Chapter 5

INTEGRATION

•*integration*, is a tool for calculating much more than areas and volumes.
•The *integral* has many applications in statistics, economics, the sciences, and engineering.
•It allows us to calculate quantities ranging from probabilities and averages to energy consumption and the forces against a dam’s floodgates.
•The idea behind integration is that we can effectively compute many quantities by breaking them into small pieces, and then summing the contributions from each small part.
•We develop the theory of the integral in the setting of area, where it most clearly reveals its nature.

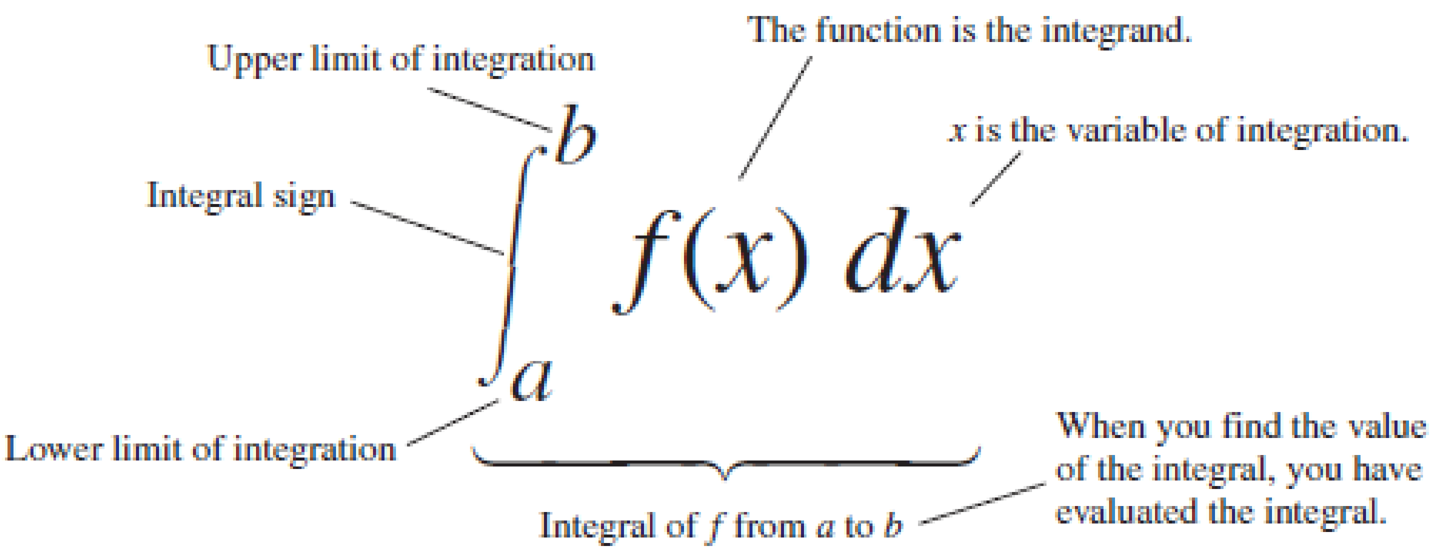
The Definite Integral

Notation and Existence of the Definite Integral

The symbol for the number I in the definition of the definite integral is

$$\int_a^b f(x)\,dx$$

which is read as "the integral from a to b of f of x dee x" or sometimes as "the integral from a to b of f of x with respect to x." The component parts in the integral symbol also have names:



Rules satisfied by definite integrals

1. *Order of Integration:* $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ — A Definition

2. *Zero Width Interval:* $\int_a^a f(x)\,dx = 0$ — Also a Definition

3. *Constant Multiple:* $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ — Any Number k

