Chapter 11: Network Functions

$$\underset{\text{Input}}{X(s)} \longrightarrow \boxed{T(s)} \longrightarrow \underset{\text{Output}}{Y(s)}$$

zero state

$$Y(s) = \underbrace{T(s)}_{\text{Natural Poles.}} \underbrace{X(s)}_{\text{Forced Poles.}}$$

$$Y(s) = \underbrace{\sum_{j=1}^{N} \frac{k_j}{s-p_j}}_{\text{Natural Poles}} + \underbrace{\sum_{l=1}^{M} \frac{k_l}{s-p_l}}_{\text{Forced Poles.}} \Rightarrow y(t) = \underbrace{\sum_{j=1}^{N} k_j e^{p_j t}}_{\text{Natural response transient}} + \underbrace{\sum_{l=1}^{M} k_l e^{p_l t}}_{\text{forced response transient + steady state}}$$

{in a stable circuit} {decay to zero} {does not decay to zero}

example

$$T(s) = \frac{V_2(s)}{V_1(s)} = \frac{2000(s+2000)}{(s+1000)(s+4000)}$$

$$V_1(t) = (20 + 15e^{-5000t})u(t) \Rightarrow V_1(s) = \frac{20}{s} + \frac{15}{s+5000} = \frac{35s+10^5}{s(s+5000)}$$

$$V_2(s) = \frac{10^4(s+2 10^3)(7s+2 10^4)}{s(s+10^3)(s+4 10^3)(s+5 10^3)} = \underbrace{\frac{k_1}{s+10^3} + \frac{k_2}{s+4 10^3}}_{\text{Natural poles}} + \underbrace{\frac{k_3}{s} + \frac{k_4}{s+5 10^3}}_{\text{Forced poles}}$$

$k_1 = -65/6$

$k_2 = 40/3$

$k_3 = 20$

$k_4 = -45/2$

$$V_2(t) = \Big[\underbrace{\frac{-65}{6}e^{-1000t} + \frac{40}{3}e^{-4000t}}_{\text{Natural response}} + \underbrace{\overset{\text{steady-state resp.}}{20} - \frac{45}{2}e^{-5000t}}_{\text{Forced response}}\Big]u(t)$$

Test Signals

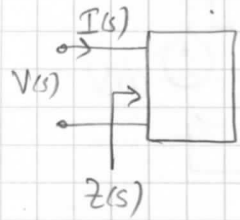
Pulse → Short (pulse duration is much shorter than the circuit response time) ~ Impulse ⇒ Impulse Response.

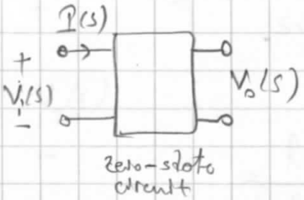
Pulse → Long (pulse duration is much longer than the circuit response time) ~ step ⇒ Step Response.

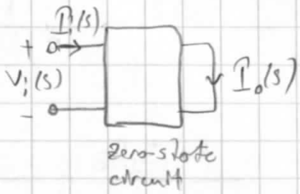
Sinusoid → Sinusoidal Steady-State ⇒ Freq. Resp.

(How circuit responds to signals with varying freq)

11.2 Network Functions of One-Port & 2-port circuits







Driving Point Impedance

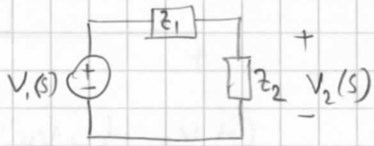
$$Z(s) = \frac{V(s)}{I(s)}$$

- Driving point impedance s-domain generalization of input resistance
- Inverse is also a Network Function

