APPLICATIONS OF DERIVATIVES

- how derivatives are used to find extreme values of functions,
- to determine and analyze the shapes of graphs,
- to calculate limits of fractions whose numerators and denominators both approach zero or infinity,

Extreme Values of Functions

- how to locate and identify extreme values of a continuous function from its derivative.

- solve a variety of *optimization problems* in which we find the optimal (best) way to do something in a given situation

- Absolute maximum and minimum values are called absolute **extrema** (plural of the Latin *extremum*).

- Absolute extrema are also called **global** extrema, to distinguish them from *local extrema*

DEFINITIONS **Absolute Maximum, Absolute Minimum**

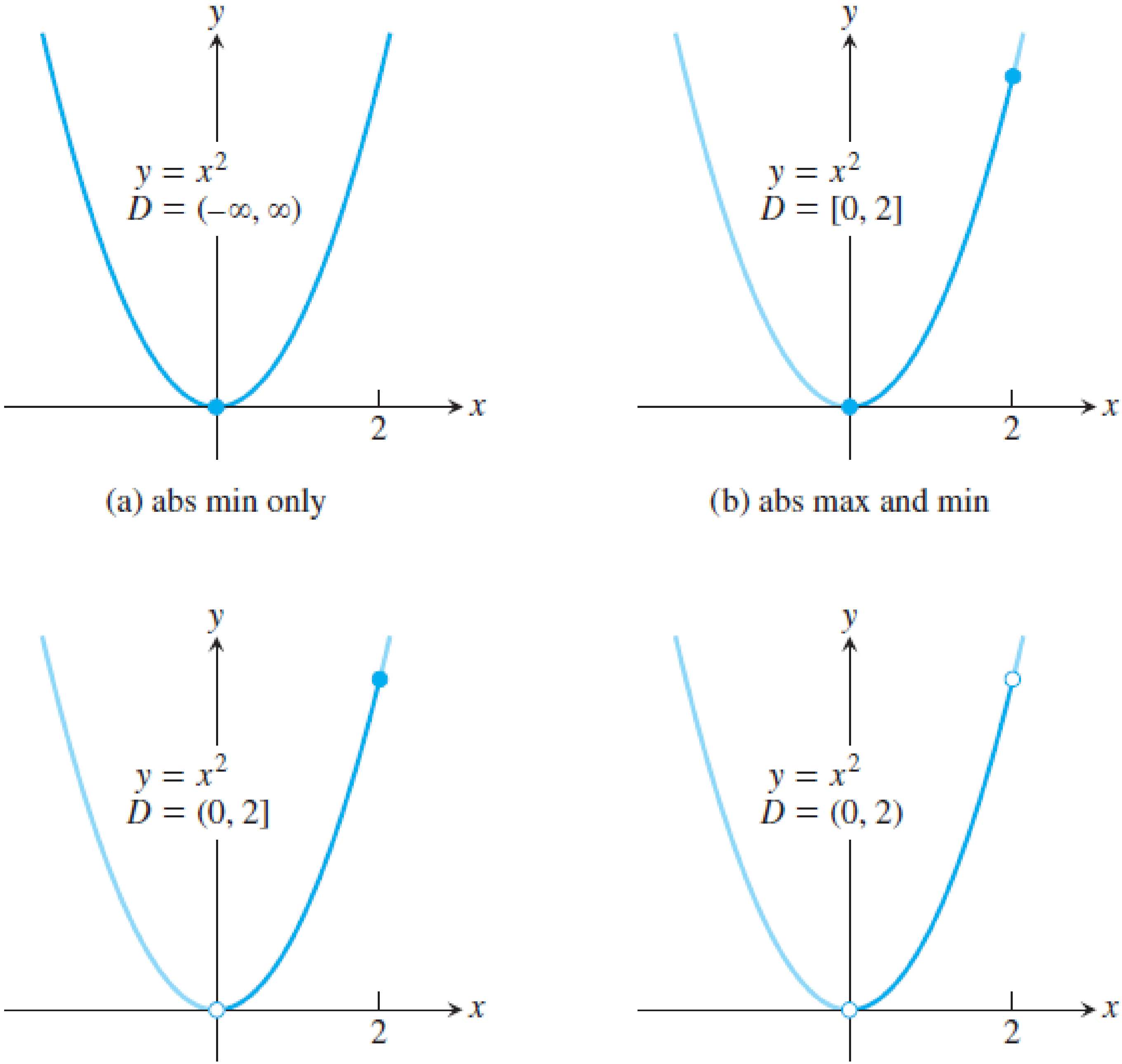
Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

Functions with the same defining rule can have different extrema, depending on the domain.

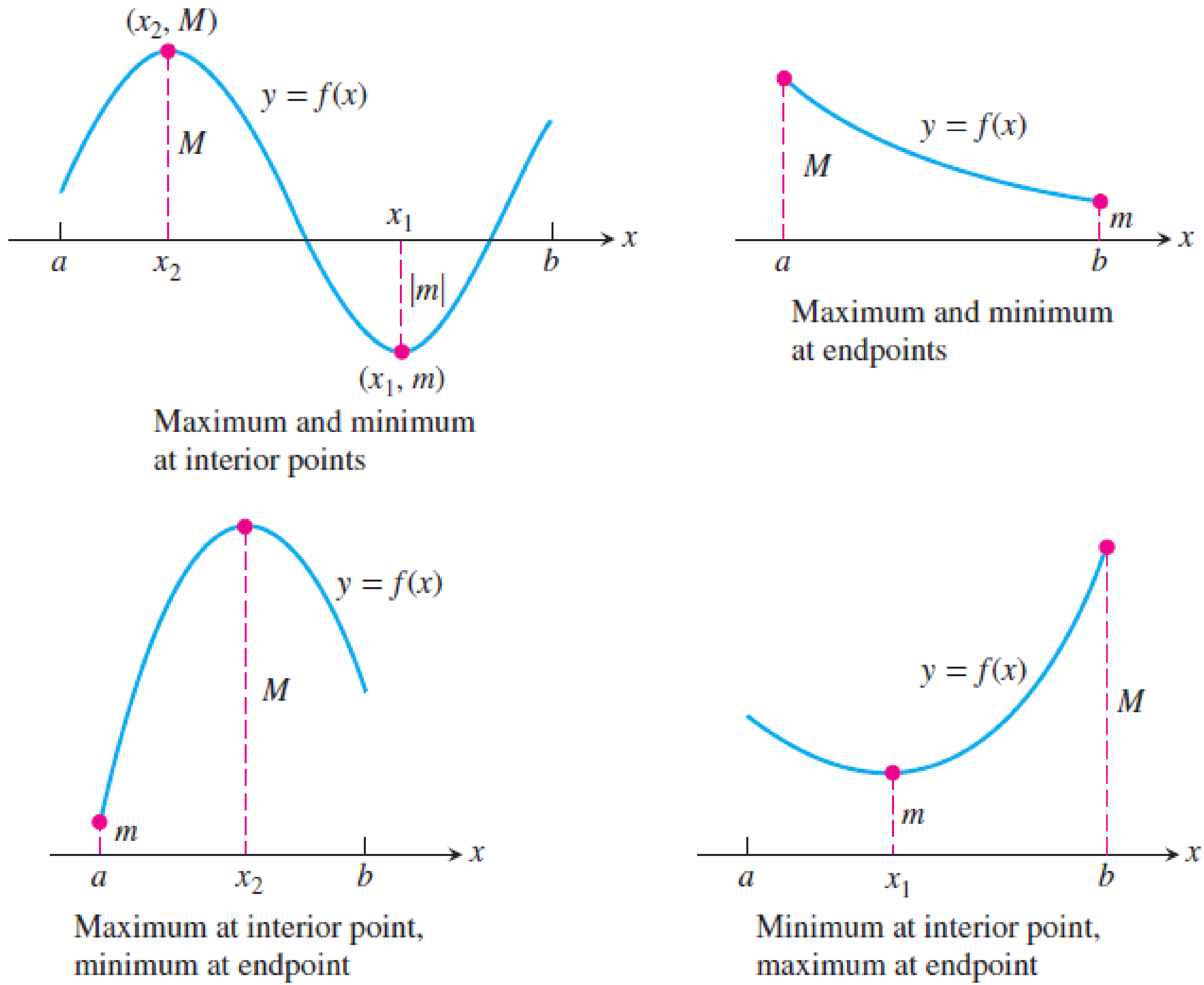


(a) abs min only

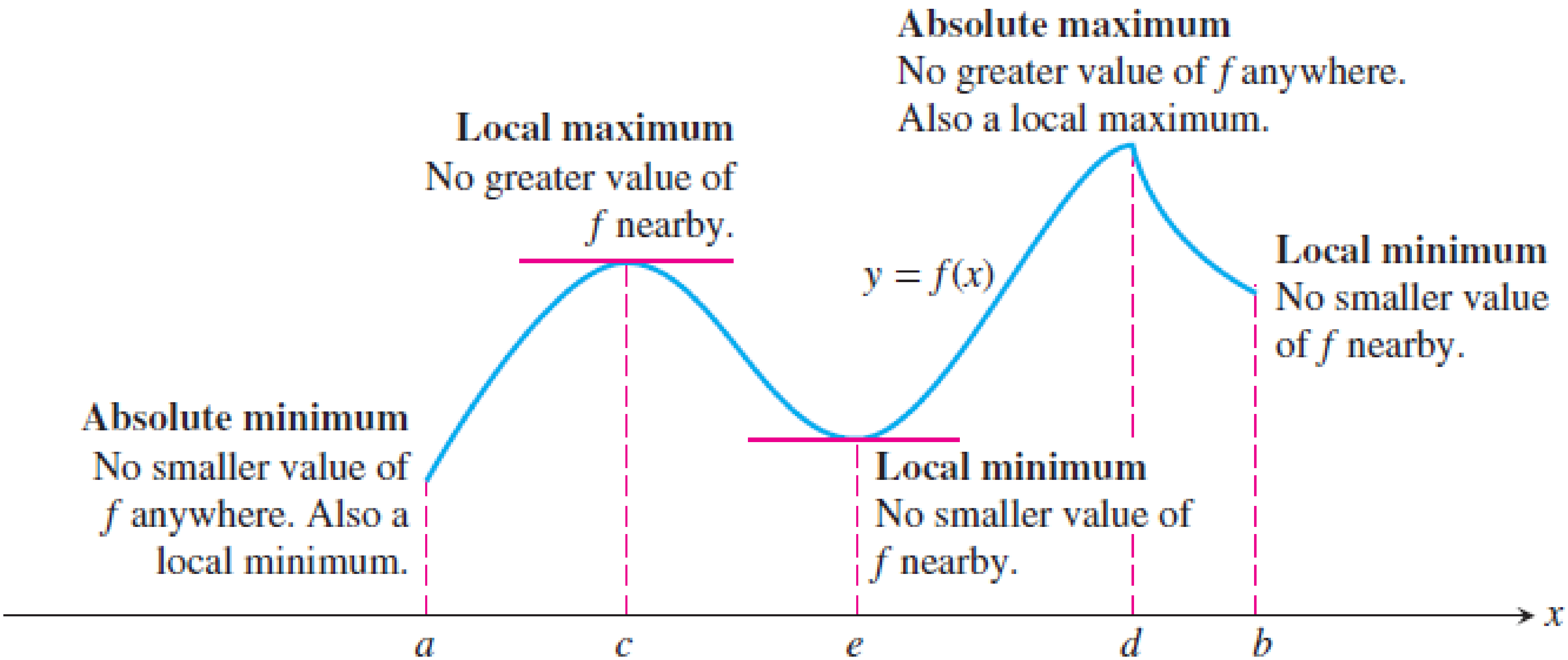
(b) abs max and min

(c) abs max only

(d) no max or min



Some possibilities for a continuous function's maximum and minimum on a closed interval $[a, b]$.



How to classify maxima and minima.

THEOREM 1 The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$

THEOREM 2 The First Derivative Theorem for Local Extreme Values

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$

places where a function f can possibly have an extreme value (local or global) are

1. interior points where $f' = 0$,

2. interior points where f' is undefined,

3. endpoints of the domain of f.

DEFINITION Critical Point

An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

How to Find the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.

EXTREMA OF A FUNCTION ON A CLOSED INTERVAL: POLYNOMIAL

Find the extrema of $f(x) = 5x^4 - 4x^3$ on the interval $[-1,2]$.

