DIFFERENTIATION

The Derivative as a Function

derivative, measures the rate at which a function changes

- we defined the slope of a curve
- one of the most important ideas in calculus
- to calculate velocity and acceleration,
- to estimate the rate of spread of a disease,
- to set levels of production so as to maximize efficiency,
- to find the best dimensions of a cylindrical can,
- to find the age of a prehistoric artifact,

and for many other applications.

The Derivative as a Function

DEFINITION **Derivative Function**

The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided the limit exists.

Calculating Derivatives from the Definition

- The process of calculating a derivative is called **differentiation**.

$$\frac{d}{dx}f(x)$$

$$\frac{d}{dx}(mx + b) = m$$

Using the definition, calculate the derivatives of the functions Then find the values of the derivatives as specified.

$$f(x) = 4 - x^2; \quad f'(-3), f'(0), f'(1)$$

$f(x) = 4 - x^2$ and $f(x + h) = 4 - (x + h)^2$

$$\frac{f(x+h) - f(x)}{h} = \frac{[4 - (x+h)^2] - (4 - x^2)}{h} = \frac{(4 - x^2 - 2xh - h^2) - 4 + x^2}{h} = \frac{-2xh - h^2}{h}$$

$$= -2x - h$$

$$f'(x) = \lim_{h \to 0} (-2x - h) = -2x; \ f'(-3) = 6, \ f'(0) = 0, \ f'(1) = -2$$

find the indicated derivatives.

$$\frac{dy}{dx} \quad \text{if} \quad y = 2x^3$$

$$y = f(x) = 2x^3 \text{ and } f(x+h) = 2(x+h)^3$$

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{2(x+h)^3 - 2x^3}{h} = \lim_{h \to 0} \frac{2(x^3 + 3x^2h + 3xh^2 + h^3) - 2x^3}{h}$$

$$= \lim_{h \to 0} \frac{6x^2h + 6xh^2 + 2h^3}{h} = \lim_{h \to 0} \frac{h(6x^2 + 6xh + 2h^2)}{h}$$

$$= \lim_{h \to 0} (6x^2 + 6xh + 2h^2) = 6x^2$$

Notations

There are many ways to denote the derivative of a function $y = f(x)$, where the independent variable is x and the dependent variable is y. Some common alternative notations for the derivative are

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_x f(x)$$

The symbols d/dx and D indicate the operation of differentiation and are called **differentiation operators**. We read dy/dx as "the derivative of y with respect to x," and df/dx and $(d/dx)f(x)$ as "the derivative of f with respect to x." The "prime" notations y' and f' come from notations that Newton used for derivatives. The d/dx notations are similar to those used by Leibniz. The symbol dy/dx should not be regarded as a ratio

To indicate the value of a derivative at a specified number $x = a$, we use the notation

$$f'(a) = \frac{dy}{dx}\bigg|_{x=a} = \frac{df}{dx}\bigg|_{x=a} = \frac{d}{dx}f(x)\bigg|_{x=a}.$$

$$f'(4) = \frac{d}{dx}\sqrt{x}\bigg|_{x=4} = \frac{1}{2\sqrt{x}}\bigg|_{x=4} = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

To evaluate an expression, we sometimes use the right bracket] in place of the vertical bar | .