Transfer Function

$$\frac{V_o(s)}{V_i(s)} = T_V(s) \qquad \frac{V_o(s)}{I_i(s)} = T_Z(s)$$

$$\frac{I_o(s)}{I_i(s)} = T_I(s) \qquad \frac{I_o(s)}{V_i(s)} = T_Y(s)$$



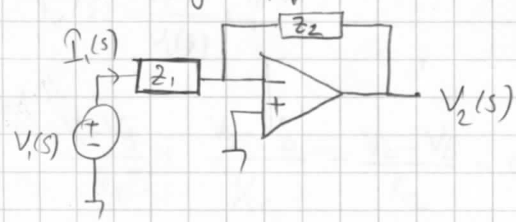
$$V_2(s) = \frac{Z_2}{Z_1 + Z_2} V_1$$

$$T_V(s) = \frac{Z_2}{Z_1 + Z_2}$$

- Inverse is not a network funct

example

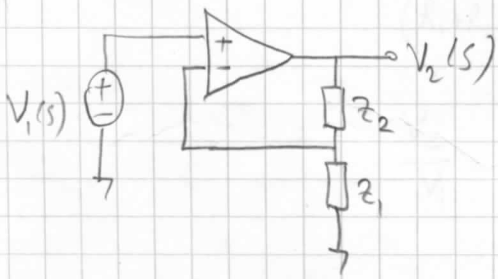
Inverting amplifier



$$V_2(s) = -\frac{Z_2}{Z_1} V_1(s) \qquad T_V(s) = -Z_2/Z_1$$

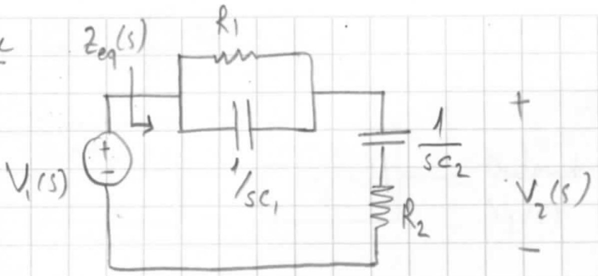
Driving point impedance $\frac{V_1(s)}{I_1(s)} = Z_1$

Non-inverting amplifier



$$V_1(s) = \frac{Z_1}{Z_1 + Z_2} V_2(s) \Rightarrow \frac{V_2(s)}{V_1(s)} = \frac{Z_1 + Z_2}{Z_2} = T_V(s)$$

example



$$Z_{eq}(s) = \frac{1}{sC_1 + \frac{1}{R_1}} + R_2 + \frac{1}{sC_2}$$

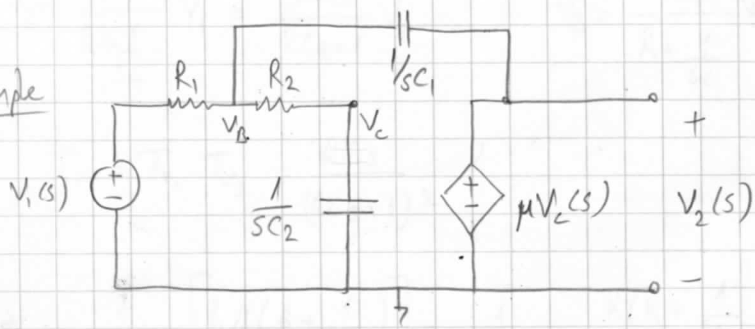
$$= \frac{R_1}{sR_1C_1 + 1} + \frac{sR_2C_2 + 1}{sC_2}$$

$$= \frac{sR_1C_1 + s^2R_1R_2C_1C_2 + 1 + sR_1C_1 + sR_2C_2}{sC_2(1 + sR_1C_1)}$$