1. Find $f'(x)$.

$$f(x) = 5x^4 - 4x^3$$
$$f'(x) = 20x^3 - 12x^2$$
$$f'(x) = 4x^2(5x - 3)$$

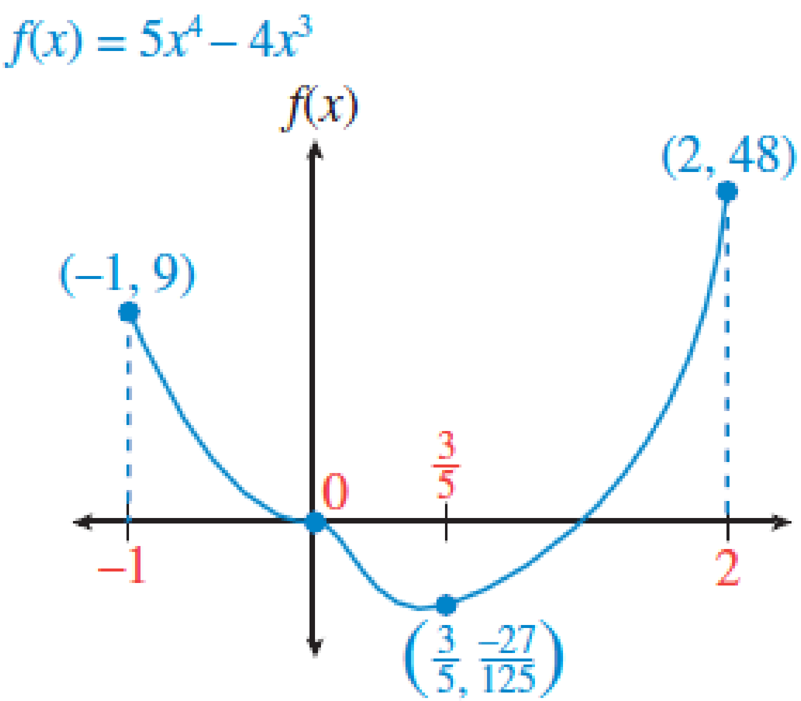
2. Set $f'(x) = 0$ and solve for x—these are the critical numbers.

$$0 = 4x^2(5x - 3)$$
$$x = 0 \qquad x = \frac{3}{5}$$

❸ Evaluate $f(x)$ at each endpoint of the interval and at each critical number.

left endpoint	critical number	critical number	right endpoint
$f(-1) = 9$	$f(0) = 0$	$f\left(\frac{3}{5}\right) = -\frac{27}{125}$	$f(2) = 48$
		minimum	maximum

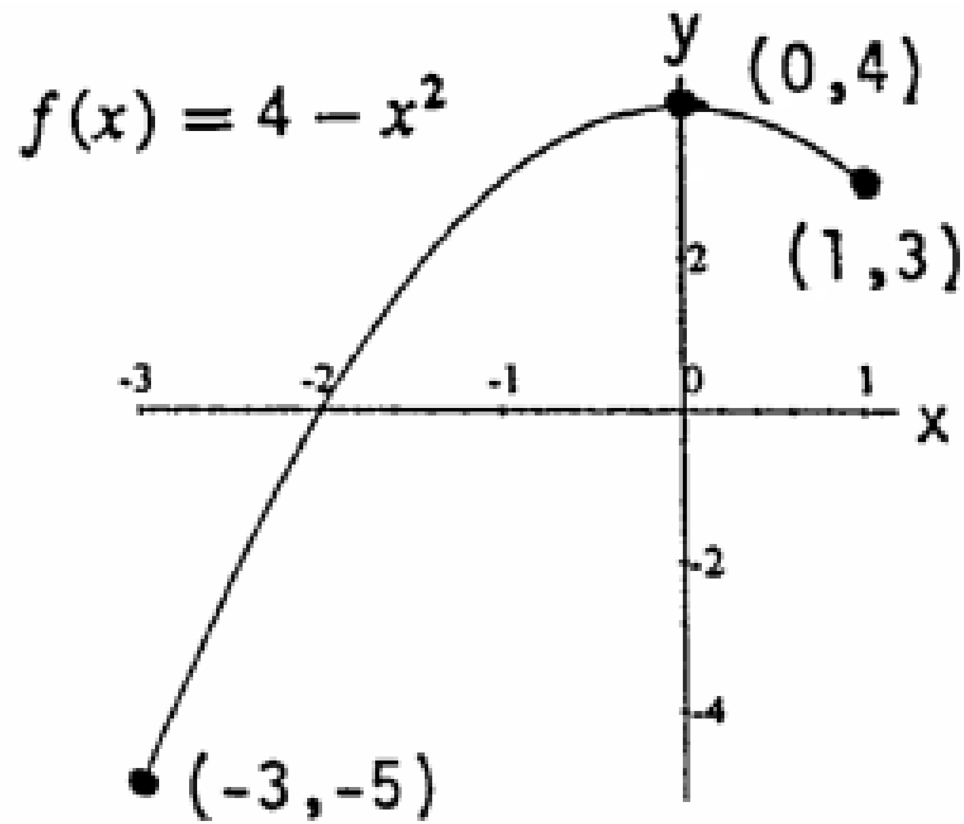
❹ State the maximum and minimum values of $f(x)$ in the interval.



Find the absolute maximum and minimum values of the function on the given interval. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

$$f(x) = 4 - x^2, \quad -3 \leq x \leq 1$$

$f(x) = 4 - x^2 \Rightarrow f'(x) = -2x \Rightarrow$ a critical point at $x = 0$; $f(-3) = -5$, $f(0) = 4$, $f(1) = 3 \Rightarrow$ the absolute maximum is 4 at $x = 0$ and the absolute minimum is -5 at $x = -3$



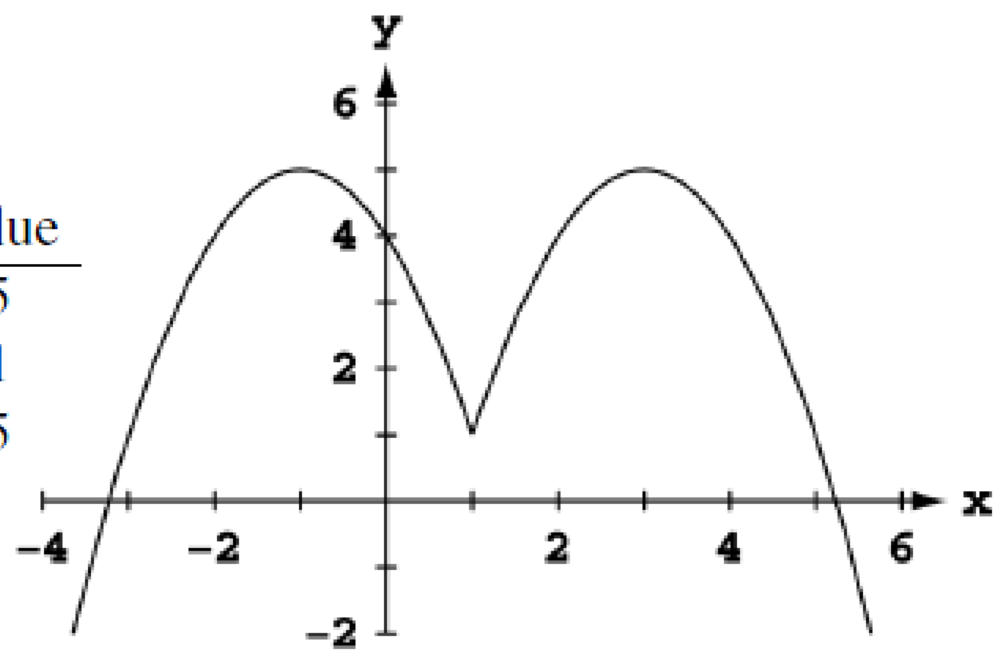
$f(x) = 4 - x^2$
y
(0,4)
(1,3)
2
-3
-2
-1
0
1
x
-2
-4
(-3,-5)

find the derivative at each critical point and determine the local extreme values

$$y = \begin{cases} -x^2 - 2x + 4, & x \le 1 \\ -x^2 + 6x - 4, & x > 1 \end{cases}$$

$$y' = \begin{cases} -2x - 2, & x < 1 \\ -2x + 6, & x > 1 \end{cases}$$

crit. pt.	derivative	extremum	value
$x = -1$	0	maximum	5
$x = 1$	undefined	local min	1
$x = 3$	0	maximum	5



[–4, 6] by [–2, 6]

First Derivative Test for Monotonic Functions

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

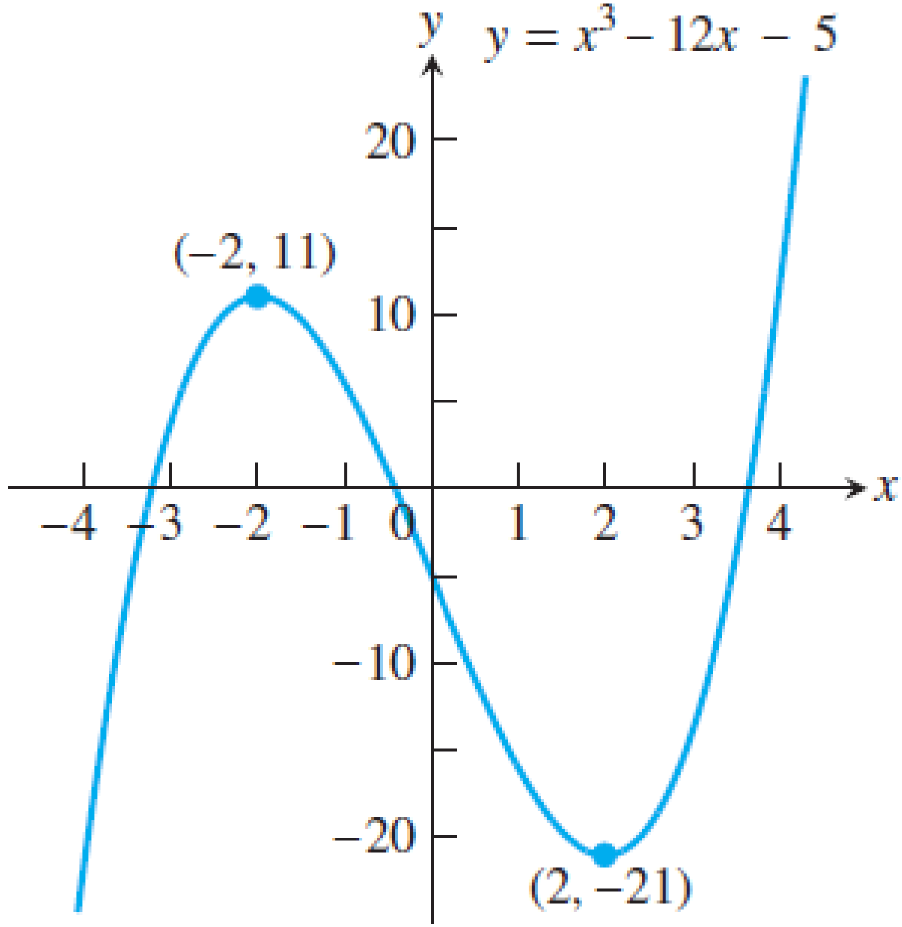
If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.

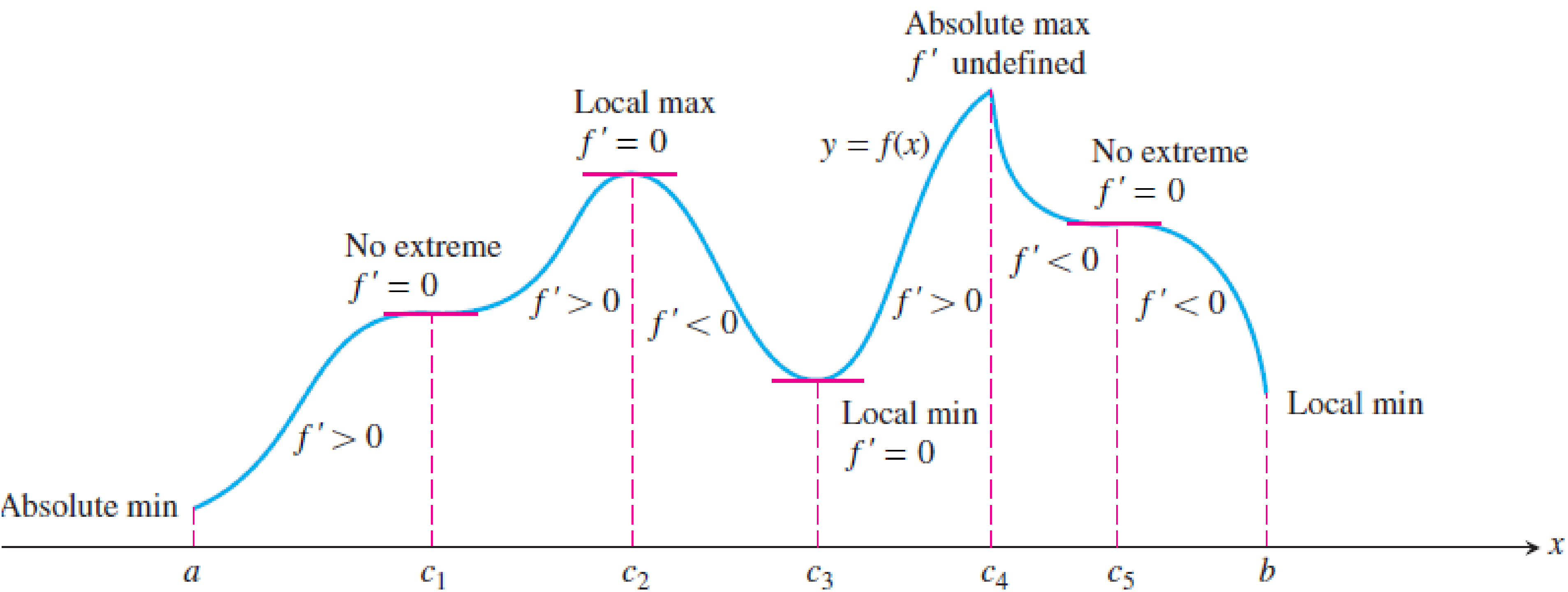
If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$

Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals increasing and decreasing.

$$\begin{aligned} f'(x) &= 3x^2 - 12 = 3(x^2 - 4) \\ &= 3(x + 2)(x - 2) \end{aligned}$$

Intervals	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
f' Evaluated	$f'(-3) = 15$	$f'(0) = -12$	$f'(3) = 15$
Sign of f'	$+$	$-$	$+$
Behavior of f	increasing	decreasing	increasing




Absolute max
f ′ undefined
Local max
f ′ = 0
y = f(x)
No extreme
f ′ = 0
No extreme
f ′ = 0
f ′ > 0
f ′ < 0
f ′ > 0
f ′ < 0
f ′ < 0
Local min
f ′ = 0
Local min
f ′ > 0
Absolute min
a
c1
c2
c3
c4
c5
b
x

First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across c from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c;
2. if f' changes from positive to negative at c, then f has a local maximum at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c.

a. Find the intervals on which the function is increasing and decreasing.

b. Then identify the function's local extreme values, if any, saying where they are taken on.

c. Which, if any, of the extreme values are absolute?

d. Support your findings with a graphing calculator or computer grapher.

$$h(x) = 2x^3 - 18x$$

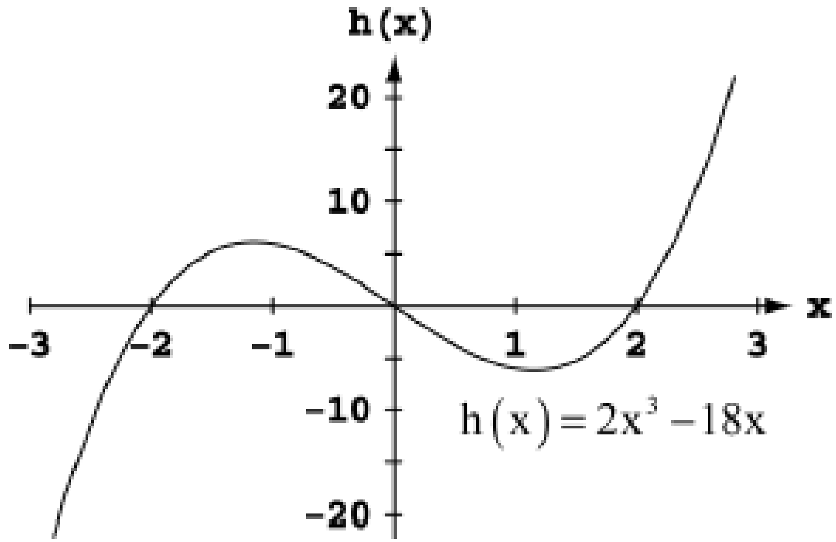
(a) $h(x) = 2x^3 - 18x \Rightarrow h'(x) = 6x^2 - 18 = 6\left(x+\sqrt{3}\right)\left(x-\sqrt{3}\right) \Rightarrow$ critical p

$$\Rightarrow h' = \underset{}{+++} \underset{-\sqrt{3}}{|} --- \underset{\sqrt{3}}{|} +++$$, increasing on $\left(-\infty, -\sqrt{3}\right)$ and $\left(\sqrt{3}, \infty\right)$,