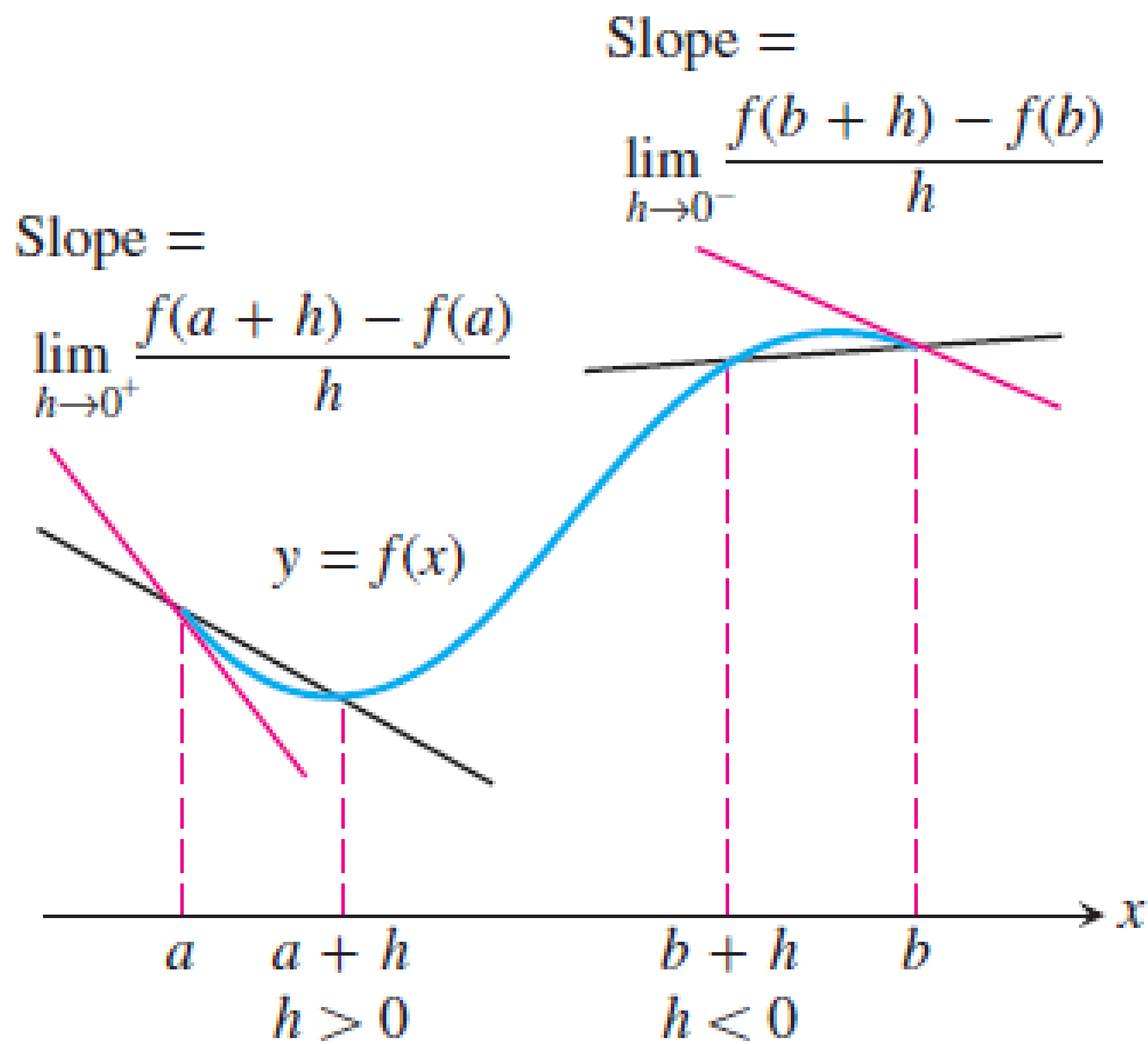
Differentiable on an Interval; One-Sided Derivatives

A function $y = f(x)$ is **differentiable** on an open interval (finite or infinite) if it has a derivative at each point of the interval. It is differentiable on a closed interval $[a, b]$ if it is differentiable on the interior (a, b) and if the limits

$$\lim_{h \to 0^+} \frac{f(a + h) - f(a)}{h} \qquad \textbf{Right-hand derivative at } a$$