Transfer function

$$T_v(s) = \frac{\frac{sR_2C_2 + 1}{sC_2}}{\frac{R_1}{1 + sR_1C_1} + \frac{sR_2C_2 + 1}{sC_2}}$$

example



$V_2 = \mu V_c$

$$T_v(s) = \frac{V_2(s)}{V_1(s)} = ?$$

Node B

$$\frac{V_1 - V_B}{R_1} + \frac{V_2 - V_B}{1/sC_1} + \frac{V_c - V_B}{R_2} = 0$$

Node C

$$\frac{V_c}{1/sC_2} + \frac{V_c - V_B}{R_2} = 0$$

$$\begin{bmatrix} -R_2 - R_1 - sR_1R_2C_1 & R_1 + \mu R_1R_2C_1 s \\ -1 & sC_2R_2 + 1 \end{bmatrix} \begin{bmatrix} V_B \\ V_c \end{bmatrix} = \begin{bmatrix} -R_2V_1 \\ 0 \end{bmatrix}$$

$$(R_1 + R_2 + sR_1R_2C_1)(sR_2C_2 + 1)V_c - (R_1 + s\mu R_1R_2C_1)V_c = R_2V_1$$

$$\frac{V_c}{V_1} = \frac{R_2}{(R_1 + R_2 + sR_1R_2C_1)(sR_2C_2 + 1) - (R_1 + s\mu R_1R_2C_1)}$$

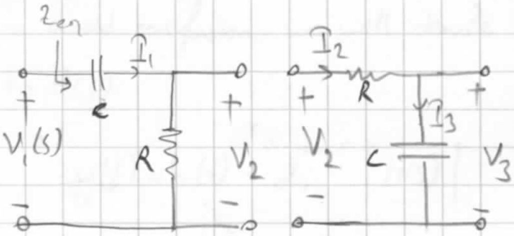
$$T_v(s) = \mu \frac{V_c}{V_1}$$

Cascade Connection

$V_1(s)$ Input $\longrightarrow \boxed{T_{v_1}(s)} \longrightarrow \boxed{T_{v_2}(s)} \longrightarrow$ Output $V_2(s)$

$$\frac{V_2(s)}{V_1(s)} = T_{v_1}(s)\, T_{v_2}(s) \;??$$

{True if there is no loading}



$$V_2 = \frac{R}{R+\frac{1}{sC}} V_1 = \frac{RCs}{RCs+1} V_1 \quad ; \quad V_3 = \frac{1/sC}{R+\frac{1}{sC}} V_2 = \frac{1}{RCs+1} V_2$$

$$T_v \overset{?}{=} T_{v_1} T_{v_2} = \frac{RCs}{(RCs+1)^2} \; ?$$

$$Z_{eq} = \frac{1}{sC} + \left[R \,//\, \left(R+\frac{1}{sC}\right)\right] = \frac{1}{sC} + \frac{R\left(R+\frac{1}{sC}\right)}{2R+\frac{1}{sC}} = \frac{1}{sC} + \frac{R(1+sRC)}{1+2sRC}$$

$$= \frac{s^2R^2C^2+3sRC+1}{sC(1+2sRC)}$$

$$I_1 = \frac{V_1}{Z_{eq}} \quad , \quad I_3 = \frac{R}{2R+\frac{1}{sC}} \frac{V_1}{Z_{eq}} = \frac{sRC}{1+2sRC} \cdot \frac{V_1}{Z_{eq}}$$

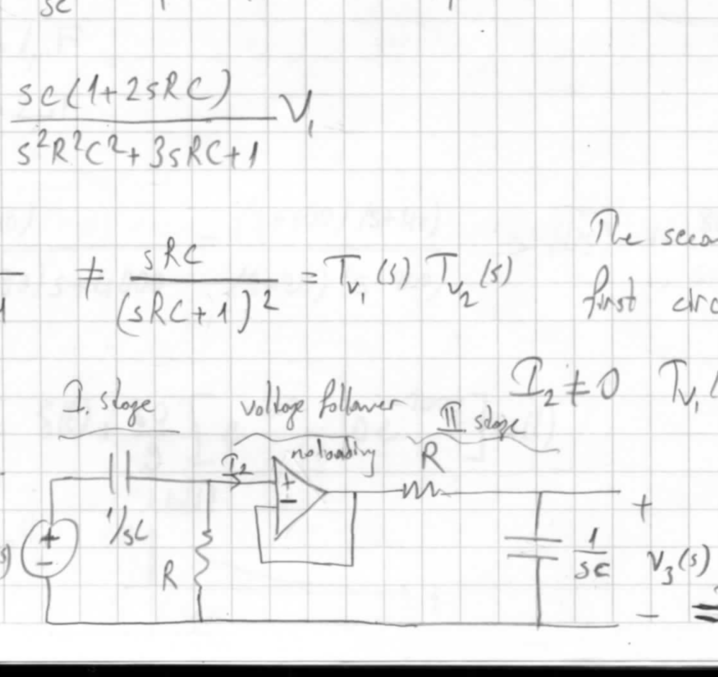
$$V_3 = \frac{1}{sC} I_3 = \frac{R}{1+2sRC} \cdot \frac{sC(1+2sRC)}{s^2R^2C^2+3sRC+1} V_1$$

