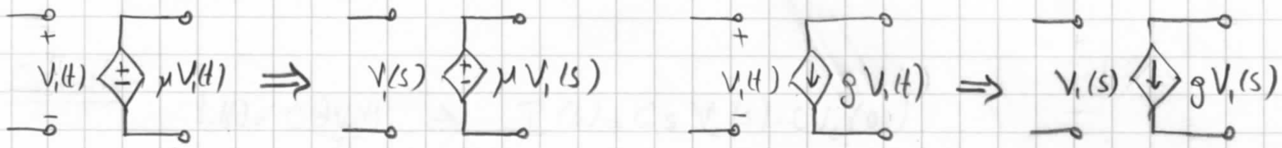
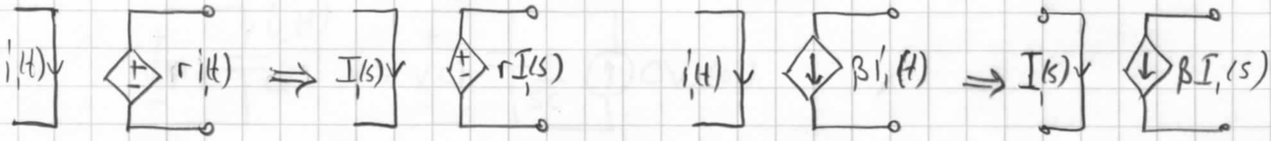
Ch 10 S-Domain Circuit Analysis

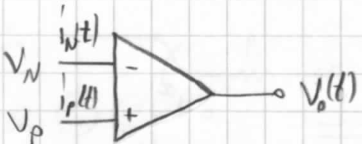
linear circuit $\xrightarrow{\mathcal{L}}$ Transformed circuit $\longrightarrow$ Solve Algebraic Eqn $\xrightarrow{\mathcal{L}^{-1}}$ Response waveform

Voltage Source $v(t) \longleftrightarrow V(s)$

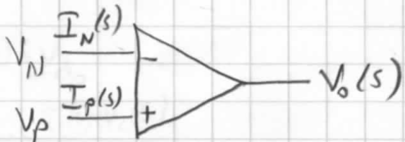
Current Source $i(t) \longleftrightarrow I(s)$







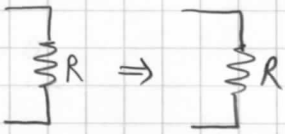
$i_N = 0 = i_P \quad v_N = v_P$



$I_N = 0 = I_P \quad V_N = V_S$

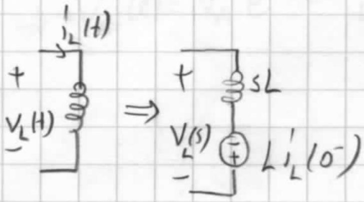
Resistor

$v_R(t) = R\, i_R(t) \;\rightarrow\; V_R(s) = R\, I_R(s)$



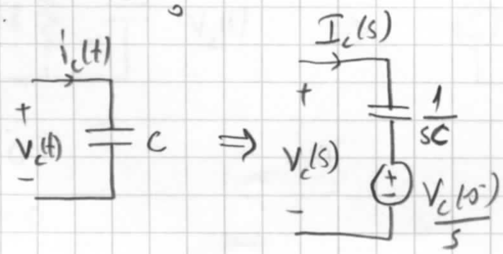
Inductor

$v_L(t) = L \dfrac{di_L}{dt} \;\Rightarrow\; V_L(s) = Ls\, I_L(s) - L\, i_L(0^-)$



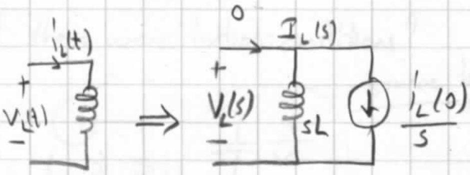
Capacitor

$v_C(t) = \dfrac{1}{C} \displaystyle\int_0^t i_C(\tau)\, d\tau + v_C(0^-) \;\Rightarrow\; V_C(s) = \dfrac{1}{C} \dfrac{I_C(s)}{s} + \dfrac{v_C(0^-)}{s}$