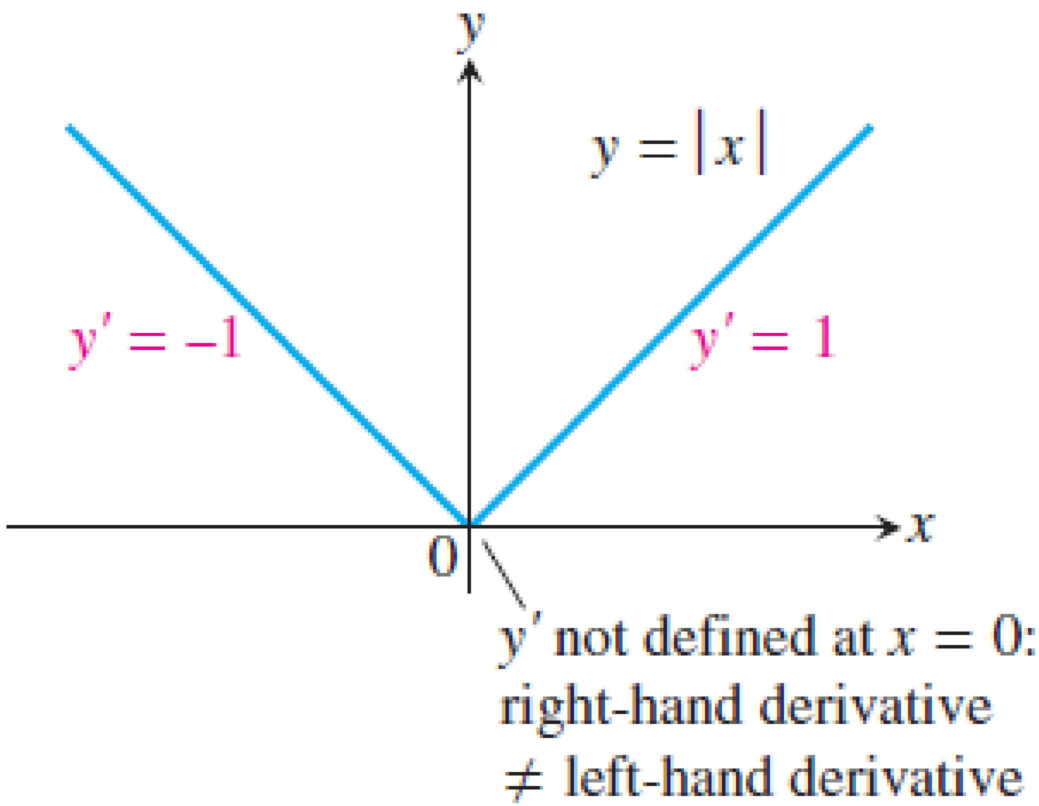
$$\lim_{h \to 0^-} \frac{f(b + h) - f(b)}{h} \qquad \textbf{Left-hand derivative at } b$$

exist at the endpoints



Derivatives at endpoints are one-sided limits.

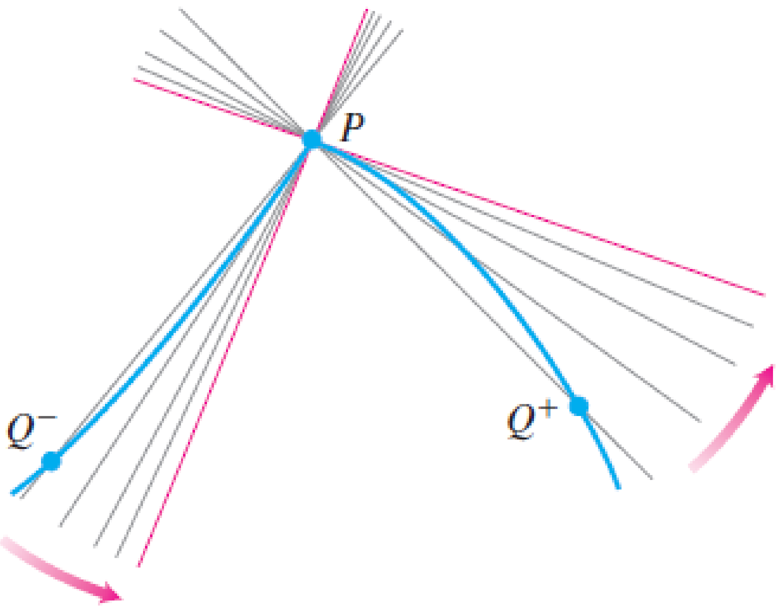
$y = |x|$ Is Not Differentiable at the Origin



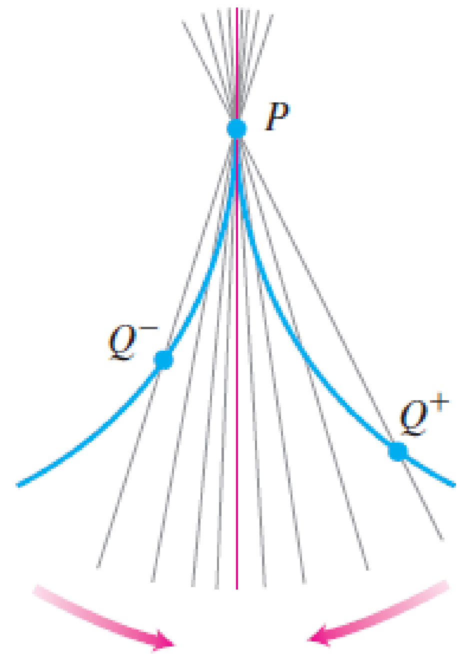
The function $y = |x|$ is not differentiable at the origin where the graph has a "corner."

When Does a Function *Not* Have a Derivative at a Point?