$$T_v = \frac{sRC}{(sRC)^2+3sRC+1} \neq \frac{sRC}{(sRC+1)^2} = T_{v_1}(s)\, T_{v_2}(s)$$

The second circuit **loads** the first circuit.

$I_2 \neq 0$ $\quad T_{v_1}(s)$ assumes $I_2(s) = 0$

this method works for such circuits.



$I_2 = 0$

$V_3(s) = T_{v_1}(s)\, T_{v_2}(s)\, V_1(s)$

11.3 Network Functions and Impulse Response

$$x(t) = \delta(t) \Rightarrow X(s) = \mathcal{L}\{\delta(t)\} = 1 \quad \text{then} \quad Y(s) = T(s)X(s) = T(s)$$

{Impulse response Transform}

To avoid confusion we will denote impulse response transform as $H(s)$

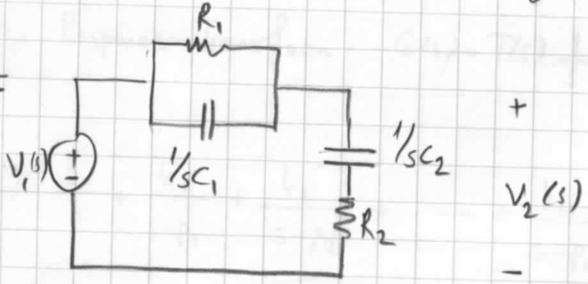
$$\underbrace{y(t) = h(t)}_{\text{impulse response}} = \mathcal{L}^{-1}\{H(s)\}$$

$$H(s) = \underbrace{\frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \cdots + \frac{k_N}{s-p_N}}_{\text{Natural Poles}}$$

$$h(t) = \left(k_1 e^{p_1 t} + k_2 e^{p_2 t} + \cdots + k_N e^{p_N t}\right) u(t)$$

for any input : $Y(s) = H(s)X(s) \longrightarrow y(t) = \mathcal{L}^{-1}\{H(s)X(s)\}$

example



$$\frac{V_2(s)}{V_1(s)} = T_v(s) = \frac{\left(s + \frac{1}{R_1C_1}\right)\left(s + \frac{1}{R_2C_2}\right)}{s^2 + \left(\frac{1}{R_2C_2} + \frac{1}{R_1C_1} + \frac{1}{R_2C_1}\right)s + \frac{1}{R_1C_1R_2C_2}}$$

let $R_1 = 10k\Omega \quad C_1 = 1\mu F$

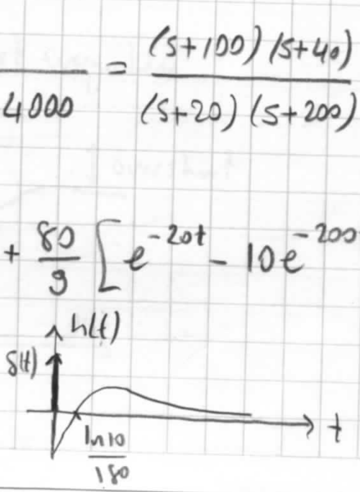
$R_2 = 12.5k\Omega \quad C_2 = 2\mu F$

$$T_v(s) = \frac{(s+100)(s+40)}{s^2 + (40+100+80)s + 4000} = \frac{(s+100)(s+40)}{(s+20)(s+200)} \Rightarrow H(s) = 1 + \frac{80/9}{s+20} - \frac{800/9}{s+200}$$