   $\int_a^b -f(x)\,dx = -\int_a^b f(x)\,dx$ — $k = -1$

4. *Sum and Difference:* $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

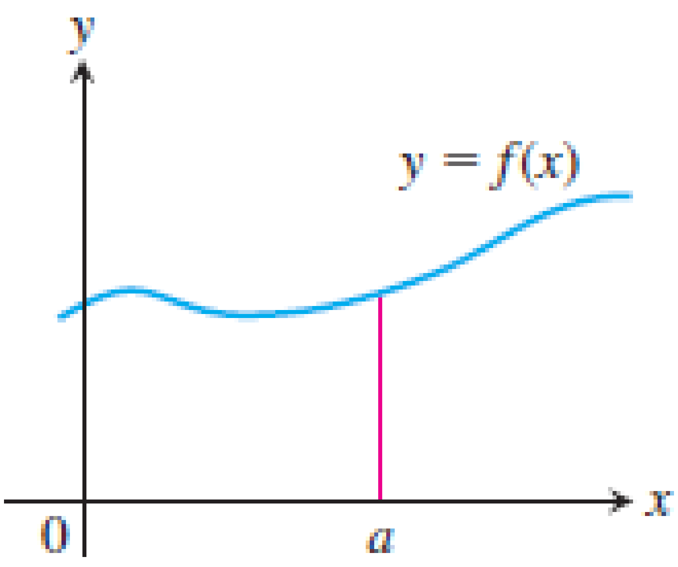
5. *Additivity:* $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

6. *Max-Min Inequality:* If f has maximum value max f and minimum value min f on $[a, b]$, then

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a).$$

7. *Domination:* $f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$

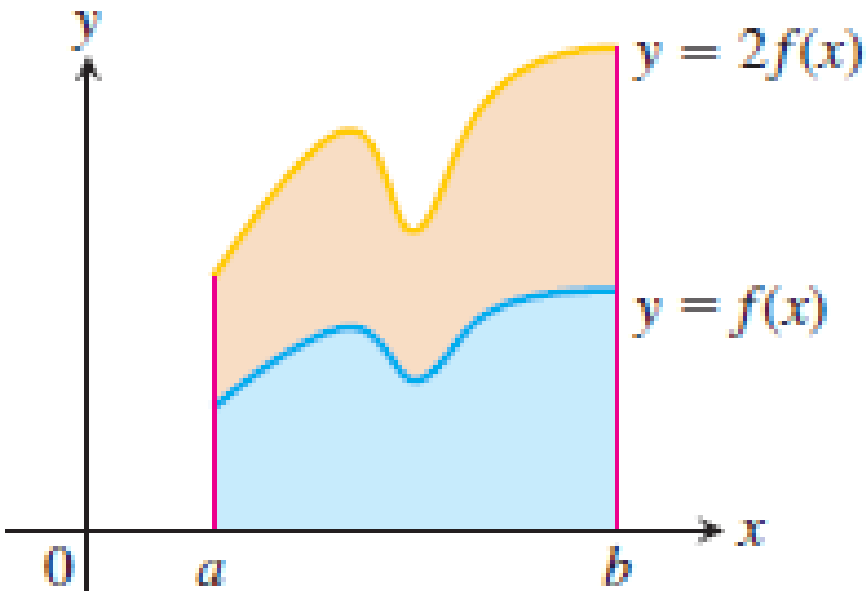
$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$ (Special Case)



(a) *Zero Width Interval:*

$$\int_a^a f(x)\,dx = 0.$$

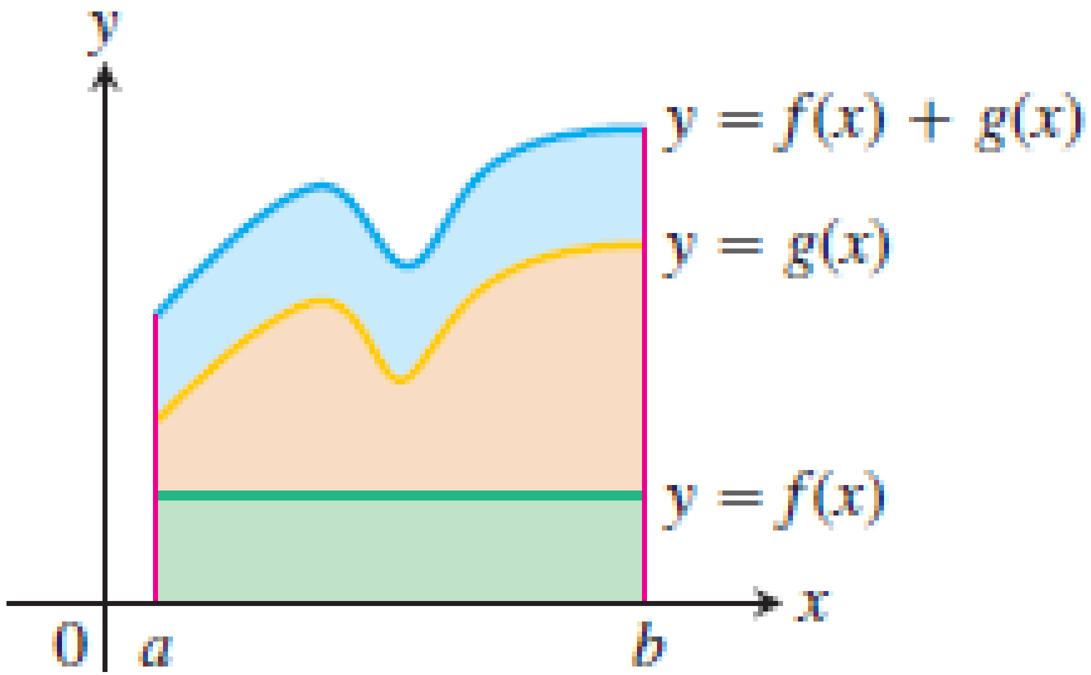
(The area over a point is 0.)