1. a *corner*, where the one-sided derivatives differ.

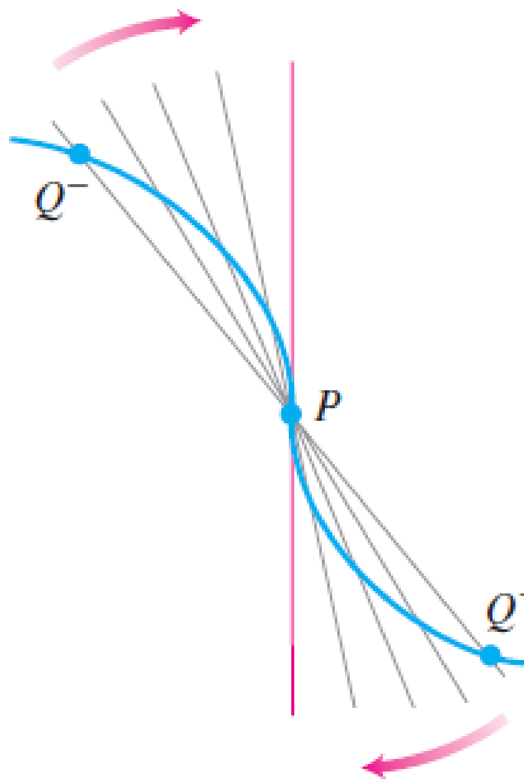


2. a *cusp*, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other.



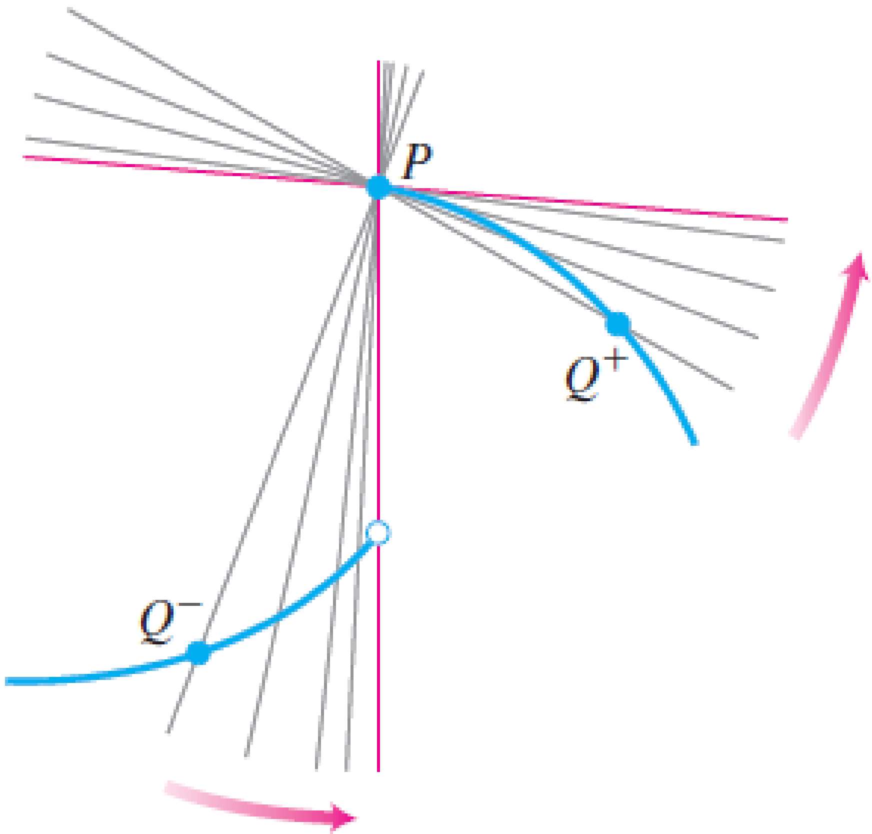
When Does a Function *Not* Have a Derivative at a Point?