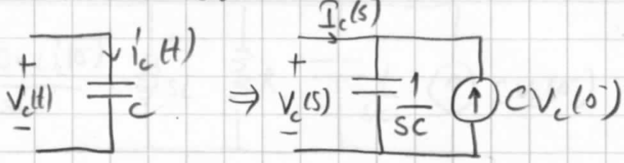


Alternatively

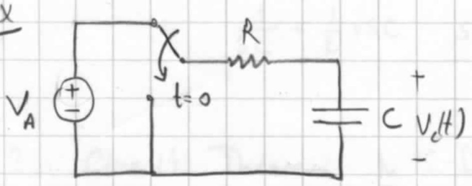
Inductor $\quad i_L(t) = \frac{1}{L}\int_0^t V_L(\tau)\,d\tau + i_L(0) \Rightarrow I_L(s) = \frac{1}{L}\frac{V_L(s)}{s} + \frac{i_L(0)}{s}$



Capacitor $\quad i_c(t) = C\frac{dV_c(t)}{dt} \Rightarrow I_c(s) = Cs\,V_c(s) - CV_c(0^-)$

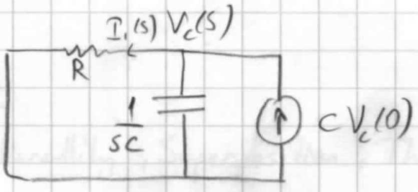


ex



$V_c(t) = ?$

$V_c(0) = V_A$



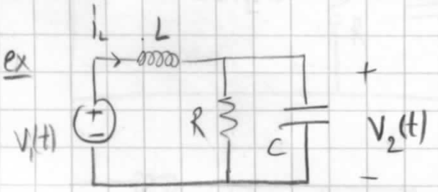
$$-CV_c(0) + sC\,V_c(s) + \frac{V_c(s)}{R} = 0$$

$$V_c(s) = \frac{CV_c(0)}{sC + \frac{1}{R}} = \frac{V_A}{s + \frac{1}{RC}} \Rightarrow V_c(t) = V_A e^{-\frac{1}{RC}t}\,u(t)$$

10.2 Basic Circuit Analysis In s-Domain

(It is the same as resistive circuits)

ex



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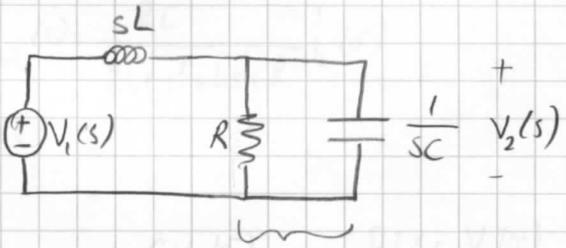


$i_L(0) = 0$

$V_L(0) = 0$

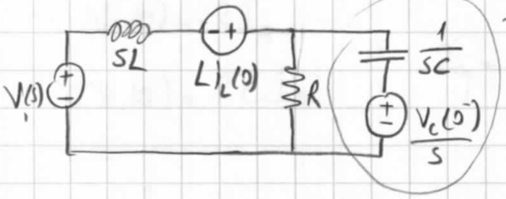
$$Z_{EQ_1} = \frac{1}{\frac{1}{R} + sC} = \frac{R}{sRC + 1}$$

find eq impedance and $V_2(s)$

$$Z_{EQ} = sL + \frac{R}{1 + sRC} = \frac{s^2RLC + sL + R}{1 + sRC}$$

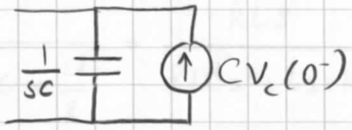
$$V_2(s) = V_1(s)\frac{\dfrac{R}{1+sRC}}{\dfrac{s^2RLC+sL+R}{1+sRC}} = V_1(s)\frac{R}{RLCs^2+Ls+R}$$

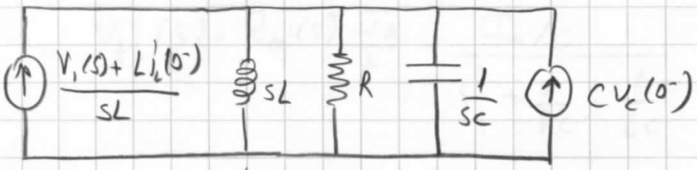
what if there were initial conditions?