(b) a local maximum is $h\left(-\sqrt{3}\right) = 12\sqrt{3}$ at $x = -\sqrt{3}$; local minimum is $h\left(\sqrt{3}\right)$

(c) no absolute extrema

(d)



a. Find the intervals on which the function is increasing and decreasing.

b. Then identify the function's local extreme values, if any, saying where they are taken on.

c. Which, if any, of the extreme values are absolute?

d. Support your findings with a graphing calculator or computer grapher.

$$f(x) = x^4 - 8x^2 + 16$$

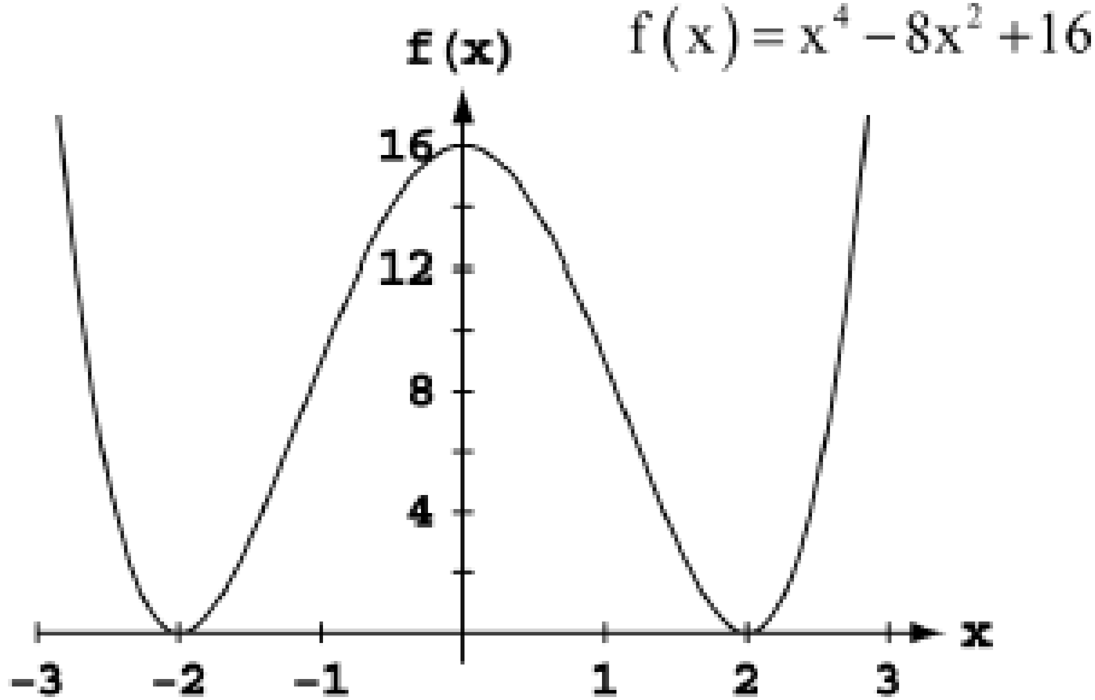
(a) $f(x) = x^4 - 8x^2 + 16 \Rightarrow f'(x) = 4x^3 - 1$

$\Rightarrow f' = \underset{-2}{---|}\underset{0}{+++|}\underset{2}{---|}+++$, i

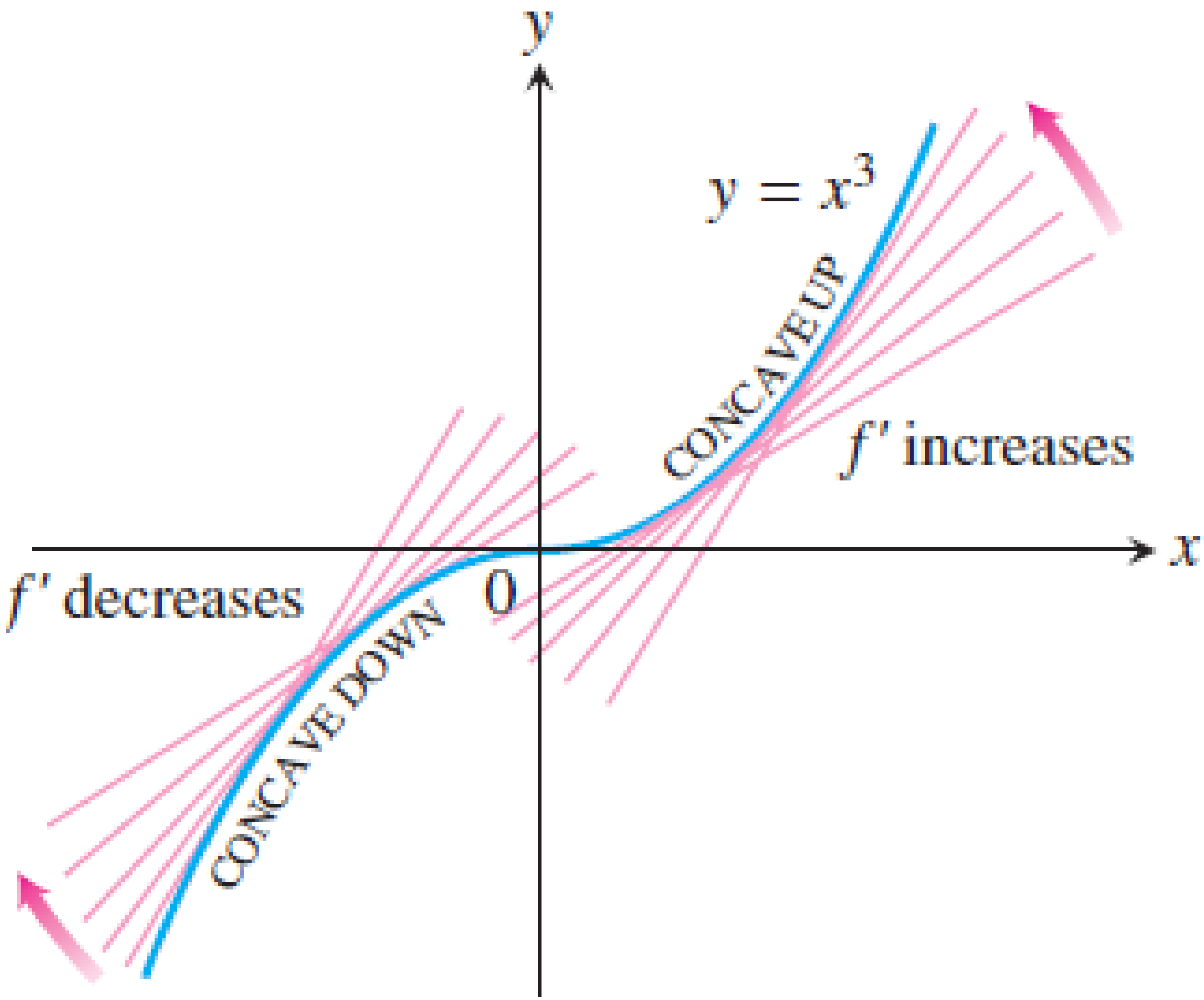
(b) a local maximum is $f(0) = 16$ at $x = 0$, lo

(c) no absolute maximum; absolute minimum

(d)



Concavity and Curve Sketching



DEFINITION Concave Up, Concave Down

The graph of a differentiable function $y = f(x)$ is

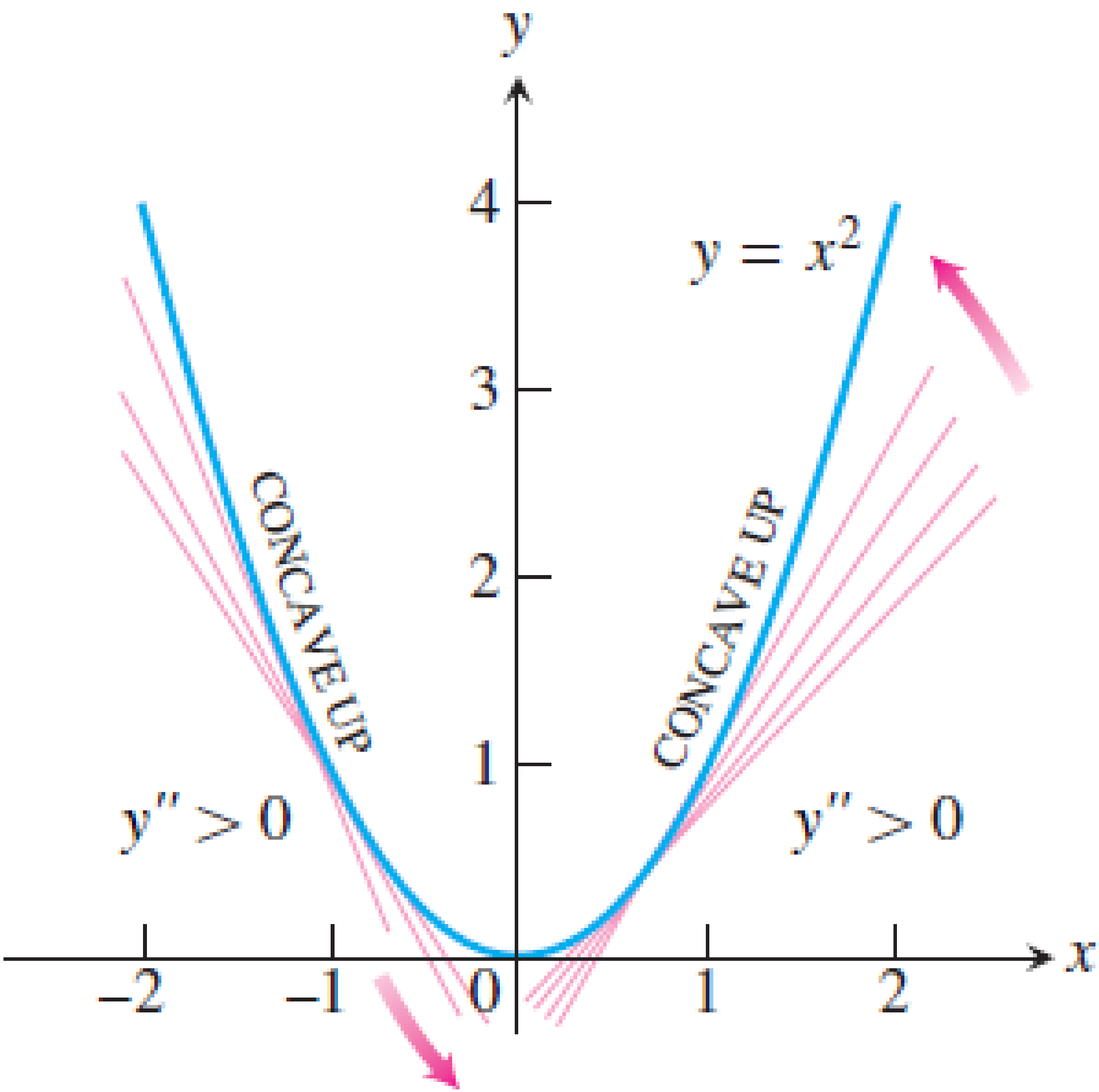
(a) **concave up** on an open interval I if f' is increasing on I

(b) **concave down** on an open interval I if f' is decreasing on I.

The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I.

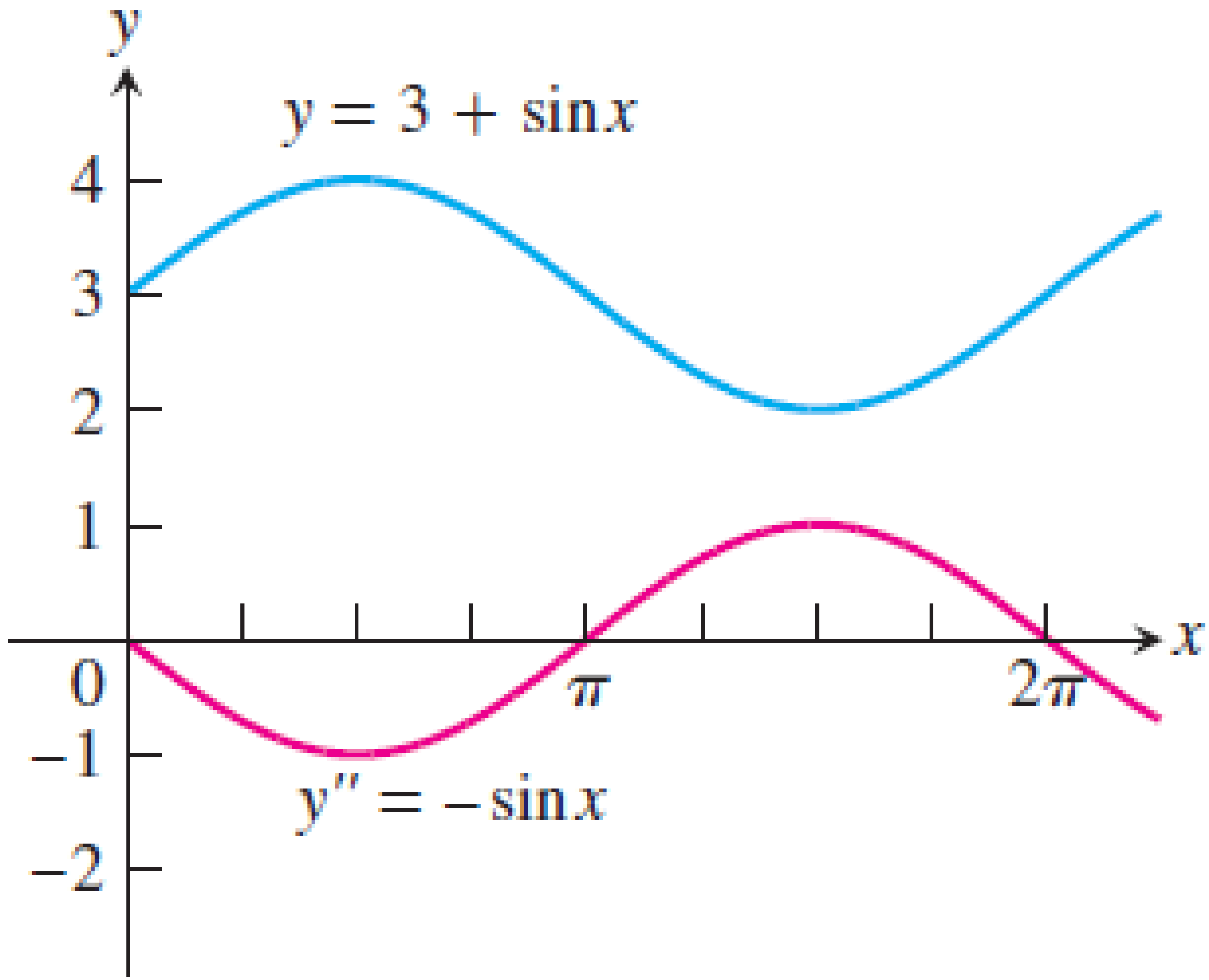
1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.