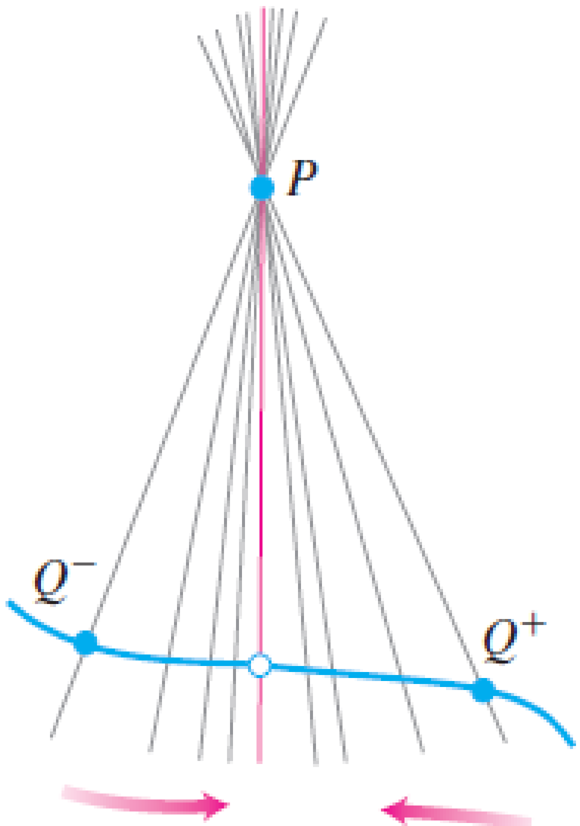
3. a *vertical tangent*, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$).



When Does a Function *Not* Have a Derivative at a Point?

4. a *discontinuity.*





Differentiable Functions Are Continuous

A function is continuous at every point where it has a derivative.

THEOREM 1 Differentiability Implies Continuity
If f has a derivative at $x = c$, then f is continuous at $x = c$.

CAUTION The converse of Theorem 1 is false. A function need not have a derivative at a point where it is continuous

The Intermediate Value Property of Derivatives

Not every function can be some function's derivative, as we see from the following theorem.

If a and b are any two points in an interval on which f is differentiable, then f' takes on every value between $f'(a)$ and $f'(b)$.

differentiate the function and find the slope of the tangent line at the given value of the independent variable.

$$y = (x + 1)^3, \quad x = -2$$

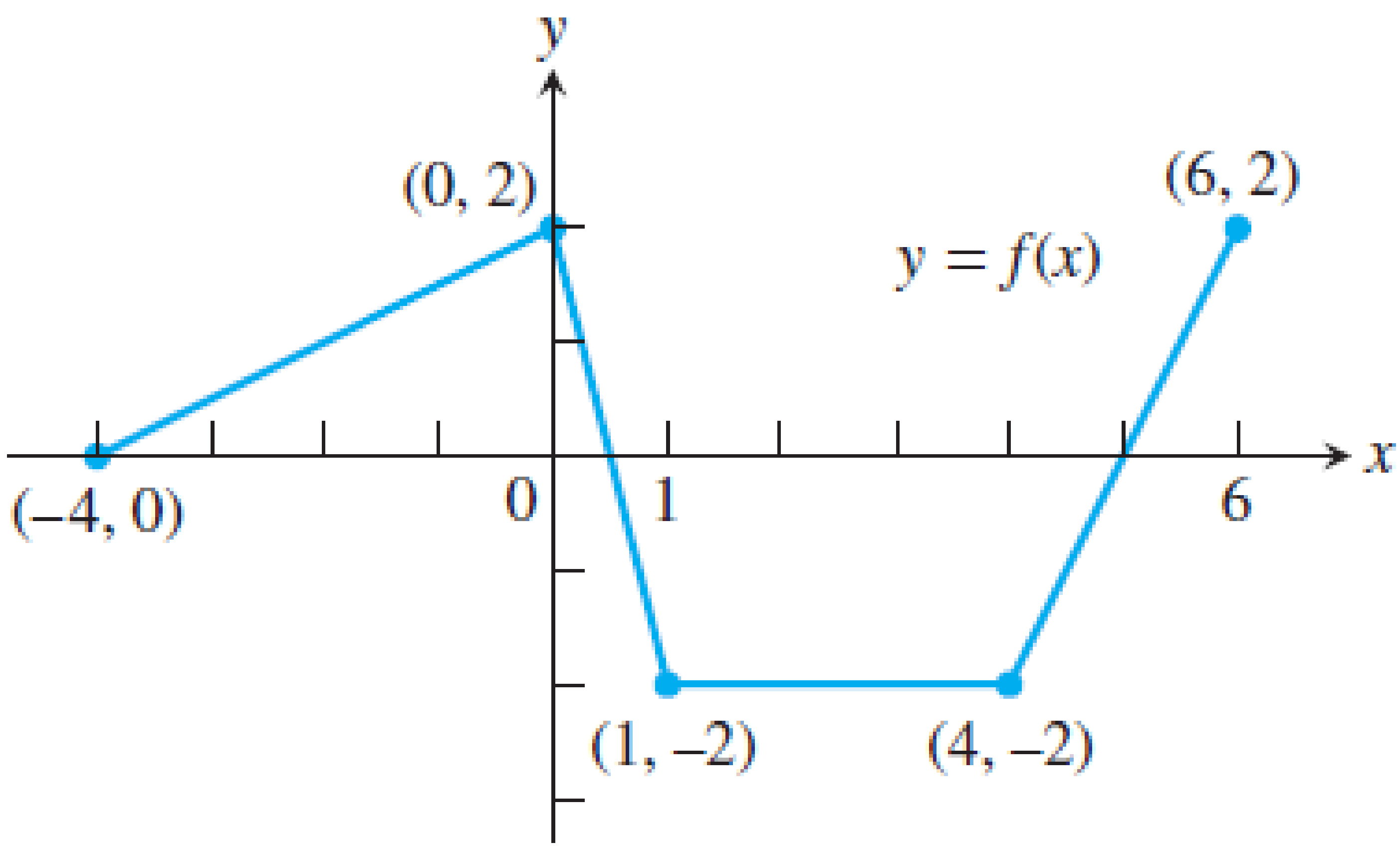
$$\frac{dy}{dx} = \lim_{h \to 0} \frac{(x + h + 1)^3 - (x + 1)^3}{h}$$