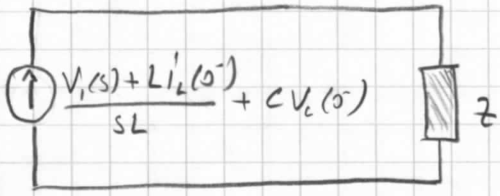
source trans.

$\equiv$





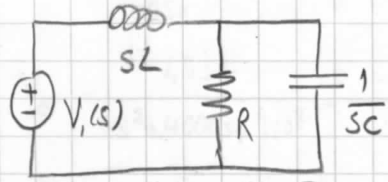
$\equiv$



$$\frac{1}{\frac{1}{sL}+\frac{1}{R}+sC} = \frac{RLs}{s^2RLC+Ls+R} = Z$$

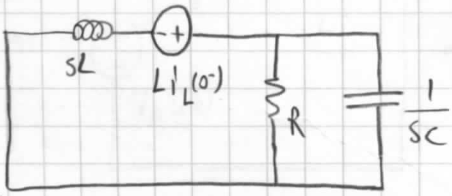
10.3. Circuit Theorems in S-Domain

(Same as resistive circuits: Proportionality, Superposition, Thevenin & Norton Eqv.)

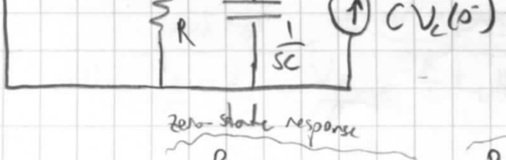


$$\underbrace{\phantom{R\ \|\ \tfrac{1}{sC}}}_{\frac{R}{1+sRC}}$$

$$V_{2,1}(s) = V_1(s)\frac{\dfrac{R}{RCs+1}}{sL+\dfrac{R}{RCs+1}} = \frac{R}{RLCs^2+Ls+R}V_1(s)$$



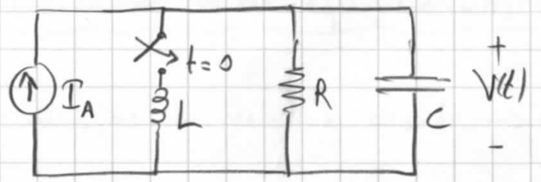
$$V_{2,2}(s) = \frac{RL}{RLCs^2+Ls+R}i_L(0^-)$$

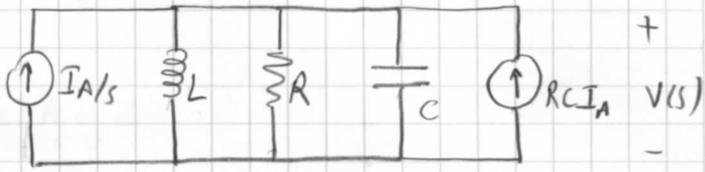


$$V_{2,3}(s) = \frac{CV_c(0^-)}{\frac{1}{R}+\frac{1}{sL}+sC} = \frac{RLCs\,V_c(0^-)}{RLCs^2+Ls+R}$$

$$V_2(s) = \underbrace{\frac{R}{RLCs^2+Ls+R}V_1(s)}_{\text{zero-state response}} + \underbrace{\frac{RL}{RLCs^2+Ls+R}i_L(0^-) + \frac{RLCs}{RLCs^2+Ls+R}V_c(0^-)}_{\text{zero-input response}}$$

ex





$$V_c(0) = RI_A$$

$$CV_c(0) = RCI_A$$

$$Z_{eq} = \frac{1}{\frac{1}{sL} + sC + \frac{1}{R}} = \frac{RLs}{RLCs^2 + Ls + R}$$

zero-state response

$$V_{zs}(s) = Z_{eq}(s)\frac{I_A}{s} = \frac{I_A/C}{s^2 + \frac{s}{RC} + \frac{1}{LC}}$$

zero-input

$$V_{zi}(s) = Z_{eq}(s)\, RCI_A = \frac{sRI_A}{s^2 + \frac{s}{RC} + \frac{1}{LC}}$$