y
4
3
2
1
0
x
–2
–1
1
2
$y = x^2$
CONCAVE UP
CONCAVE UP
$y'' > 0$
$y'' > 0$

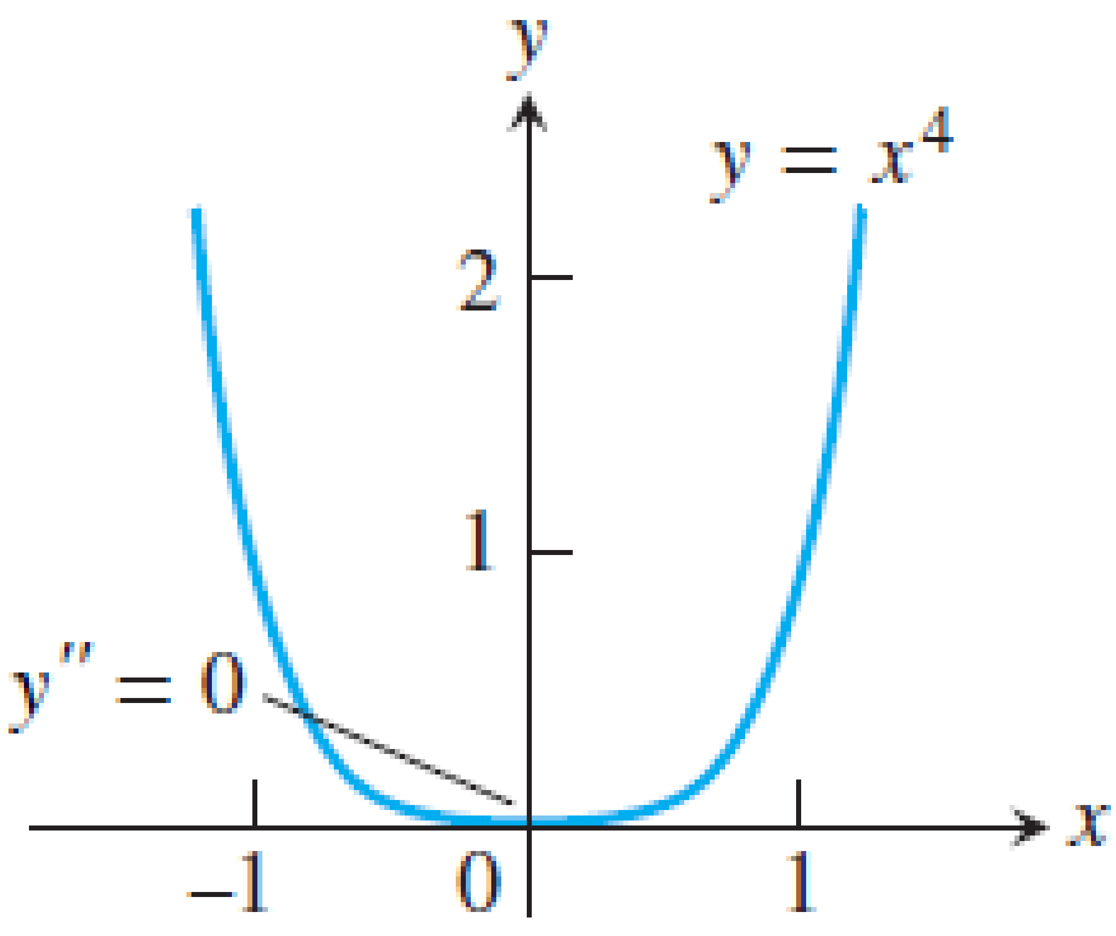
DEFINITION **Point of Inflection**

A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.



An Inflection Point May Not Exist Where $y'' = 0$

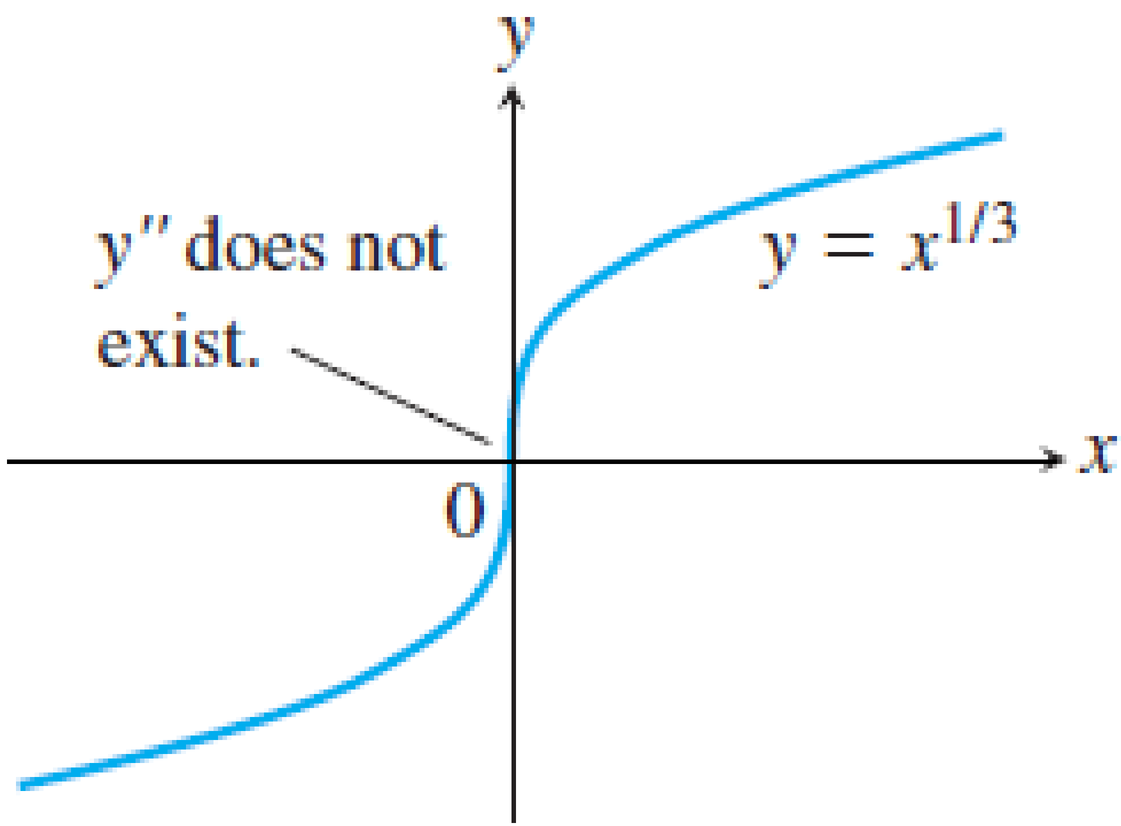
The curve $y = x^4$ has no inflection point at $x = 0$ (Even though $y'' = 12x^2$ is zero there, it does not change sign. ■



An Inflection Point May Occur Where y'' Does Not Exist

The curve $y = x^{1/3}$ has a point of inflection at $x = 0$ but y'' does not exist there.

$$y'' = \frac{d^2}{dx^2}\left(x^{1/3}\right) = \frac{d}{dx}\left(\frac{1}{3}x^{-2/3}\right) = -\frac{2}{9}x^{-5/3}.$$

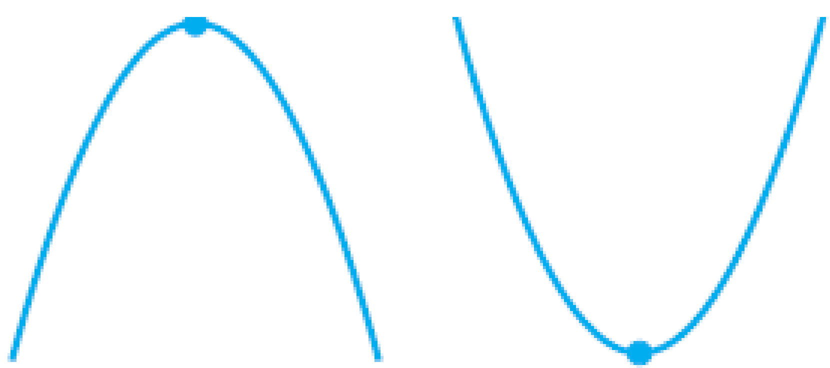


Second Derivative Test for Local Extrema

Instead of looking for sign changes in f' at critical points, we can sometimes use the following test to determine the presence and character of local extrema.

THEOREM 5 **Second Derivative Test for Local Extrema**
Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.



$f' = 0, f'' < 0$
$\Rightarrow$ local max

$f' = 0, f'' > 0$
$\Rightarrow$ local min

Using f' and f'' to Graph f

Sketch a graph of the function

$$f(x) = x^4 - 4x^3 + 10$$

using the following steps.

(a) Identify where the extrema of f occur.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing.

(c) Find where the graph of f is concave up and where it is concave down.

(d) Sketch the general shape of the graph for f.

(e) Plot some specific points, such as local maximum and minimum points, points of inflection, and intercepts. Then sketch the curve.

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$$

the first derivative is zero at $x = 0$ and $x = 3$.

Intervals	$x < 0$	$0 < x < 3$	$3 < x$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

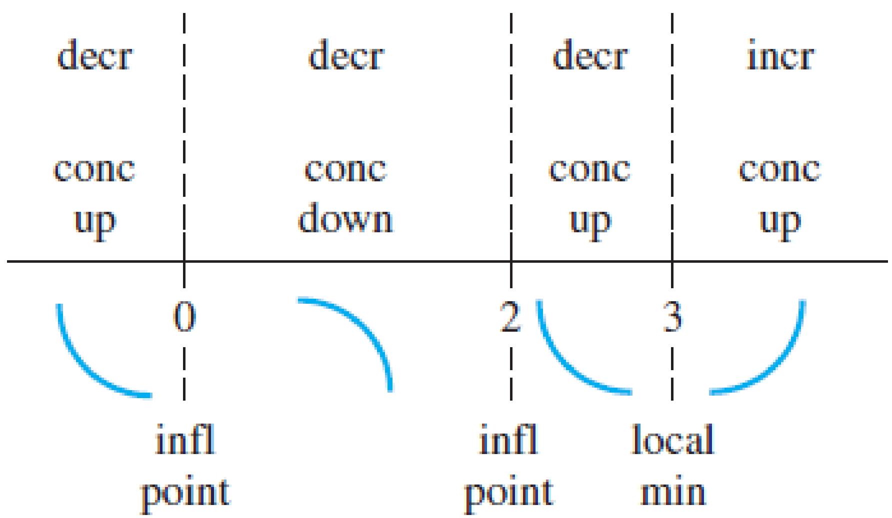
(c) $f''(x) = 12x^2 - 24x = 12x(x - 2)$ is zero at $x = 0$ and $x = 2$.

Intervals	$x < 0$	$0 < x < 2$	$2 < x$
Sign of f'	$+$	$-$	$+$
Behavior of f	concave up	concave down	concave up

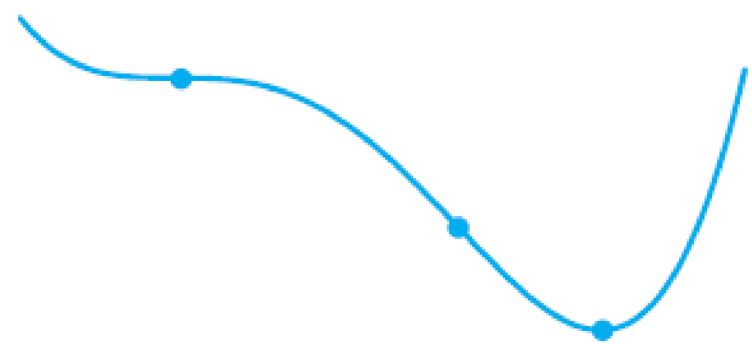
(d) Summarizing the information in the two tables above, we obtain