(b) *Constant Multiple:*

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx.$$

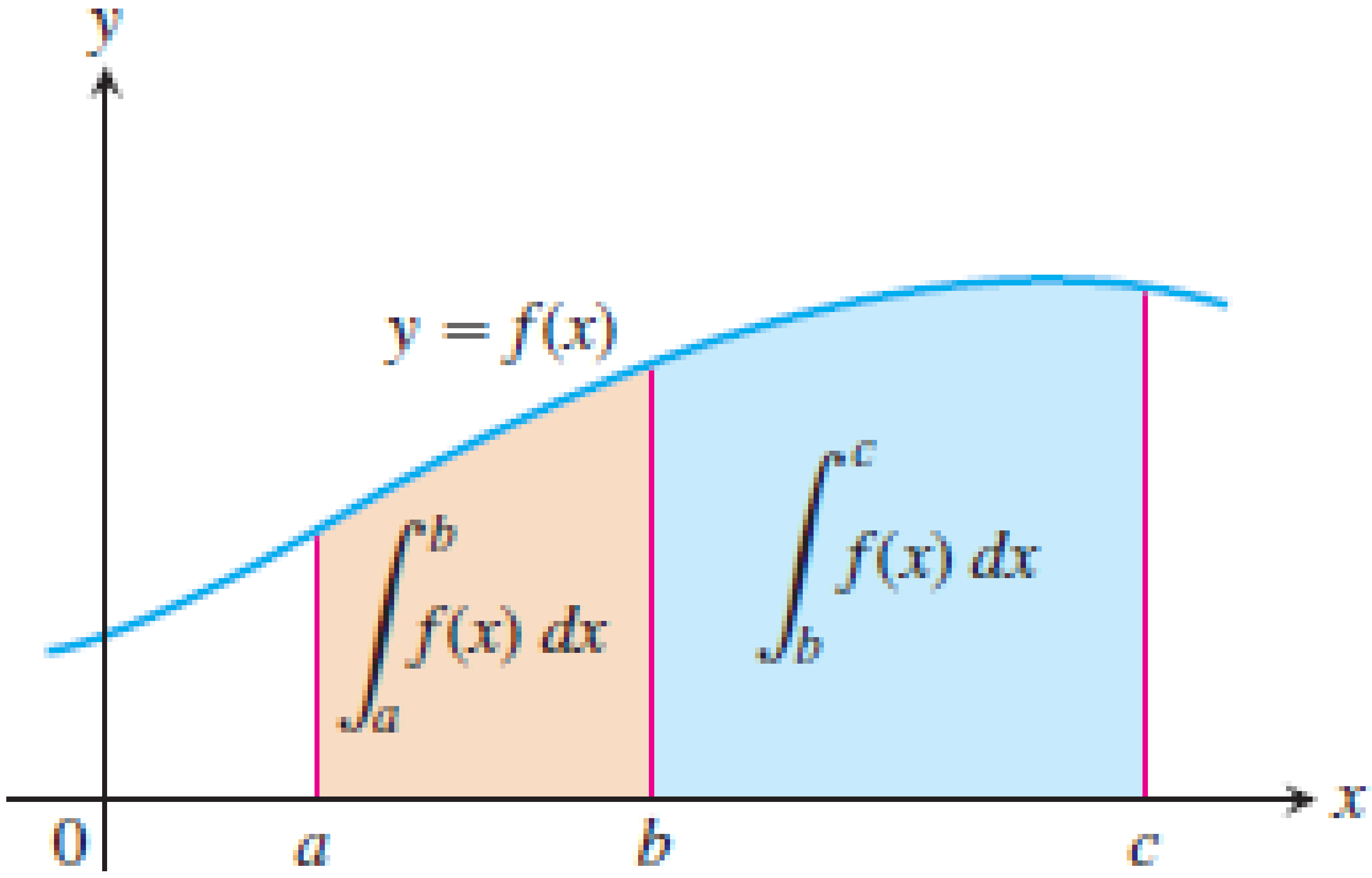
(Shown for $k = 2$.)