$$= \lim_{h \to 0} \frac{(x + 1)^3 + 3(x + 1)^2 h + 3(x + 1)h^2 + h^3 - (x + 1)^3}{h}$$

$$= \lim_{h \to 0} [3(x + 1)^2 + 3(x + 1)h + h^2]$$

$$= 3(x + 1)^2; \; m = \left.\frac{dy}{dx}\right|_{x=-2} = 3$$

The graph in the accompanying figure is made of line segments joined end to end. At which points of the interval $[-4, 6]$ is f' not defined? Give reasons for your answer.



f' is not defined at $x = 0, 1, 4$. At these points, the left-hand and right-hand derivatives do not agree.

Differentiation Rules

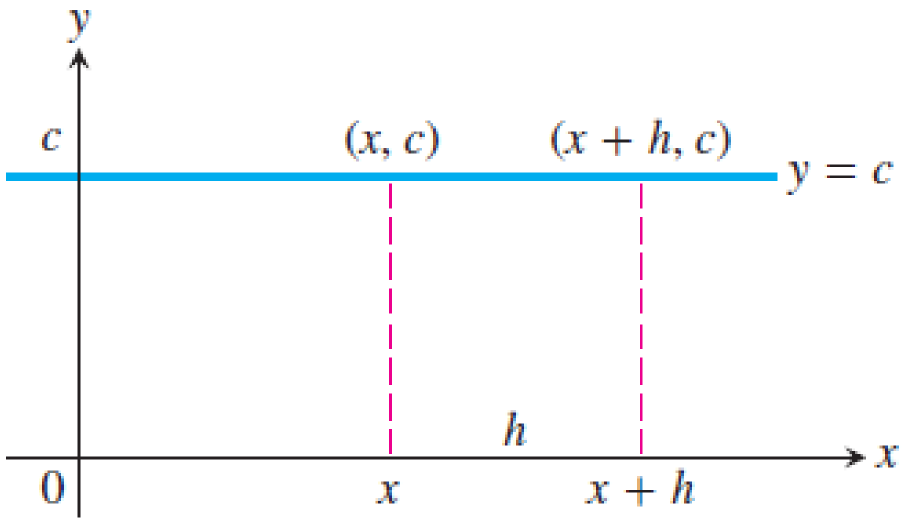
Powers, Multiples, Sums, and Differences

The first rule of differentiation is that the derivative of every constant function is zero.

RULE 1 Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$



The second rule tells how to differentiate x^n if n is a positive integer.