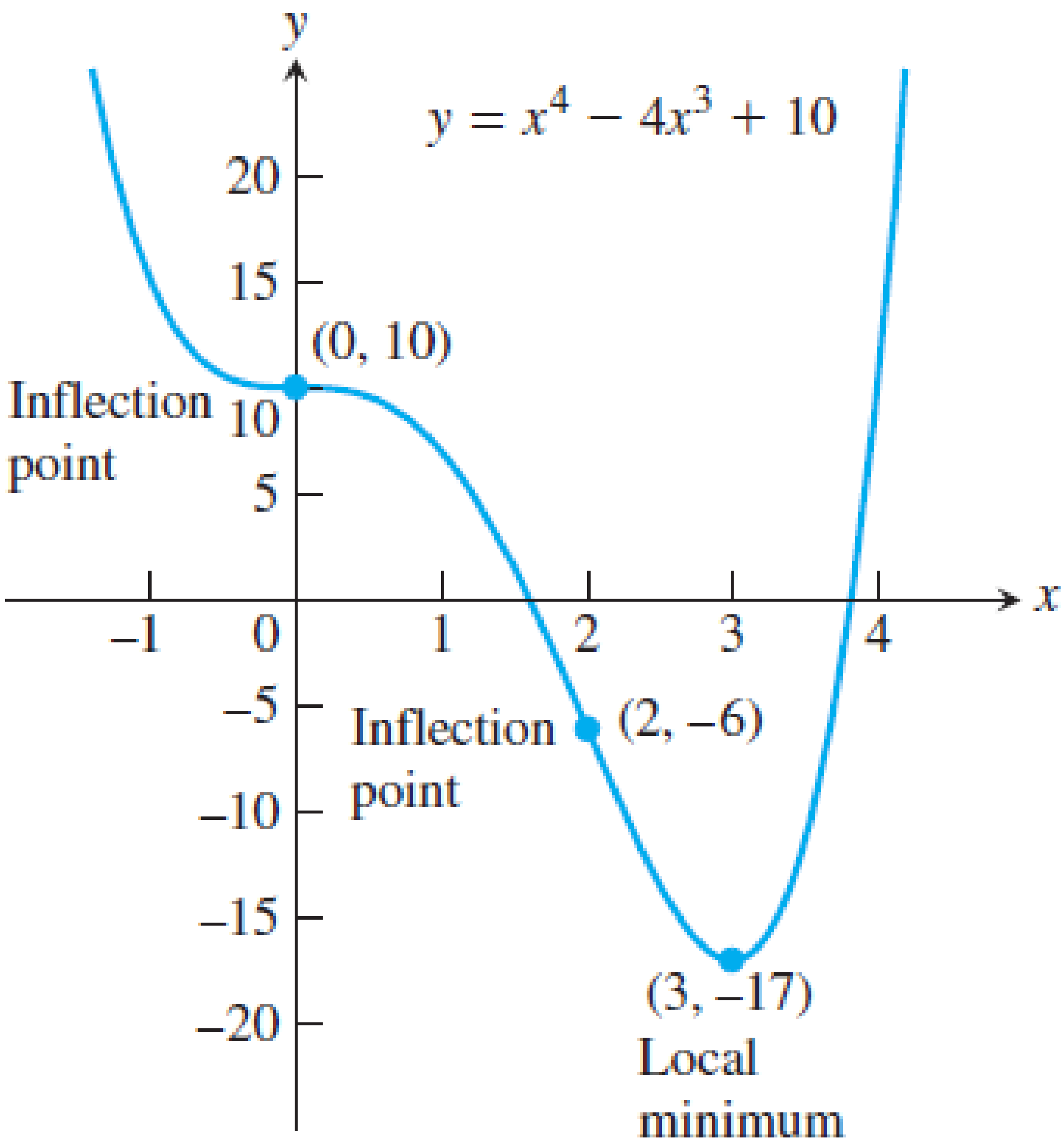
$x < 0$	$0 < x < 2$	$2 < x < 3$	$3 < x$
decreasing concave up	decreasing concave down	decreasing concave up	increasing concave up

The general shape of the curve is



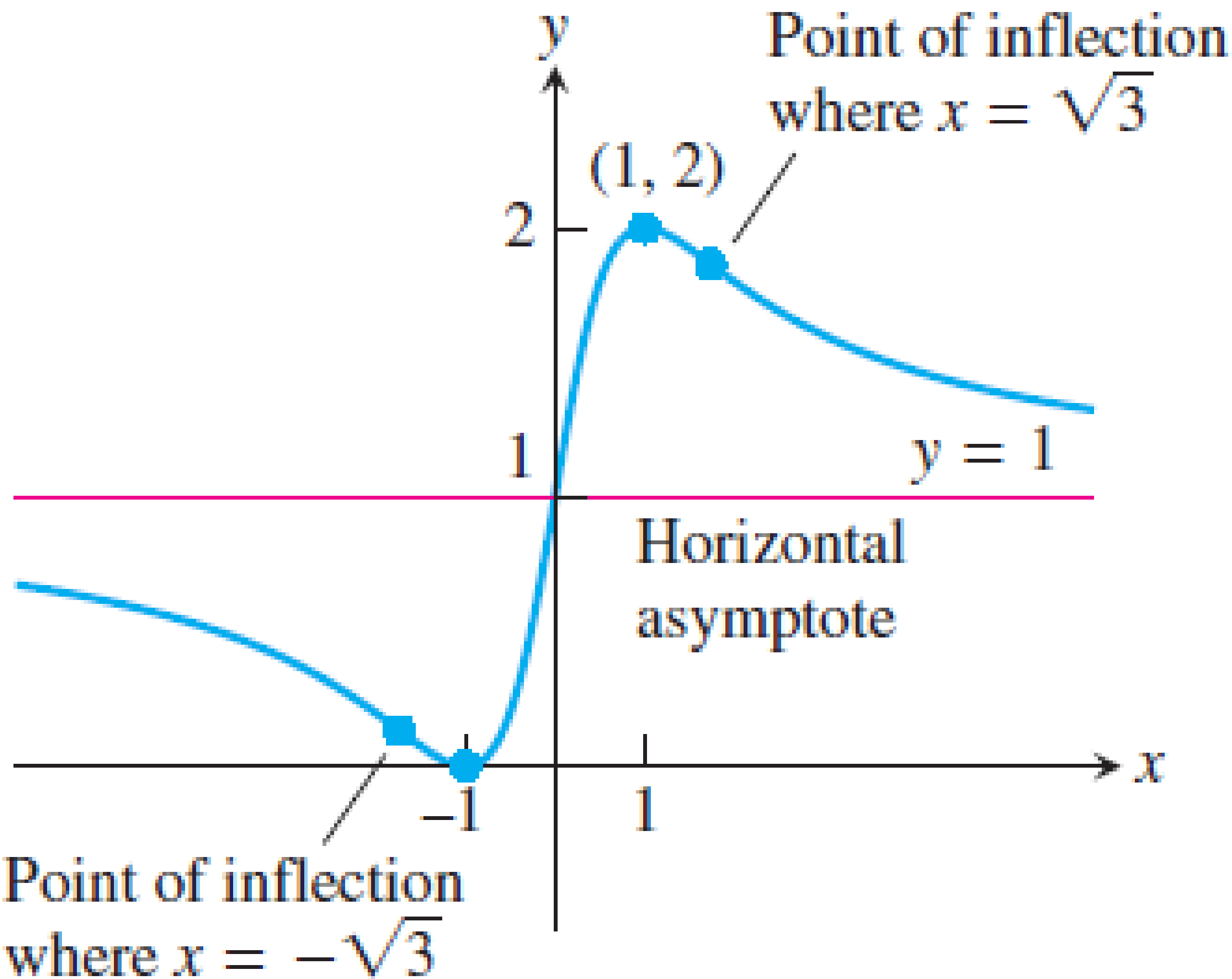
General shape.

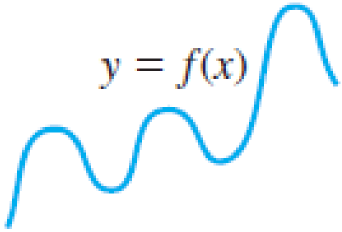
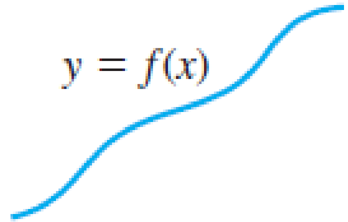
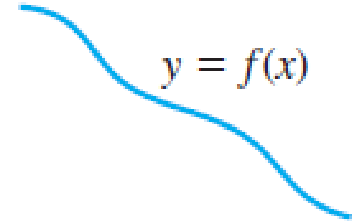
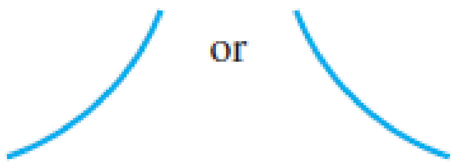
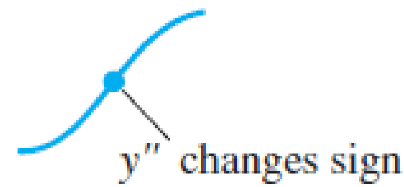
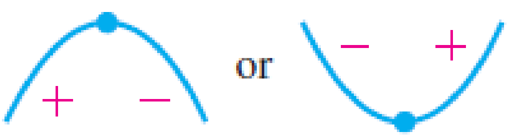
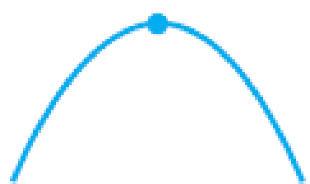



y
$y = x^4 - 4x^3 + 10$
20
15
(0, 10)
Inflection point
10
5
x
–1
0
1
2
3
4
–5
Inflection point
(2, –6)
–10
–15
(3, –17)
–20
Local minimum

1. Identify the domain of f and any symmetries the curve may have.
2. Find y' and y''.
3. Find the critical points of f, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve.

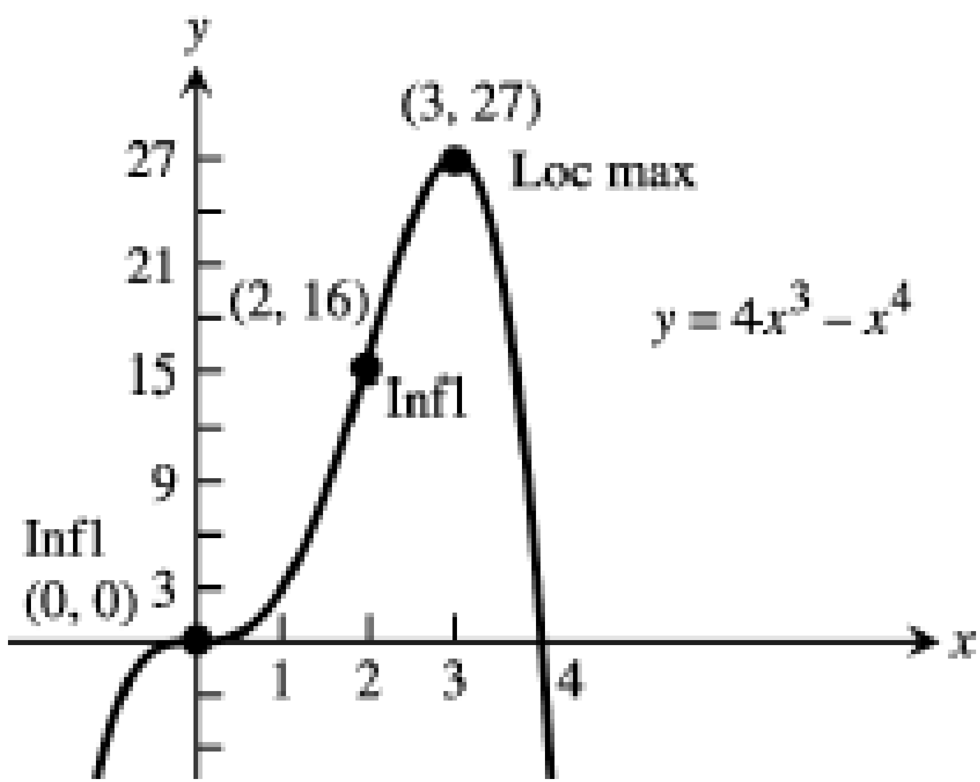
Sketch the graph of $f(x) = \dfrac{(x + 1)^2}{1 + x^2}$.



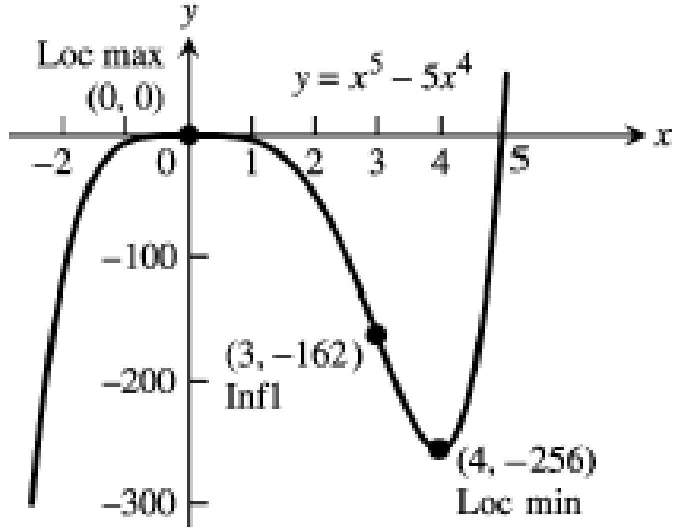
 Differentiable $\Rightarrow$ smooth, connected; graph may rise and fall	 $y' > 0 \Rightarrow$ rises from left to right; may be wavy	 $y' < 0 \Rightarrow$ falls from left to right; may be wavy
 $y'' > 0 \Rightarrow$ concave up throughout; no waves; graph may rise or fall	 $y'' < 0 \Rightarrow$ concave down throughout; no waves; graph may rise or fall	 Inflection point
 y' changes sign $\Rightarrow$ graph has local maximum or local minimum	 $y' = 0$ and $y'' < 0$ at a point; graph has local maximum	 $y' = 0$ and $y'' > 0$ at a point; graph has local minimum

Use the steps of the graphing procedure to graph the equation. Include the coordinates of any local **extreme** points and **inflection** points.