Impulse Response:

$$h(t) = \delta(t) + \frac{80}{9}\left[e^{-20t} - 10e^{-200t}\right]u(t)$$

asymptotically stable

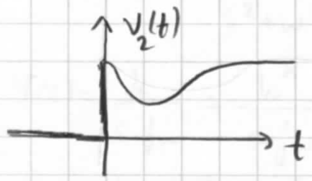


For a step input

$$V_2(s) = T_V(s)\frac{1}{s} = \frac{1}{s} + \frac{80/9}{s(s+20)} - \frac{800/9}{s(s+200)} = \frac{1}{s} + \frac{A}{s} + \frac{B}{s+20} + \frac{C}{s} + \frac{D}{s+200}$$

$$A = \frac{4}{9}, \quad B = -\frac{4}{9}, \quad C = -\frac{4}{9}, \quad D = \frac{4}{9}$$

$$V_2(s) = \frac{1}{s} - \frac{4/9}{s+20} + \frac{4/9}{s+200} \Rightarrow v_2(t) = u(t) - \frac{4}{9}e^{-20t}u(t) + \frac{4}{9}e^{-200t}u(t)$$



Step Response

zero-state response of the circuit output when the input is unit step.

Step Response Transform $G(s) = T(s)\frac{1}{s}$; step response waveform $g(t) = \mathcal{L}^{-1}\left\{\frac{T(s)}{s}\right\}$

$$G(s) = \underbrace{\frac{k_0}{s}}_{\text{Forced pole}} + \underbrace{\frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} + \dots + \frac{k_N}{s-p_N}}_{\text{Natural Poles.}} \qquad g(t) = k_0 u(t) + \left(\sum_{i=1}^{N} k_i e^{p_i t}\right)u(t)$$