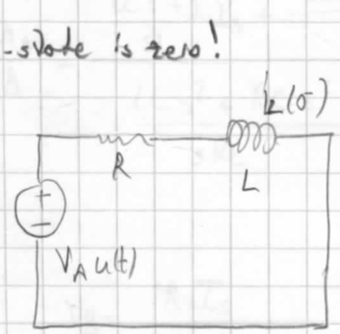
$I_A = 1mA,\ L = 2H,\ R = 1{,}5k\Omega,\ C = \frac{1}{6}\mu F$

$$V_{zs} = \frac{6000}{s^2 + \frac{s}{250}10^6 + 3\,10^6} = \frac{6000}{(s+1000)(s+3000)} = \frac{3}{s+1000} - \frac{3}{s+3000} \Rightarrow V_{zs}(t) = \left(3e^{-1000t} - 3e^{-3000t}\right)u(t)$$

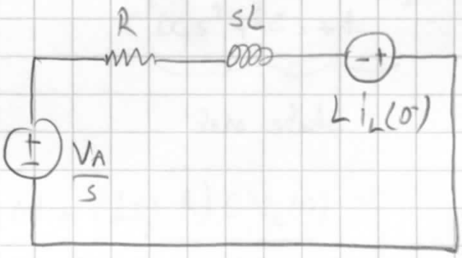
$$V_{zi} = \frac{1{,}5s}{s^2 + 4000s + 3\,10^6} = \frac{-0{,}75}{s+1000} + \frac{2{,}25}{s+3000} \Rightarrow V_{zi}(t) = \left(-0{,}75e^{-1000t} + 2{,}25e^{-3000t}\right)u(t)$$

steady-state is zero!

example



$\overset{\mathcal{L}}{\Longrightarrow}$



$$I(s) = \frac{\frac{V_A}{s} + Li_L(0^-)}{R + sL} = \frac{V_A/L}{s(s + R/L)} + \frac{i_L(0^-)}{s + R/L} = \frac{V_A/R}{s} + \frac{-V_A/R}{s + R/L} + \frac{i_L(0^-)}{s + R/L}$$

$$i(t) = \underbrace{\overbrace{\frac{V_A}{R}u(t)}^{\text{forced resp}} \overbrace{- \frac{V_A}{R}e^{-\frac{R}{L}t}u(t)}^{\text{natural response}}}_{\text{zero-state resp.}} + \underbrace{i_L(0^-)e^{-\frac{R}{L}t}u(t)}_{\text{zero-input resp.}}$$

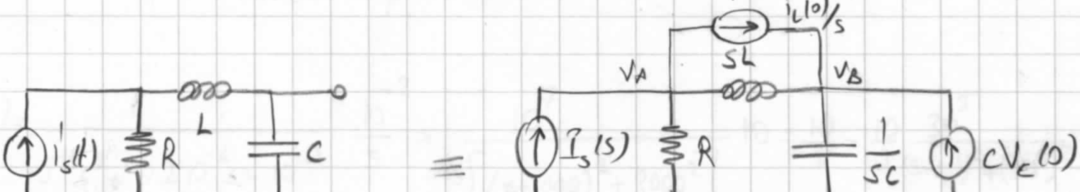
zero-state resp. has both forced & natural resp. components.

Thévenin & Norton Equ. Circuits

Same as in resistive circuits $\rightarrow$ replace R by Z

- $V_T(s) = Z_T(s)\, I_N(s)$
- $Z_T(s) = Z_N(s)$

example



Node A:
$$\frac{V_A}{R} + \frac{i_L(0)}{s} + \frac{V_A - V_B}{sL} - I_s(s) = 0$$

Node B:
$$-\frac{i_L(0)}{s} + \frac{V_B - V_A}{sL} + sCV_B - CV_c(0) = 0$$

$$\begin{bmatrix} \frac{1}{R} + \frac{1}{sL} & -\frac{1}{sL} \\ -\frac{1}{sL} & \frac{1}{sL} + sC \end{bmatrix} \begin{bmatrix} V_A \\ V_B \end{bmatrix} = \begin{bmatrix} I_s(s) - i_L(0)/s \\ CV_c(0) + i_L(0)/s \end{bmatrix}$$

$$V_A = \frac{\Delta_A}{\Delta} = \frac{\frac{LCs^2+1}{sL} I_s + \frac{-LCs\,i_L(0) + CV_c(0)}{sL}}{\frac{LCs^2 + RCs^2 + 1}{sRL}} = \underbrace{\frac{RLCs^2+1}{LCs^2+RCs+1} I_s}_{\text{zero state}} + \underbrace{\frac{-RLCs\,i_L(0) + RCV_c(0)}{LCs^2+RCs+1}}_{\text{zero input}}$$

similarly:
$$V_B = \underbrace{\frac{R I_s}{LCs^2 + RCs + 1}}_{\text{zero state}} + \underbrace{\frac{L i_L(0) + (Ls + R)CV_c(0)}{LCs^2 + RCs + 1}}_{\text{zero input}}$$