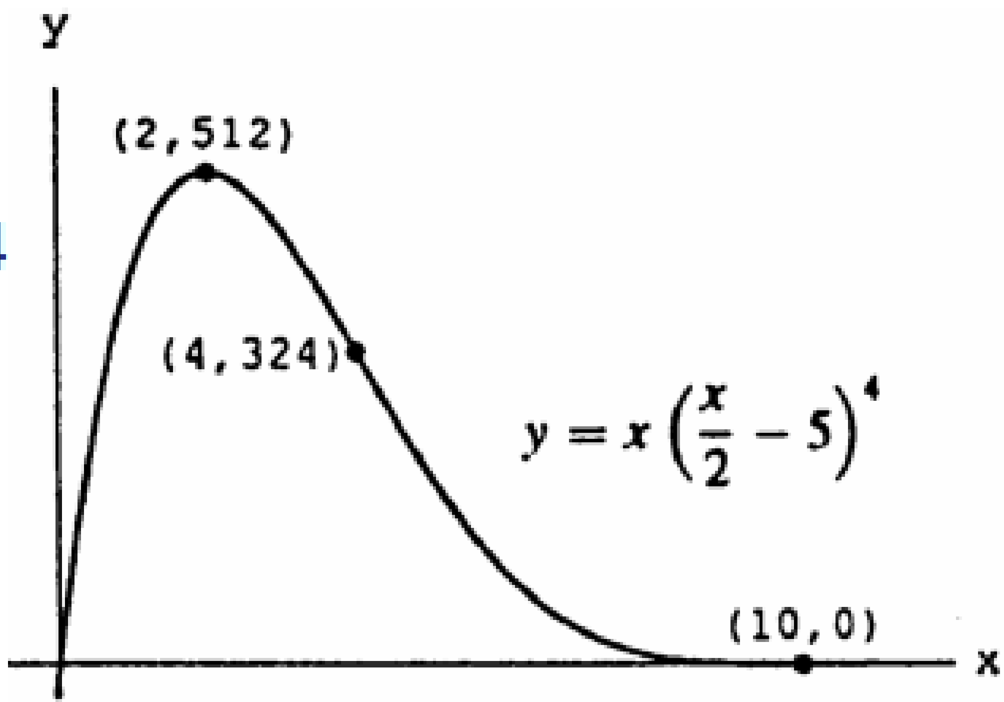
$$y = 4x^3 - x^4 = x^3(4 - x)$$



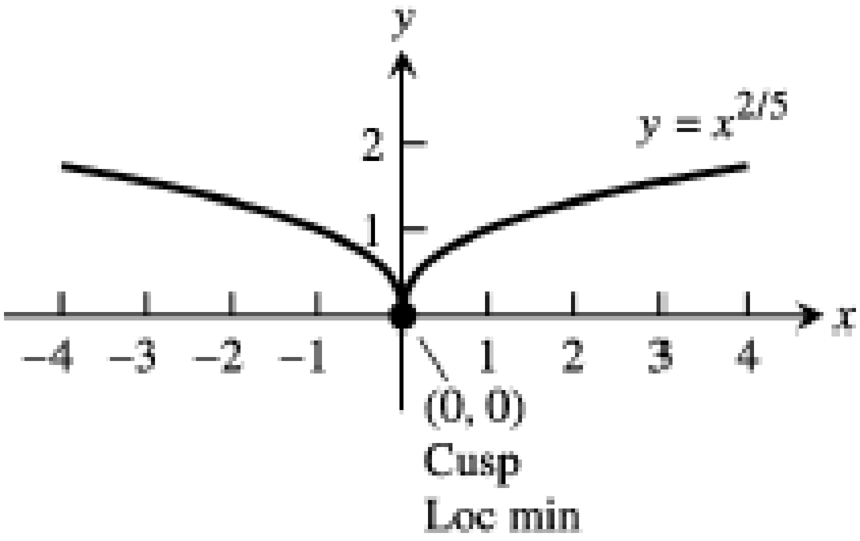
$$y = x^5 - 5x^4$$



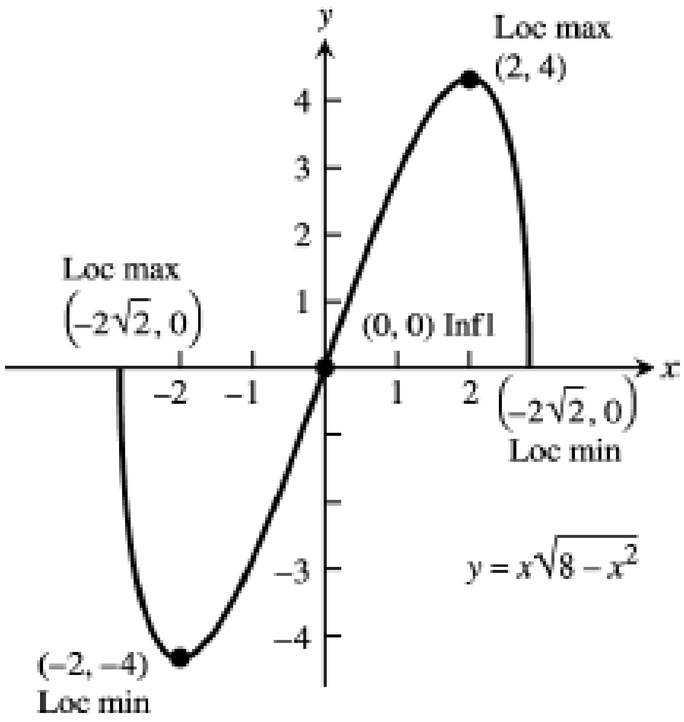
$$y = x\left(\frac{x}{2} - 5\right)^4$$



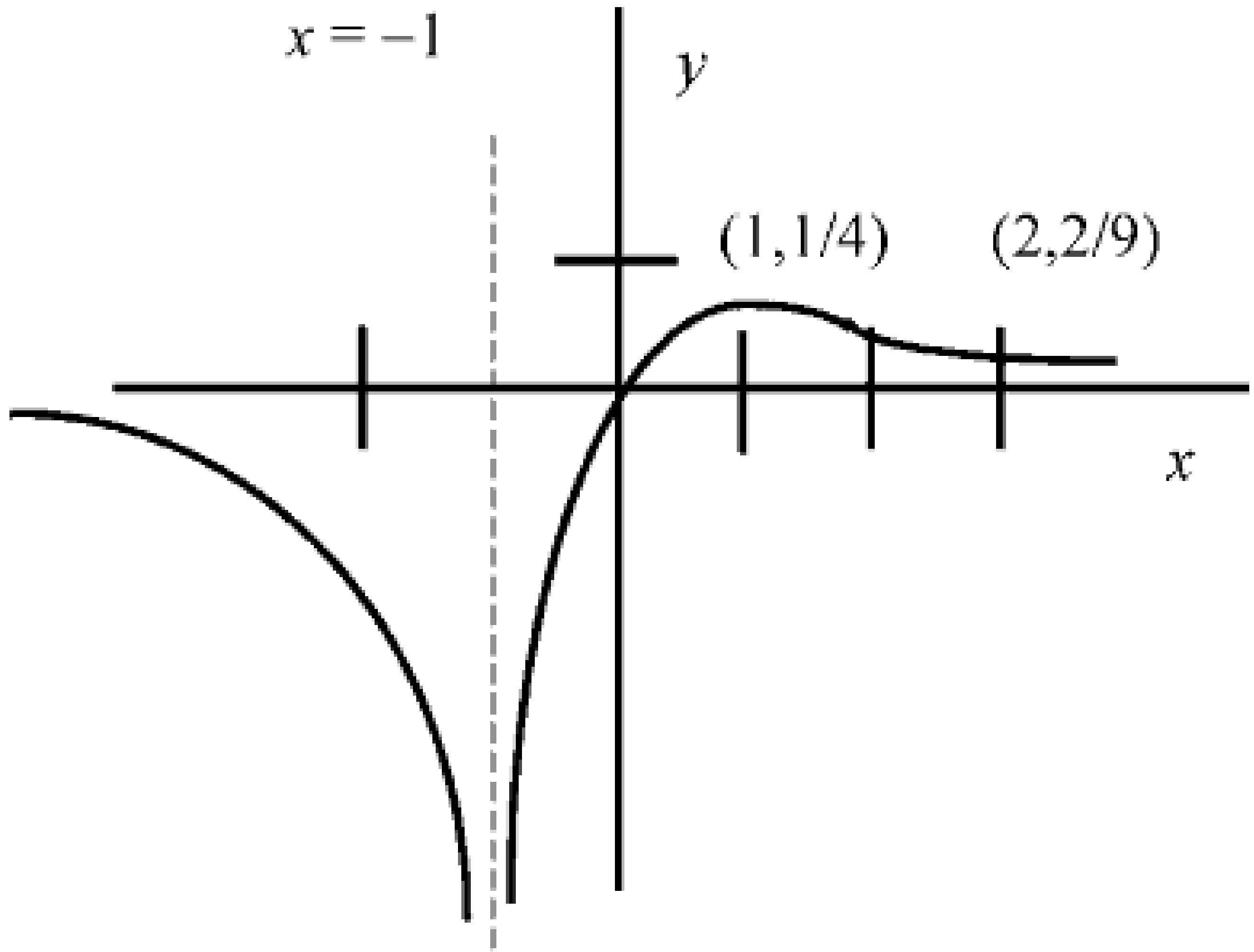
$$y = x^{2/5}$$

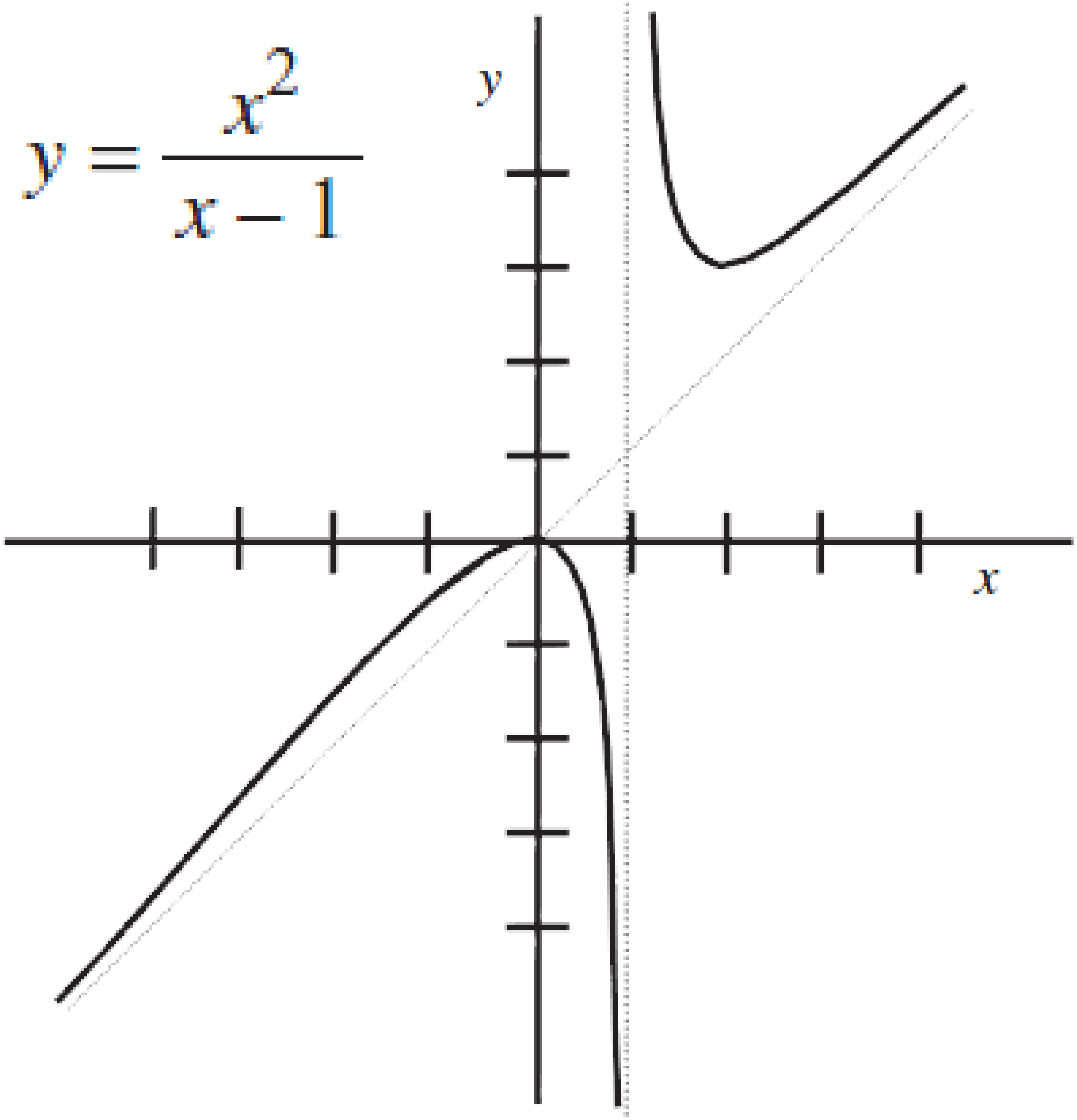


$$y = x\sqrt{8 - x^2}$$

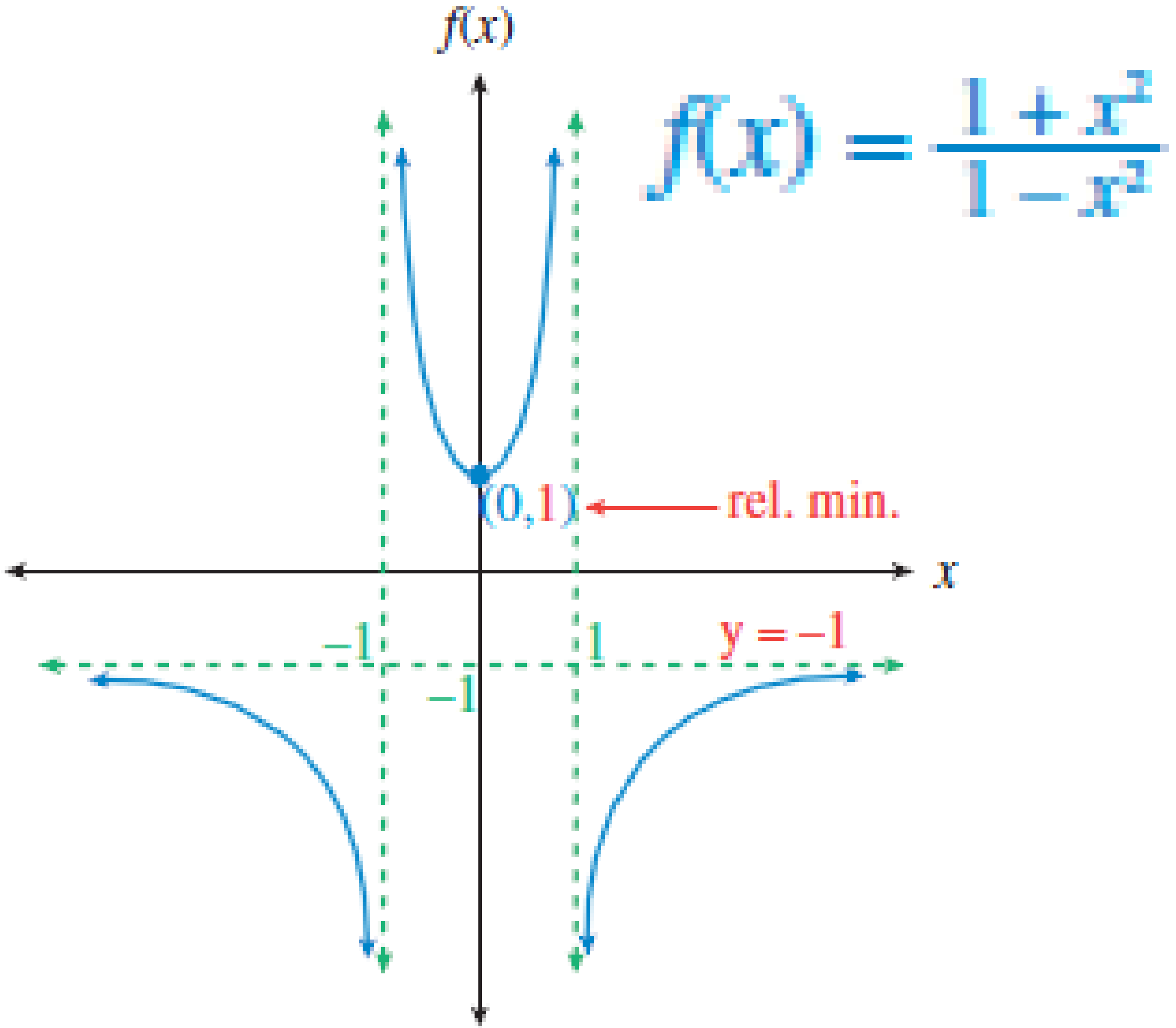


Sketch the graph of $y = x^2/(x - 1)$.





