$ to be in $\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$, there must exist scalars a, b, and c such that

$$a(2,1,0,3)+b(3,-1,5,2)+c(-1,0,2,1)=(2,3,-7,3)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2a & + & 3b & - & 1c & = & 2 \\ 1a & - & 1b & + & 0c & = & 3 \\ 0a & + & 5b & + & 2c & = & -7 \\ 3a & + & 2b & + & 1c & = & 3 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. This system is consistent (its only solution is $a=2,\ b=-1,\ c=-1$), therefore $(2,3,-7,3)$ is in $\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$.

(b) The vector $(0,0,0,0)$ is obviously in $\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$ since

$$0(2,1,0,3)+0(3,-1,5,2)+0(-1,0,2,1)=(0,0,0,0)$$

(c) In order for the vector $(1,1,1,1)$ to be in $\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$, there must exist scalars a, b, and c such that

$$a(2,1,0,3)+b(3,-1,5,2)+c(-1,0,2,1)=(1,1,1,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2a & + & 3b & - & 1c & = & 1 \\ 1a & - & 1b & + & 0c & = & 1 \\ 0a & + & 5b & + & 2c & = & 1 \\ 3a & + & 2b & + & 1c & = & 1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. This system is inconsistent therefore $(1,1,1,1)$ is not in $\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$.

(d) In order for the vector $(-4,6,-13,4)$ to be in $\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$, there must exist scalars a, b, and c such that

$$a(2,1,0,3)+b(3,-1,5,2)+c(-1,0,2,1)=(-4,6,-13,4)$$

Equating corresponding components on both sides yields the linear system

$$\begin{array}{rcrcrcr} 2a & + & 3b & - & 1c & = & -4 \\ 1a & - & 1b & + & 0c & = & 6 \\ 0a & + & 5b & + & 2c & = & -13 \\ 3a & + & 2b & + & 1c & = & 4 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. This system is consistent (its only solution is $a=3,\ b=-3,\ c=1$), therefore $(-4,6,-13,4)$ is in $\operatorname{span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$.

4.3.6. **(a)** The set $\{\mathbf{v}_1, \mathbf{v}_3\}$ can be shown to be linearly independent since $a(-1,2,3) + b(-3,6,0) = (0,0,0)$ has only the trivial solution $a = b = 0$. Therefore the three vectors do not lie on the same line (even though the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ *are* collinear).

(b) Any subset of two vectors chosen from these three vectors can be shown to be linearly independent (e.g., $a(2,-1,4) + b(4,2,3) = (0,0,0)$ has only the trivial solution $a = b = 0$). Therefore the three vectors do not lie on the same line.

(An alternate way to show this would be to demonstrate that the three vectors form a linearly independent set, therefore they do not even lie on the same plane, so that they cannot possibly lie on the same line.)

(c) Each subset of two vectors chosen from these three vectors can be shown to be linearly dependent since $-1\mathbf{v}_1 + 2\mathbf{v}_2 = \mathbf{0}$, $1\mathbf{v}_1 + 2\mathbf{v}_3 = \mathbf{0}$, and $1\mathbf{v}_2 + 1\mathbf{v}_3 = \mathbf{0}$. Therefore all three vectors lie on the same line.

4.3.8. **(a)** The vector equation $a(1,2,3,4) + b(0,1,0,-1) + c(1,3,3,3) = (0,0,0,0)$ can be rewritten as a homogeneous linear system by equating the corresponding components on both sides

$$\begin{array}{rcrcrcr} 1a & + & 0b & + & 1c & = & 0 \\ 2a & + & 1b & + & 3c & = & 0 \\ 3a & + & 0b & + & 3c & = & 0 \\ 4a & - & 1b & + & 3c & = & 0 \end{array}$$

The augmented matrix of this system has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ therefore a general solution of the system is

$$a = -t,\ b = -t,\ c = t$$

Since the system has nontrivial solutions, the given set of vectors is linearly dependent.

(b) In the general solution we obtained in part (a), let the parameter t have a nonzero value, e.g., $t = 1$. Then $a = -1$, $b = -1$, and $c = 1$ so that $-\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$. This can be solved for each of the three vectors: $\mathbf{v}_1 = -\mathbf{v}_2 + \mathbf{v}_3$, $\mathbf{v}_2 = -\mathbf{v}_1 + \mathbf{v}_3$, and $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$.

4.4.4. **(a)** We look at the system

$$\begin{aligned} 2c_1 + 3c_2 + 1c_3 &= 0 \\ -4c_1 + 2c_2 + 6c_3 &= 0 \\ 1c_1 - 1c_2 - 2c_3 &= 0 \end{aligned}$$

, to see if the vectors are linearly independent.