RULE 2 Power Rule for Positive Integers

If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Interpreting Rule 2

f	x	x^2	x^3	x^4	$\cdots$
f'	1	$2x$	$3x^2$	$4x^3$	$\cdots$

RULE 3 Constant Multiple Rule

If u is a differentiable function of x, and c is a constant,

$$\frac{d}{dx}(cu) = c\frac{du}{dx}$$

$$\frac{d}{dx}(3x^2) = 3 \cdot 2x = 6x$$

RULE 4 Derivative Sum Rule

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}$$

Derivative of a Sum

$$y = x^4 + 12x$$

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) + \frac{d}{dx}(12x)$$

$$= 4x^3 + 12$$

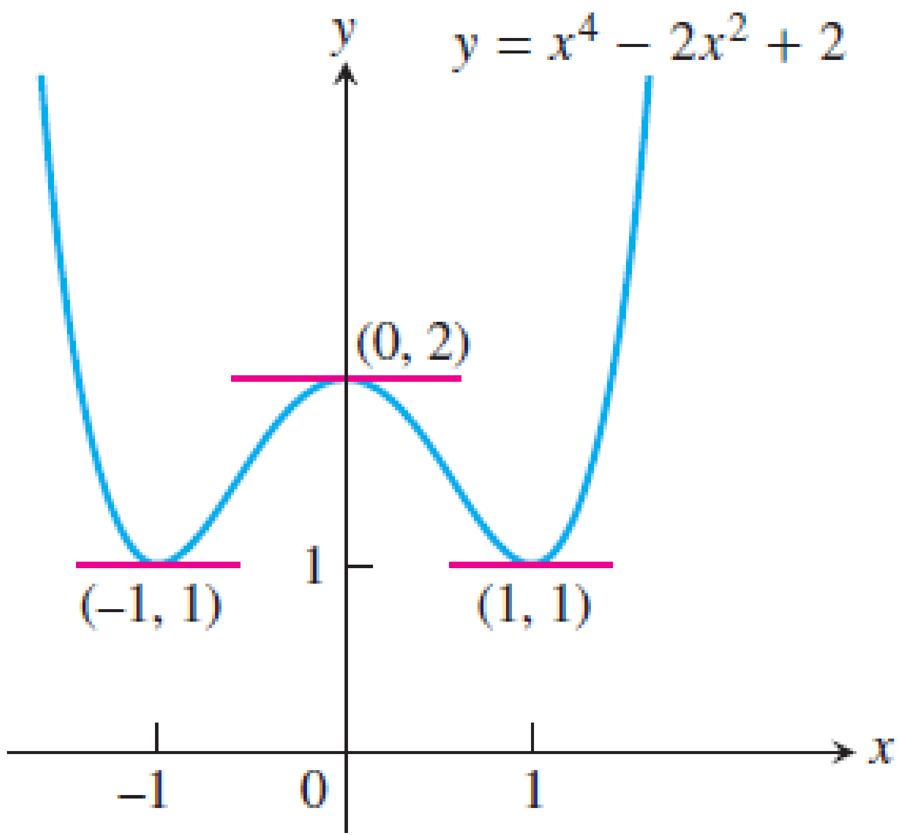
Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

horizontal tangents, if any, occur where the slope dy/dx is zero.

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 - 2x^2 + 2) = 4x^3 - 4x.$$

$\frac{dy}{dx} = 0$ for x:

$$4x^3 - 4x = 0$$
$$4x(x^2 - 1) = 0$$
$$x = 0, 1, -1.$$



RULE 5 Derivative Product Rule

If u and v are differentiable at x, then so is their product uv,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

In *prime notation*, $(uv)' = uv' + vu'$

In function notation,

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x).$$

Which way is easier?

From the Product Rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$\begin{aligned}\frac{d}{dx}\left[\left(x^2+1\right)\left(x^3+3\right)\right] &= (x^2+1)(3x^2) + (x^3+3)(2x) \\ &= 3x^4 + 3x^2 + 2x^4 + 6x \\ &= 5x^4 + 3x^2 + 6x.\end{aligned}$$

or

$$y = (x^2+1)(x^3+3) = x^5 + x^3 + 3x^2 + 3$$

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x.$$

RULE 6 Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

$$f(t) = \frac{t^2 - 1}{t^2 + t - 2} = \frac{(t-1)(t+1)}{(t+2)(t-1)} = \frac{t+1}{t+2},\ t \neq 1 \Rightarrow$$

$$f'(t) = \frac{(t+2)(1) - (t+1)(1)}{(t+2)^2} = \frac{t+2-t-1}{(t+2)^2} = \frac{1}{(t+2)^2}$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = (-1)x^{-2} = -\frac{1}{x^2}$$

$$\frac{d}{dx}\left(\frac{4}{x^3}\right) = 4\frac{d}{dx}(x^{-3}) = 4(-3)x^{-4} = -\frac{12}{x^4}$$