$y = \dfrac{x^2}{x-1}$
y
x

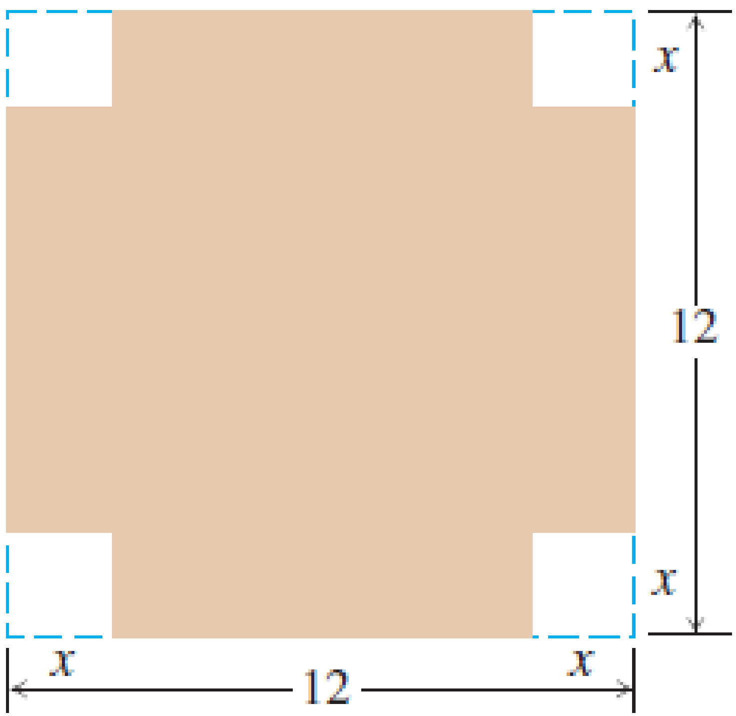


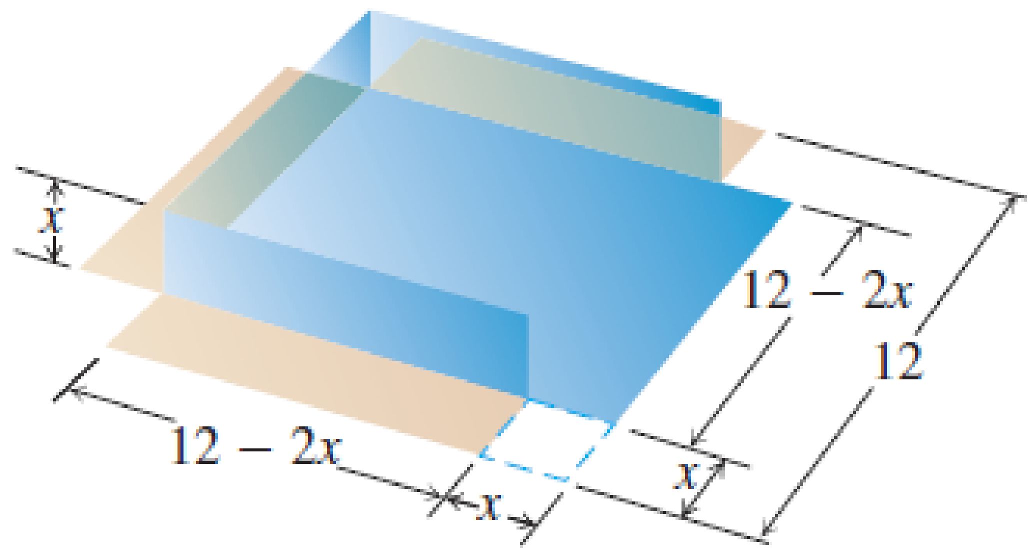
f(x)
f(x) = (1 + x^2)/(1 − x^2)
(0,1)
rel. min.
x
−1
1
y = −1
−1

Applied Optimization Problems

Fabricating a Box

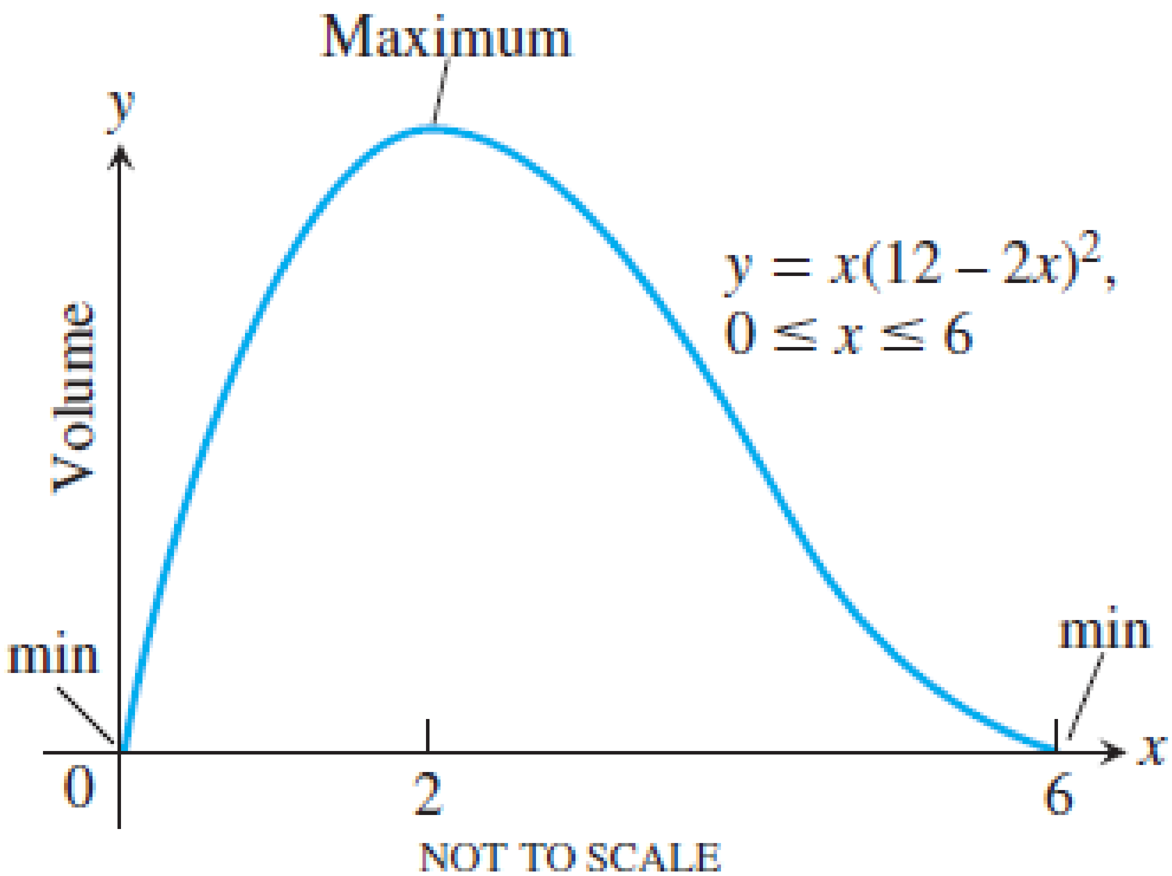
An open-top box is to be made by cutting small congruent squares from the corners of a 12-in.-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?





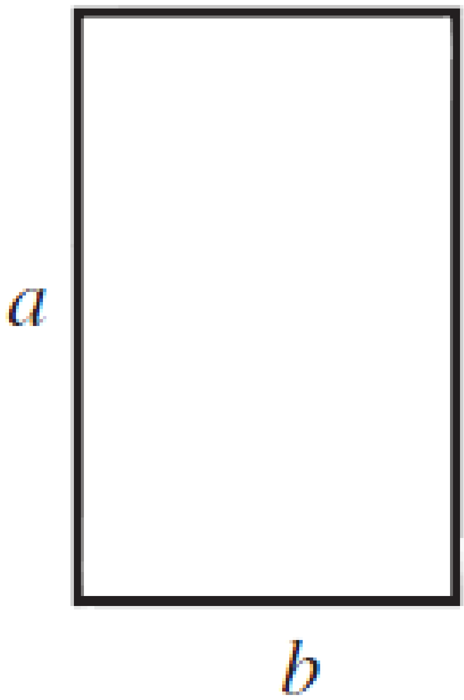
Critical-point value: $\quad V(2) = 128$

Endpoint values: $\quad V(0) = 0, \qquad V(6) = 0.$

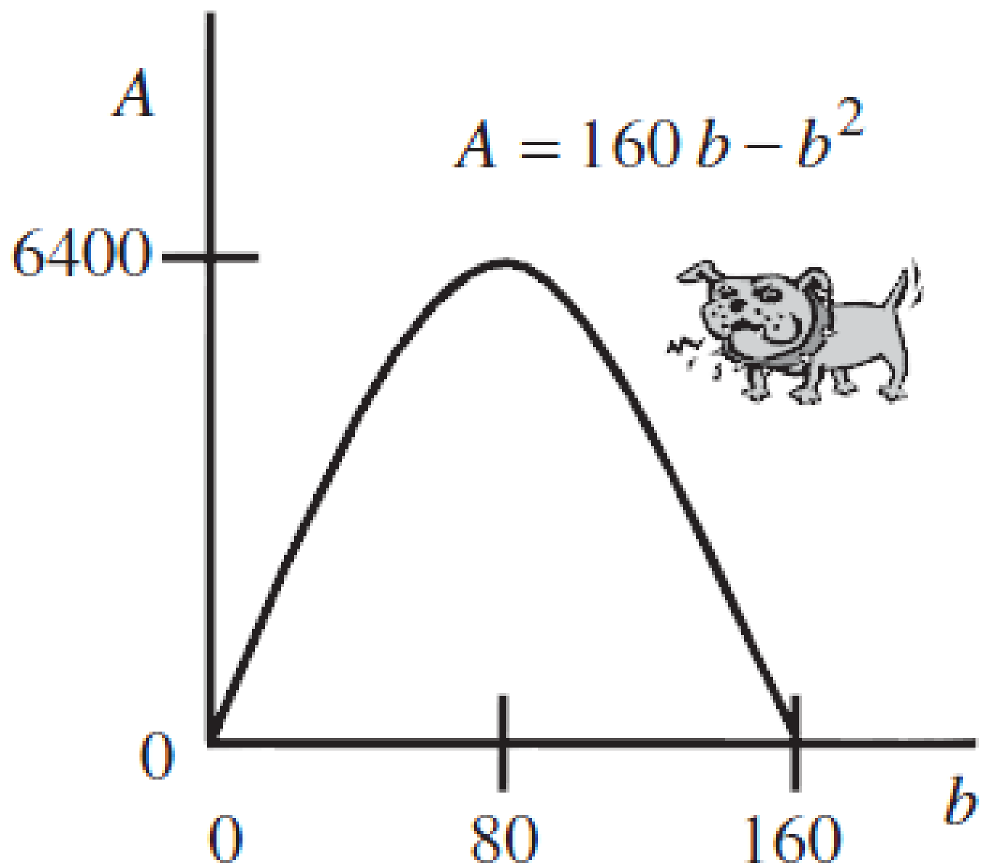


Solving Applied Optimization Problems

1. *Read the problem.* Read the problem until you understand it. What is given? What is the unknown quantity to be optimized?
2. *Draw a picture.* Label any part that may be important to the problem.
3. *Introduce variables.* List every relation in the picture and in the problem as an equation or algebraic expression, and identify the unknown variable.
4. *Write an equation for the unknown quantity.* If you can, express the unknown as a function of a single variable or in two equations in two unknowns. This may require considerable manipulation.
5. *Test the critical points and endpoints in the domain of the unknown.* Use what you know about the shape of the function's graph. Use the first and second derivatives to identify and classify the function's critical points.



A rectangular area is to be enclosed with 320 ft of fence What dimensions of rectangle give the maximum area?



What are the dimensions of the soup can of greatest volume that can be made with 50 square inches of tin? (The entire can, including the top and bottom, are made of tin.) And what's its volume?