This augmented matrix $\begin{pmatrix} 2 & 3 & 1 & 0 \\ -4 & 2 & 6 & 0 \\ 1 & -1 & -2 & 0 \end{pmatrix}$ row reduces to $\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ which tells us that $t(2 - 4x + x^2) - t(3 + 2x - x^2) + t(1 + 6x - 2x^2) = 0$ for any scalar t. Since the vectors are linearly dependent, they cannot form a basis.

(b) The matrix $A = \begin{pmatrix} 3 & 0 & 2 \\ 2 & 1 & -4 \\ -1 & 5 & 1 \end{pmatrix}$ has determinant 85 ($\neq 0$), so the polynomials form a linearly independent set. Also, this tells us that the system $A\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ will have a solution for any **b**, so the polynomials span P_2. Thus the vectors form a basis for P_2.

(c) Vectors $\mathbf{p}_1 = 1 + x + x^2$, $\mathbf{p}_2 = x + x^2$, and $\mathbf{p}_3 = x^2$ are linearly independent if the vector equation $c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{0}$ has only the trivial solution.

For these vectors to span P_2, it must be possible to express every vector $\mathbf{p} = a_0 + a_1x + a_2x^2$ in P_2 as

$$c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{p}$$

By grouping the terms on the left hand sides as

$$c_1(1 + x + x^2) + c_2(x + x^2) + c_3(x^2) = c_1 + (c_1 + c_2)x + (c_1 + c_2 + c_3)x^2$$

these two equations can be rewritten as linear systems

$$\begin{aligned} c_1 \qquad\qquad &= 0 \\ c_1 + c_2 \qquad &= 0 \\ c_1 + c_2 + c_3 &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} c_1 \qquad\qquad &= a_0 \\ c_1 + c_2 \qquad &= a_1 \\ c_1 + c_2 + c_3 &= a_2 \end{aligned}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_0, a_1, and a_2. Therefore the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly independent and span P_2 so that they form a basis for P_2.

(d)

$$c_1(-4 + x + 3x^2) + c_2(6 + 5x + 2x^2) + c_3(8 + 4x + x^2) =$$
$$(-4c_1 + 6c_2 + 8c_3) + (c_1 + 5c_2 + 4c_3)x + (3c_1 + 2c_2 + c_3)x^2$$

these two equations can be rewritten as linear systems

$$\begin{aligned} -4c_1 + 6c_2 + 8c_3 &= 0 \\ c_1 + 5c_2 + 4c_3 &= 0 \\ 3c_1 + 2c_2 + c_3 &= 0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} -4c_1 + 6c_2 + 8c_3 &= a_0 \\ c_1 + 5c_2 + 4c_3 &= a_1 \\ 3c_1 + 2c_2 + c_3 &= a_2 \end{aligned}$$

Since the coefficient matrix of both systems has determinant $\begin{vmatrix} -4 & 6 & 8 \\ 1 & 5 & 4 \\ 3 & 2 & 1 \end{vmatrix} = -26 \neq 0$, it follows from parts (b), (e), and (g) of Theorem 2.3.8 that the homogeneous system has only the trivial solution and the nonhomogeneous system is consistent for all real values a_0, a_1, and a_2. Therefore the vectors $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly independent and span P_2 so that they form a basis for P_2.

4.4.5. The equations $c_1\begin{bmatrix}3 & 6\\ 3 & -6\end{bmatrix}+c_2\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}+c_3\begin{bmatrix}0 & -8\\ -12 & -4\end{bmatrix}+c_4\begin{bmatrix}1 & 0\\ -1 & 2\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$ and $c_1\begin{bmatrix}3 & 6\\ 3 & -6\end{bmatrix}+c_2\begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}+c_3\begin{bmatrix}0 & -8\\ -12 & -4\end{bmatrix}+c_4\begin{bmatrix}1 & 0\\ -1 & 2\end{bmatrix}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ generate the systems

$$\begin{aligned}3c_1 \qquad\qquad\quad +c_4 &= 0\\ 6c_1-c_2\ -8c_3 \qquad &= 0\\ 3c_1-c_2-12c_3\ -c_4 &= 0\\ -6c_1 \qquad -4c_3+2c_4 &= 0\end{aligned} \quad \text{and} \quad \begin{aligned}3c_1 \qquad\qquad\quad +c_4 &= a\\ 6c_1-c_2\ -8c_3 \qquad &= b\\ 3c_1-c_2-12c_3\ -c_4 &= c\\ -6c_1 \qquad -4c_3+2c_4 &= d\end{aligned}$$

which have the same coefficient matrix. Since

$\det\begin{bmatrix}3 & 0 & 0 & 1\\ 6 & -1 & -8 & 0\\ 3 & -1 & -12 & -1\\ -6 & 0 & -4 & 2\end{bmatrix}=48\neq 0$ the matrices are linearly independent and also span M_{22}, so they are a basis.