<u>Numerical values</u> $R = 1k\Omega$, $L = 0{,}5\,H$, $C = 0{,}2\,\mu F$, $i_s(t) = 10u(t)$ mA

$$V_A(s) = \frac{10^{-7}s^2+1}{10^{-3}\,0{,}5\,0{,}2\,10^{-6}s^2 + 0{,}2\,10^{-6}s + 10^{-3}}\,\frac{10^{-2}}{s} = 10\,\frac{s^2+10^7}{s\left[(s+1000)^2+9\,10^6\right]} = \frac{10}{s} - \frac{20}{3}\,\frac{3000}{(s+1000)^2+3000^2}$$

$$V_A(t) = \underbrace{10u(t)}_{\text{forced resp.}} - \underbrace{20e^{-1000t}\left[\frac{1}{3}\sin(3000t)\right]}_{\text{Natural resp.}}u(t)$$

$$V_B(s) = \frac{1}{10^{-10}s^2 + 0{,}2\,10^{-6}s + 10^{-3}}\,\frac{10^{-2}}{s} = \frac{10^7}{s\left[(s+1000)^2+3000^2\right]}\,10 = \frac{10}{s} - \frac{10}{3}\,\frac{3\,10^3}{(s+10^3)^2+(3\,10^3)^2} - 10\,\frac{s+10^3}{(s+10^3)^2+(3\,10^3)^2}$$

$$V_B(t) = 10u(t) - 10e^{-1000t}\left[\frac{1}{3}\sin 3000t\right]u(t) - 10e^{-1000t}\left[\cos 3000t\right]u(t)$$

Standard form $s^2 + 2\zeta\omega_0 s + \omega_0^2 = s^2 + \frac{R}{L}s + \frac{1}{LC}$ $\Rightarrow \omega_0 = \frac{1}{\sqrt{LC}}$, $\zeta = \frac{R/L}{2\frac{1}{\sqrt{LC}}} = \frac{R}{2}\sqrt{\frac{C}{L}}$

$\zeta > 1$: overdamped

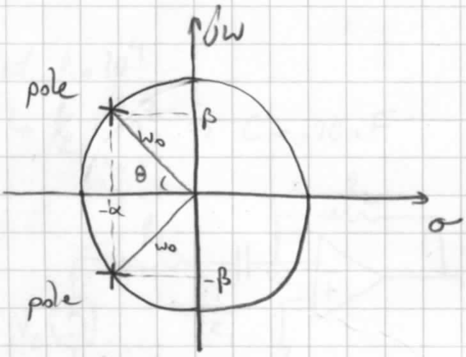
$\zeta = 1$: critically damped

$\zeta < 1$: underdamped $\Rightarrow s = -\alpha \mp j\beta$ $(s+\alpha)^2+\beta^2 = 0$

poles at $s_{1,2} = \omega_0\left(-\zeta \pm \sqrt{\zeta^2-1}\right)$

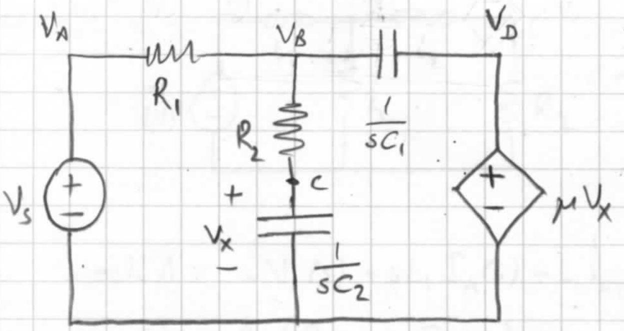
$$s^2 + 2\alpha s + \alpha^2 + \beta^2 = s^2 + 2\zeta\omega_0 s + \omega_0^2$$

$$\Rightarrow \omega_0^2 = \alpha^2 + \beta^2 \;;\; \zeta = \frac{\alpha}{\sqrt{\alpha^2+\beta^2}}$$