. Use the given information to relate r and h.

$$Surface\ Area = \overbrace{2\pi r^2}^{\text{top and bottom}} + \overbrace{2\pi rh}^{\text{lateral area}}$$

$$50 = 2\pi r^2 + 2\pi rh$$

$$25 = \pi r^2 + \pi rh$$

. Solve for h and substitute to create a function of one variable.

$$\pi r h = 25 - \pi r^2 \qquad V = \pi r^2 h$$

$$h = \frac{25}{\pi r} - r \qquad V(r) = \pi r^2\left(\frac{25}{\pi r} - r\right)$$

$$= 25r - \pi r^3$$

Find the critical numbers of $V(r)$.

$$V(r) = 25r - \pi r^3$$

$$V'(r) = 25 - 3\pi r^2$$

$$0 = 25 - 3\pi r^2$$

$$r^2 = \frac{25}{3\pi}$$

$$r = \pm\sqrt{\frac{25}{3\pi}}$$

≈ 1.63 inches (You can reject the negative

Evaluate the volume at the critical number.

$$V(1.63) = 25 \cdot 1.63 - \pi(1.63)^3$$

≈ 27.14 cubic inches

THEOREM 7 L'Hôpital's Rule (Stronger Form)

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

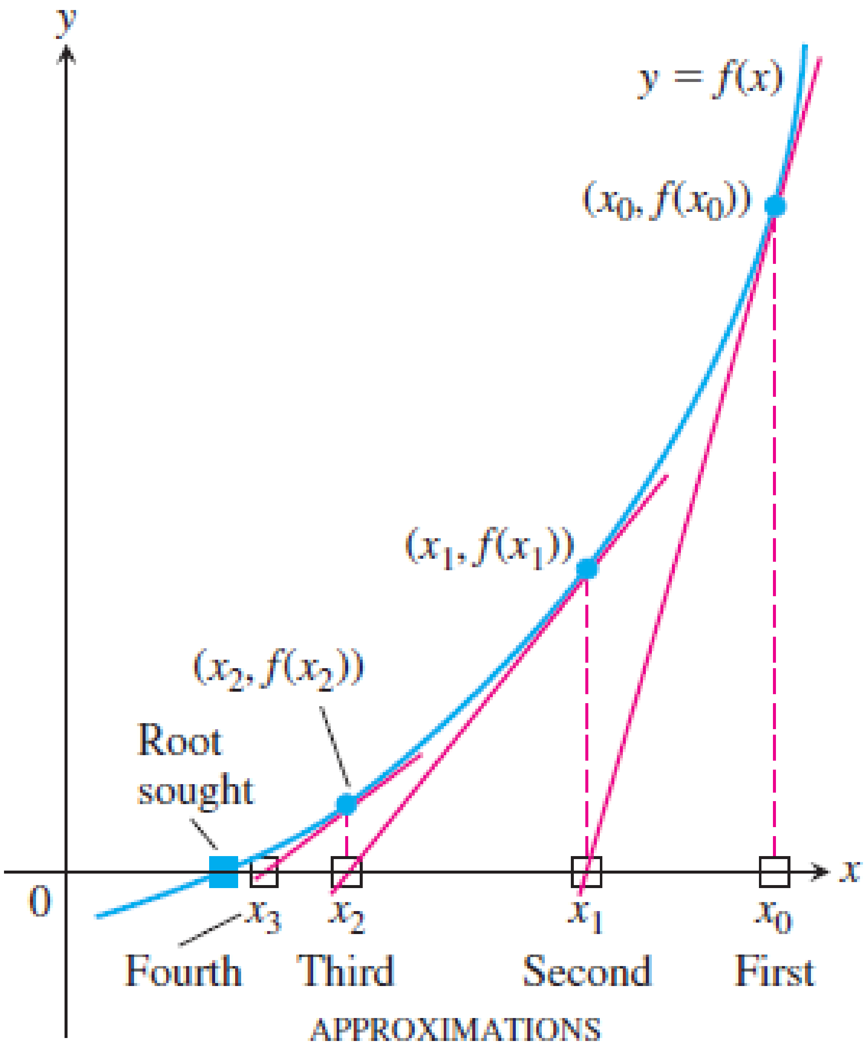
assuming that the limit on the right side exists.

$$\begin{aligned}
\lim_{x \to 0} \frac{x - \sin x}{x^3} &= \lim_{x \to 0} \frac{1 - \cos x}{3x^2} \\
&= \lim_{x \to 0} \frac{\sin x}{6x} \\
&= \lim_{x \to 0} \frac{\cos x}{6} = \frac{1}{6}
\end{aligned}$$

Newton’s Method

- solving equations. using the quadratic root Formula
- more complicated formulas to solve cubic or quartic equations
- Niels Abel showed that no simple formulas exist to solve polynomials of degree equal to five.
- In this section we study a numerical method, called *Newton’s method* or the *Newton–Raphson method,*
- which is a technique to approximate the solution

The initial estimate, x_0, may be found by graphing or just plain guessing. The method then uses the tangent to the curve $y = f(x)$ at $(x_0, f(x_0))$ to approximate the curve, calling the point x_1 where the tangent meets the x-axis The number x_1 is usually a better approximation to the solution than is x_0. The point x_2 where the tangent to the curve at $(x_1, f(x_1))$ crosses the x-axis is the next approximation in the sequence. We continue on,



$$y = f(x_n) + f'(x_n)(x - x_n).$$

$$0 = f(x_n) + f'(x_n)(x - x_n)$$

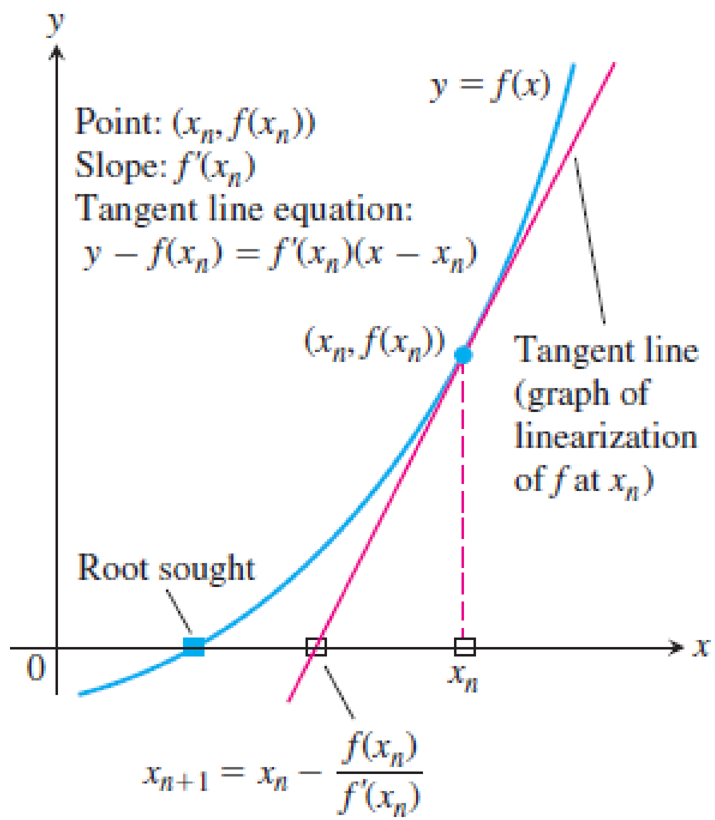
$$-\frac{f(x_n)}{f'(x_n)} = x - x_n$$

$$x = x_n - \frac{f(x_n)}{f'(x_n)}$$

Procedure for Newton's Method

1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.
2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad \text{if } f'(x_n) \neq 0 \tag{1}$$



Finding the Square Root of 2

Find the positive root of the equation

$$f(x) = x^2 - 2 = 0.$$

With $f(x) = x^2 - 2$ and $f'(x) = 2x$, Equation (1) becomes

$$\begin{aligned} x_{n+1} &= x_n - \frac{x_n{}^2 - 2}{2x_n} \\ &= x_n - \frac{x_n}{2} + \frac{1}{x_n} \\ &= \frac{x_n}{2} + \frac{1}{x_n}. \end{aligned}$$

on

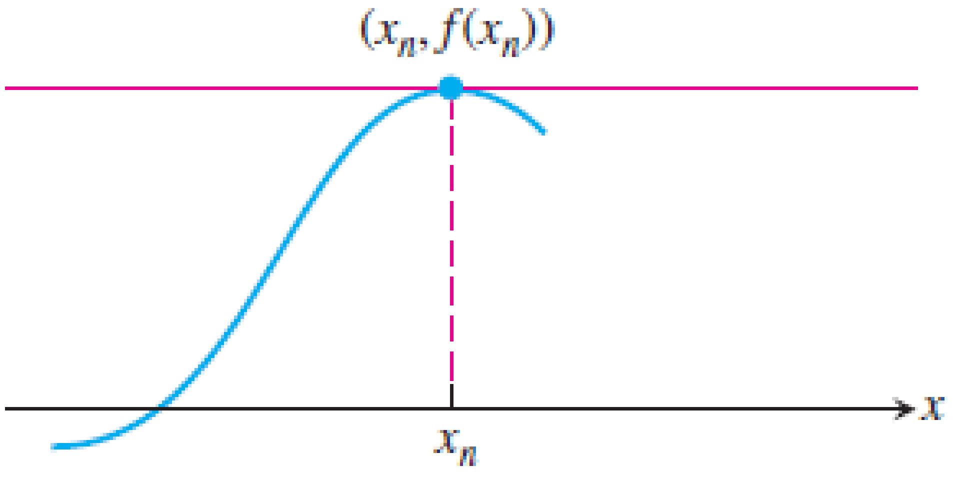
$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$$

starting value $x_0 = 1$, we get the results in the first column of the following table. (To five decimal places, $\sqrt{2} = 1.41421$.)

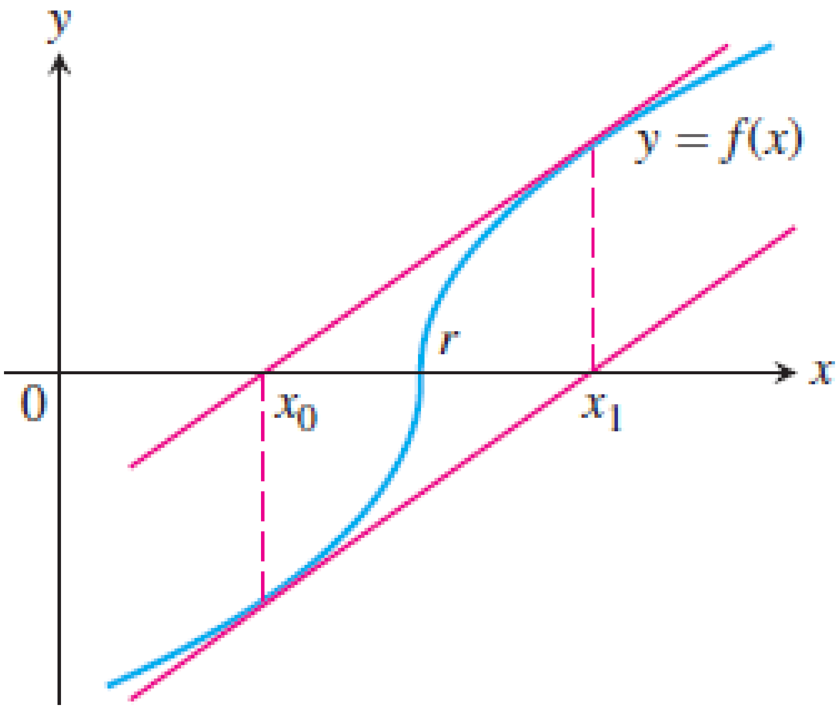
	Error	Number of correct digits
$x_0 = 1$	-0.41421	1
$x_1 = 1.5$	0.08579	1
$x_2 = 1.41667$	0.00246	3
$x_3 = 1.41422$	0.00001	5

■

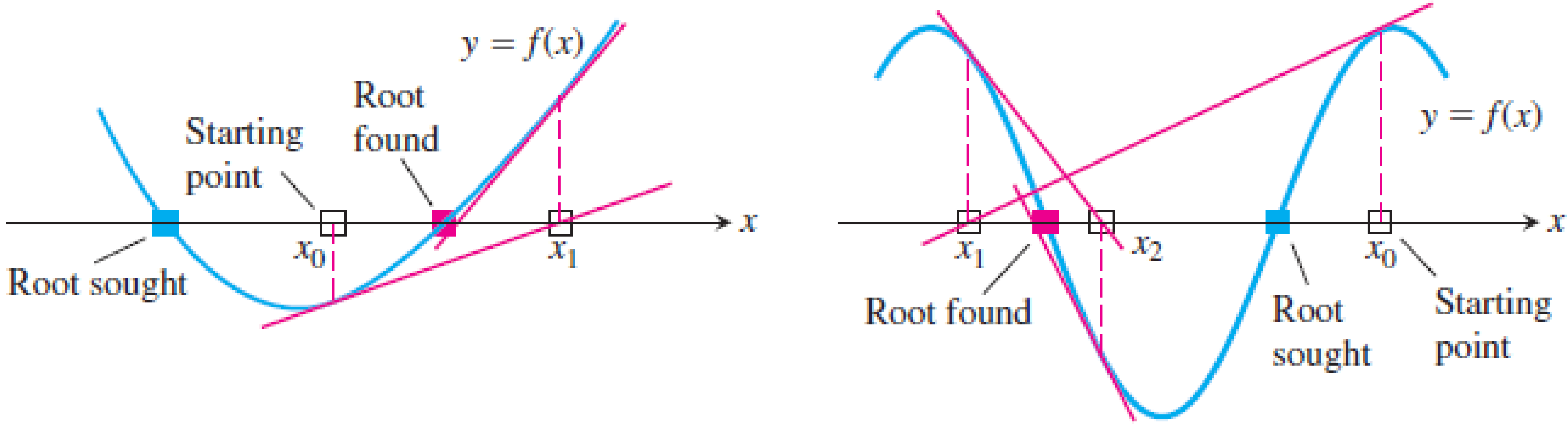
But Things Can Go Wrong



If $f'(x_n) = 0$, there is no intersection point to define x_{n+1}.



Newton's method fails to converge. You go from x_0 to x_1 and back to x_0, never getting any closer to r.



If you start too far away, Newton's method may miss the root you want.

Antiderivatives

DEFINITION Antiderivative

A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Find an antiderivative for each of the following functions

(a) $f(x) = 2x$

(b) $g(x) = \cos x$

(c) $h(x) = 2x + \cos x$

(a) $F(x) = x^2$

(b) $G(x) = \sin x$

(c) $H(x) = x^2 + \sin x$

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

	Function	General antiderivative
1.	x^n	$\dfrac{x^{n+1}}{n+1} + C, \quad n \neq -1, n$ rational
2.	$\sin kx$	$-\dfrac{\cos kx}{k} + C, \quad k$ a constant, $k \neq 0$
3.	$\cos kx$	$\dfrac{\sin kx}{k} + C, \quad k$ a constant, $k \neq 0$
4.	$\sec^2 x$	$\tan x + C$
5.	$\csc^2 x$	$-\cot x + C$
6.	$\sec x \tan x$	$\sec x + C$
7.	$\csc x \cot x$	$-\csc x + C$

		Function	General antiderivative
1.	*Constant Multiple Rule:*	$kf(x)$	$kF(x) + C$, k a constant
2.	*Negative Rule:*	$-f(x)$	$-F(x) + C$,
3.	*Sum or Difference Rule:*	$f(x) \pm g(x)$	$F(x) \pm G(x) + C$

Indefinite Integrals

A special symbol is used to denote the collection of all antiderivatives of a function f.

DEFINITION Indefinite Integral, Integrand

The set of all antiderivatives of f is the **indefinite integral** of f with respect to x, denoted by

$$\int f(x)\,dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.