4.4.8. **(a)** Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(1,0)=c_1(1,-1)+c_2(1,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{aligned}c_1 + c_2 &= 1\\ -c_1 + c_2 &= 0\end{aligned}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix}1 & 0 & \frac{1}{2}\\ 0 & 1 & \frac{1}{2}\end{bmatrix}$. The solution of the linear system is $c_1=\frac{1}{2}$, $c_2=\frac{1}{2}$, therefore the coordinate vector is $(\mathbf{w})_S=(\frac{1}{2},\frac{1}{2})$.

(b) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obtain

$$(0,1)=c_1(1,-1)+c_2(1,1)$$

Equating corresponding components on both sides yields the linear system

$$\begin{aligned}c_1 + c_2 &= 0\\ -c_1 + c_2 &= 1\end{aligned}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix}1 & 0 & -\frac{1}{2}\\ 0 & 1 & \frac{1}{2}\end{bmatrix}$. The solution of the linear system is $c_1=-\frac{1}{2}$, $c_2=\frac{1}{2}$, therefore the coordinate vector is $(\mathbf{w})_S=(-\frac{1}{2},\frac{1}{2})$.

(c) Expressing $\mathbf{w}$ as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$ we obviously have

$(1,1)=0(1,-1)+1(1,1)$ therefore the coordinate vector is $(\mathbf{w})_S=(0,1)$.

4.4.10. (a) Since $\mathbf{p} = 4\mathbf{p}_1 + (-3)\mathbf{p}_2 + 1\mathbf{p}_3$ we conclude that the coordinate vector is $(\mathbf{p})_S = (4, -3, 1)$.

(b) Expressing $\mathbf{p}$ as a linear combination of $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ we obtain

$$2 - x + x^2 = c_1(1 + x) + c_2(1 + x^2) + c_3(x + x^2)$$

Grouping the terms on the right hand side according to powers of x yields

$$2 - x + x^2 = (c_1 + c_2) + (c_1 + c_3)x + (c_2 + c_3)x^2$$

For this equality to hold for all real x, the coefficients associated with the same power of x on both sides must match. This leads to the linear system

$$\begin{array}{rcrcrcr} c_1 & + & c_2 & & & = & 2 \\ c_1 & & & + & c_3 & = & -1 \\ & & c_2 & + & c_3 & = & 1 \end{array}$$

whose augmented matrix has the reduced row echelon form $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{bmatrix}$. The solution is $c_1 = 0,\ c_2 = 2,\ c_3 = -1,$ therefore the coordinate vector is $(\mathbf{p})_S = (0, 2, -1)$.

4.5.4. The augmented matrix of the linear system $\begin{bmatrix} 1 & -3 & 1 & 0 \\ 2 & -6 & 2 & 0 \\ 3 & -9 & 3 & 0 \end{bmatrix}$ has the reduced row echelon form $\begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_1 = 3s - t,\ x_2 = s,\ x_3 = t$. In vector form

$$(x_1, x_2, x_3) = (3s - t, s, t) = s(3, 1, 0) + t(-1, 0, 1)$$

therefore the solution space is spanned by the vectors $\mathbf{v}_1 = (3, 1, 0)$ and $\mathbf{v}_2 = (-1, 0, 1)$. These vectors are linearly independent since neither of them is a scalar multiple of the other (Theorem 4.3.2(c)). We conclude that $\mathbf{v}_1$ and $\mathbf{v}_2$ form a basis for the solution space and that the dimension of the solution space is 2.

4.5.12. (a) Any of the three standard basis vectors can be used as $\mathbf{v}_3$ to complete a basis for R^3. For example, if $\mathbf{v}_3 = \mathbf{e}_3 = (0, 0, 1)$, the augmented system $\begin{pmatrix} 1 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 3 & 1 & 0 \end{pmatrix}$ row reduces to

$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, so the only solution to $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{e}_3 = \mathbf{0}$ is the trivial solution so

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_3\}$ forms a basis for R^3.

(b) Any of the three standard basis vectors can be used as $\mathbf{v}_3$ to complete a basis for R^3. For example, if $\mathbf{v}_3 = \mathbf{e}_2 = (0, 1, 0)$, the augmented system $\begin{pmatrix} 5 & 1 & 0 & 0 \\ 3 & -1 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}$ row reduces to

$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, so the only solution to $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{e}_2 = \mathbf{0}$ is the trivial solution so

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_2\}$ forms a basis for R^3.