$\cos\theta = \zeta$ & ω_0 → polar coordinates

α, β → rectangular coordinates example



Node Analysis

1. Assign node voltages
2. From voltage sources

$$V_A(s) = V_s(s)$$
$$V_D(s) = \mu V_x$$

3. For dependent sources substitute node voltages

$$V_D(s) = \mu V_x = \mu V_c(s)$$

4. Write KCL eqns at other nodes.

Node B: $$\frac{V_B - V_A}{R_1} + \frac{V_B - V_c}{R_2} + \frac{V_B - \mu V_c}{1/sC_1} = 0$$

Node C: $$\frac{V_c}{1/sC_2} + \frac{V_c - V_B}{R_2} = 0$$

$$\begin{bmatrix} sC_1 + \frac{1}{R_1} + \frac{1}{R_2} & -\mu sC_1 - \frac{1}{R_2} \\ -\frac{1}{R_2} & sC_2 + \frac{1}{R_2} \end{bmatrix} \begin{bmatrix} V_B \\ V_C \end{bmatrix} = \begin{bmatrix} V_s/R_1 \\ 0 \end{bmatrix}$$

design $w_0 = 1000$ rad/s, $\zeta = 0.5$

$C_1 = C_2 = C$, $R_1 = R_2 = R$

let's find natural poles

$$\Delta(s) = s^2 C^2 + \frac{C}{R} s + \frac{2C}{R} s + \frac{2}{R^2} - \mu \frac{C}{R} s - \frac{1}{R^2}$$

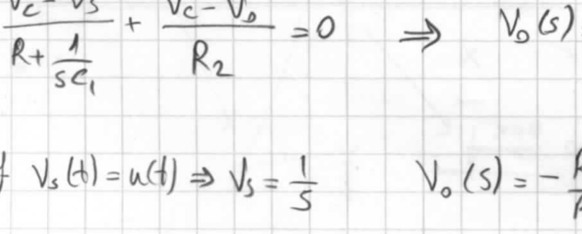
$$\frac{\Delta(s)}{C^2} = s^2 + \frac{(3-\mu)}{RC} s + \frac{1}{R^2 C^2} = s^2 + 2\zeta w_0 s + w_0^2 \Rightarrow w_0 = \frac{1}{RC} = 1000$$

$$\frac{3-\mu}{RC} = 2\zeta w_0 = 1000 \Rightarrow \mu = 2$$

let $R = 10^4$

$\hookrightarrow \frac{1}{C} = 10^7 \Rightarrow C = 100\,nF$

example



$V_c = 0$

$I_N = 0$

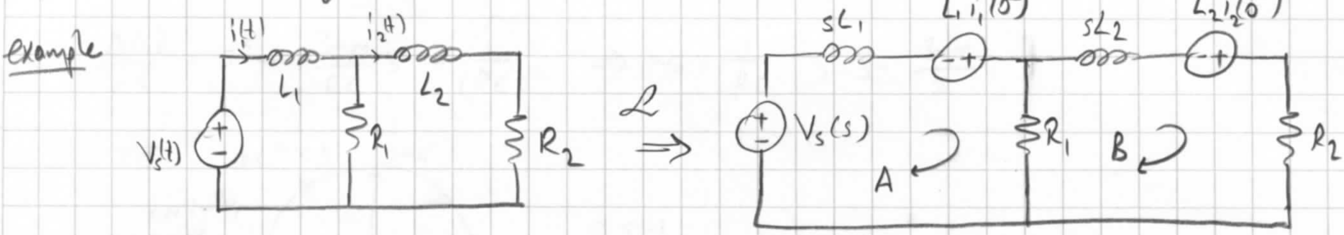
$I_p = 0$

$$\frac{V_c - V_s}{R + \frac{1}{sC_1}} + \frac{V_c - V_o}{R_2} = 0 \Rightarrow V_o(s) = -\frac{sC_1 R_2}{sC_1 R_1 + 1} V_s = -\frac{R_2}{R_1} \frac{s}{s + \frac{1}{R_1 C_1}} V_s$$

If $V_s(t) = u(t) \Rightarrow V_s = \frac{1}{s}$ $$V_o(s) = -\frac{R_2}{R_1} \underbrace{\frac{1}{s + \frac{1}{R_1 C_1}}}_{\text{Natural pole}} \Rightarrow V_o(t) = \underbrace{-\frac{R_2}{R_1} e^{-t/R_1 C_1} u(t)}_{\text{Steady-State} = 0}$$