When the circuit is stable $\Rightarrow$ natural response decay to zero

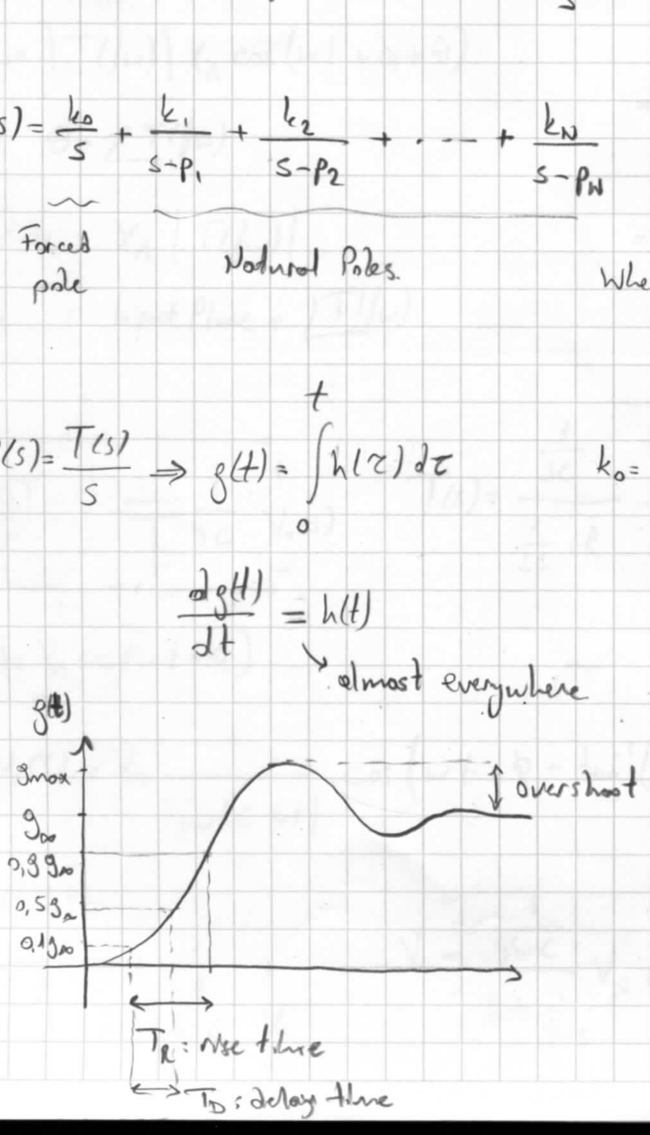
and

k_0 : DC steady-state response

$$G(s) = \frac{T(s)}{s} \Rightarrow g(t) = \int_0^t h(\tau)\,d\tau \qquad k_0 = sG(s)\Big|_{s=0} = T(0)$$

$$\frac{dg(t)}{dt} = h(t)$$

almost everywhere



11.5 Network Functions and Sinusoidal Steady-State Response

$$x(t) = X_A \cos(\omega t + \phi)$$

$$x(t) = X_A(\cos \omega t \cos\phi - \sin \omega t \sin\phi)$$

$$X(s) = X_A\left[\cos\phi \frac{s}{s^2+\omega^2} - \sin\phi \frac{\omega}{s^2+\omega^2}\right] = X_A \frac{s\cos\phi - \omega\sin\phi}{s^2+\omega^2}$$

$$Y(s) = X_A \frac{s\cos\phi - \omega \sin\phi}{s^2+\omega^2} T(s)$$

$$Y(s) = \underbrace{\frac{k}{s-j\omega} + \frac{k^*}{s+j\omega}}_{\text{forced poles}} + \underbrace{\sum_i \frac{k_i}{s-p_i}}_{\text{Natural resp.}}$$

- If the circuit is stable, natural resp. decays to zero

$y_{ss}(t) = k e^{j\omega t} + k^* e^{-j\omega t}$: steady-state response

$$k = (s-j\omega) X_A \frac{s\cos\phi - \omega\sin\phi}{s^2+\omega^2} T(s) \Big|_{s=j\omega}$$

$$= X_A \frac{j\omega\cos\phi - \omega\sin\phi}{j2\omega} T(j\omega)$$

$$= X_A \frac{\cos\phi + j\sin\phi}{2} T(j\omega) = \frac{1}{2} X_A e^{j\phi} T(j\omega)$$

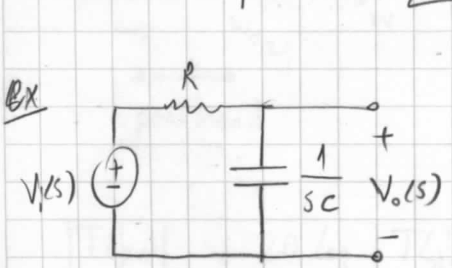
$$y_{ss}(t) = |T(j\omega)| X_A \cos(\omega t + \phi + \theta)$$

where $\theta = \angle T(j\omega)$

Amplitude $= X_A |T(j\omega)|$

Phase : Input Phase $+ \angle T(j\omega)$

Ex

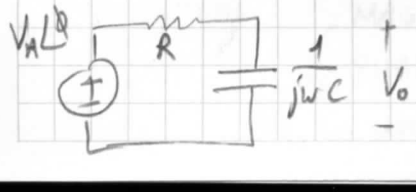


$$T(s) = \frac{\frac{1}{sC}}{\frac{1}{sC}+R} = \frac{1}{sRC+1}$$

$$v_1(t) = V_A \cos(\omega t + \phi)$$

$$V_o(s) = V_A \frac{1}{|j\omega RC+1|} \cos\left(\omega t + \phi - \tan^{-1}(RC\omega)\right)$$

using phasors



$$V_o = \frac{\frac{1}{j\omega C}}{R+\frac{1}{j\omega C}} V_A \angle\phi = \frac{V_A}{j\omega RC+1} \angle\phi = \frac{V_A}{|j\omega RC+1|} \angle \phi - \tan^{-1}(RC\omega)$$