Second- and Higher-Order Derivatives

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x{}^2 f(x).$$

If $y = x^6$, then $y' = 6x^5$ and we have

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\left(6x^5\right) = 30x^4$$

Finding Higher Derivatives

The first four derivatives of $y = x^3 - 3x^2 + 2$ are

First derivative: $y' = 3x^2 - 6x$

Second derivative: $y'' = 6x - 6$

Third derivative: $y''' = 6$

Fourth derivative: $y^{(4)} = 0.$

the fifth and later derivatives all being zero.

find the first and second derivatives.

$$r = \frac{1}{3s^2} - \frac{5}{2s}$$

$$r = \frac{1}{3} s^{-2} - \frac{5}{2} s^{-1} \Rightarrow$$

$$\frac{dr}{ds} = -\frac{2}{3} s^{-3} + \frac{5}{2} s^{-2} = \frac{-2}{3s^3} + \frac{5}{2s^2}$$

$$\frac{d^2r}{ds^2} = 2s^{-4} - 5s^{-3} = \frac{2}{s^4} - \frac{5}{s^3}$$

Which way is easier?

(a) by applying the Product Rule and

(b) by multiplying the factors to produce a sum of simpler terms to differentiate.

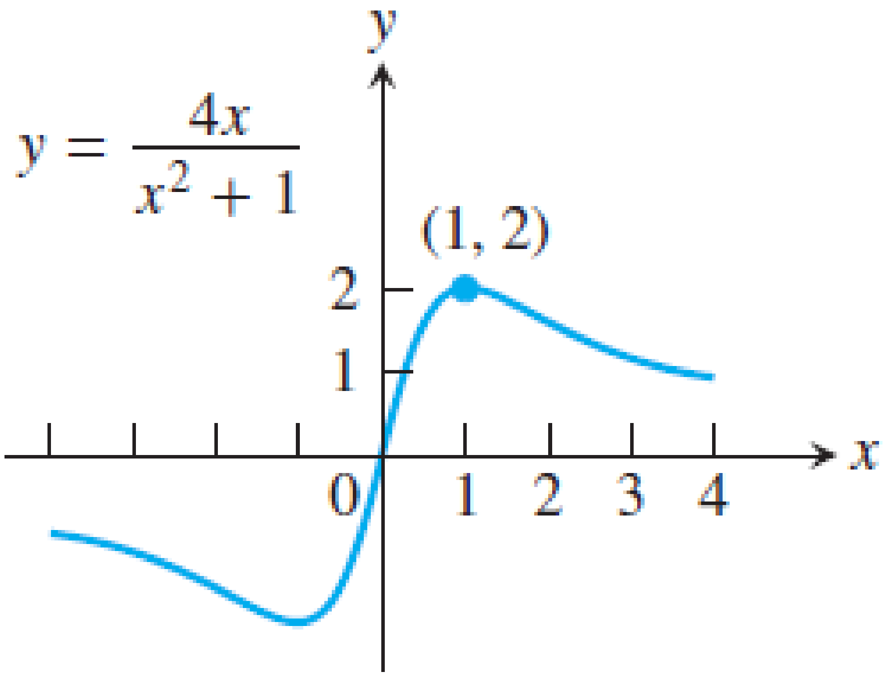
$$y = (3 - x^2)(x^3 - x + 1)$$

$$y = (x^2 + 1)\left(x + 5 + \frac{1}{x}\right)$$

a. Horizontal tangents Find equations for the horizontal tangents to the curve $y = x^3 - 3x - 2$. Also find equations for the lines that are perpendicular to these tangents at the points of tangency.

b. Smallest slope What is the smallest slope on the curve? At what point on the curve does the curve have this slope? Find an equation for the line that is perpendicular to the curve's tangent at this point.

Find the tangents to *Newton's serpentine* (graphed here) at the origin and the point (1, 2).



The body's reaction to medicine The reaction of the body to a dose of medicine can sometimes be represented by an equation of the form

$$R = M^2\left(\frac{C}{2} - \frac{M}{3}\right),$$

where C is a positive constant and M is the amount of medicine absorbed in the blood. If the reaction is a change in blood pressure, R is measured in millimeters of mercury. If the reaction is a change in temperature, R is measured in degrees, and so on.

Find dR/dM. This derivative, as a function of M, is called the sensitivity of the body to the medicine.

Cylinder pressure If gas in a cylinder is maintained at a constant temperature T, the pressure P is related to the volume V by a formula of the form

$$P = \frac{nRT}{V - nb} - \frac{an^2}{V^2},$$

in which a, b, n, and R are constants. Find dP/dV.