Mesh Current Analysis

example



mesh A: $-V_s(s) + sL_1 I_A(s) - L_1 i_{L_1}(0^-) + R_1\left(I_A(s) - I_B(s)\right) = 0$

mesh B: $R_1\left(I_B(s) - I_A(s)\right) + sL_2 I_B(s) - L_2 i_2(0^-) + R_2 I_B(s) = 0$

$$\begin{bmatrix} sL_1 + R_1 & -R_1 \\ -R_1 & R_1 + sL_2 + R_2 \end{bmatrix} \begin{bmatrix} I_A(s) \\ I_B(s) \end{bmatrix} = \begin{bmatrix} V_s(s) + L_1 i_{L_1}(0^-) \\ L_2 i_2(0^-) \end{bmatrix}$$

Summary

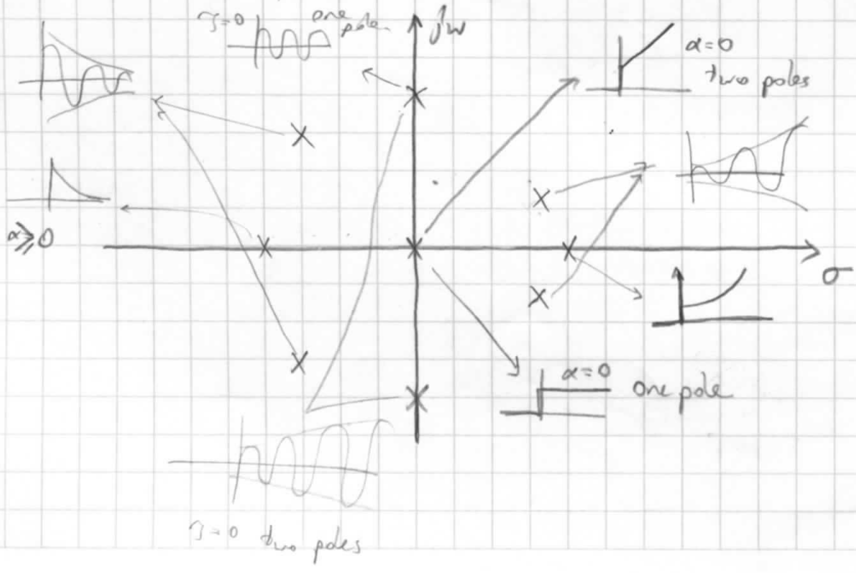
$Y(s)$ = output → real poles ⇒ exponential, damped ramp

$Y(s)$ = output → complex conjugate ⇒ damped sinusoid.

Natural Poles: zeros of the circuit determinant
→ natural response

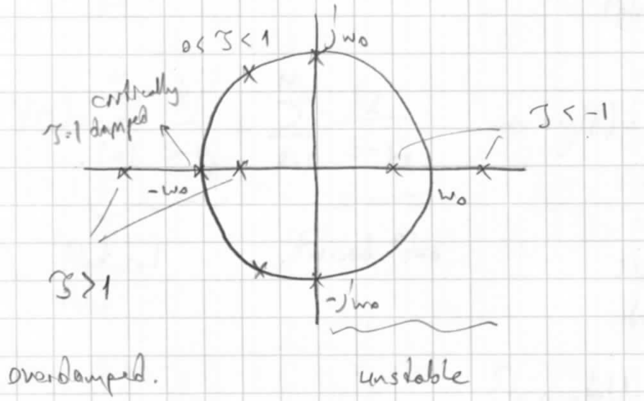
Forced Poles: poles of the input
→ forced response

Stability: If natural response decays to zero as $t \to \infty$
⇒ Natural poles are in the left hand plane (LHP)



Op-Amp Circuit Example

$$\frac{\Delta(s)}{C^2} = s^2 - \frac{3-\mu}{RC} s + \frac{1}{(RC)^2} \Rightarrow \omega_0 = \frac{1}{RC} \quad , \zeta = \frac{3-\mu}{2}$$



$\zeta_1 \zeta_2 = \omega_0^2$

In a circuit of passive resistors, capacitors and inductors $\zeta \geq 0 \Rightarrow$ stable.

However for the above OPAMP circuit

$\mu > 3 \Rightarrow \zeta < 0 \Rightarrow$ unstable.