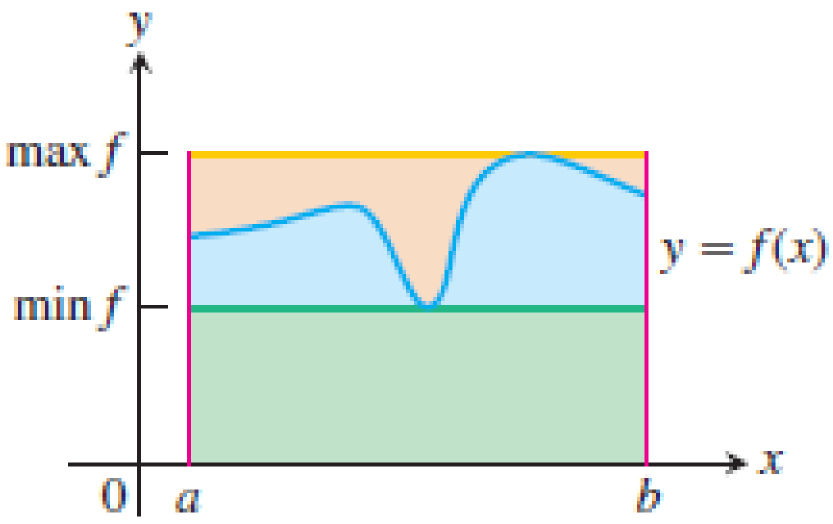
(c) *Sum:*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

(Areas add)

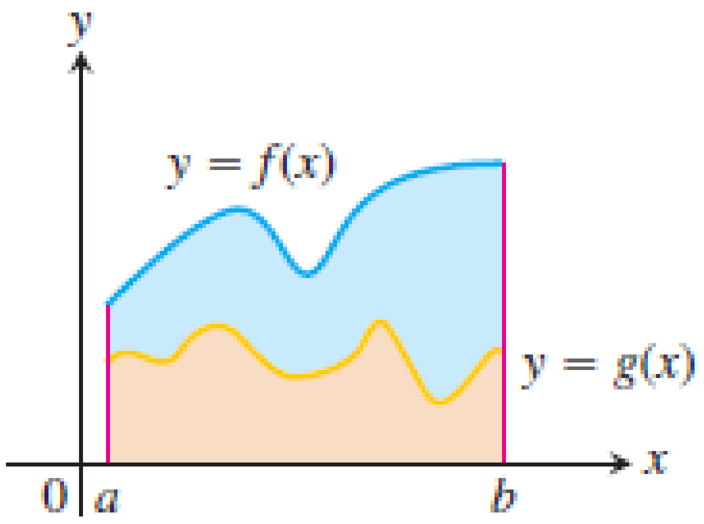


(d) *Additivity for definite integrals:*



(e) *Max-Min Inequality:*

$$\min f \cdot (b-a) \le \int_a^b f(x)\,dx \le \max f \cdot (b-a)$$



(f) *Domination:*

$$f(x) \ge g(x) \text{ on } [a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$$

Suppose that f and h are integrable and that

$$\int_1^9 f(x)\,dx = -1, \quad \int_7^9 f(x)\,dx = 5, \quad \int_7^9 h(x)\,dx = 4.$$

a. $\int_1^9 -2f(x)\,dx$

b. $\int_7^9 [f(x) + h(x)]\,dx$

c. $\int_7^9 [2f(x) - 3h(x)]\,dx$

d. $\int_9^1 f(x)\,dx$

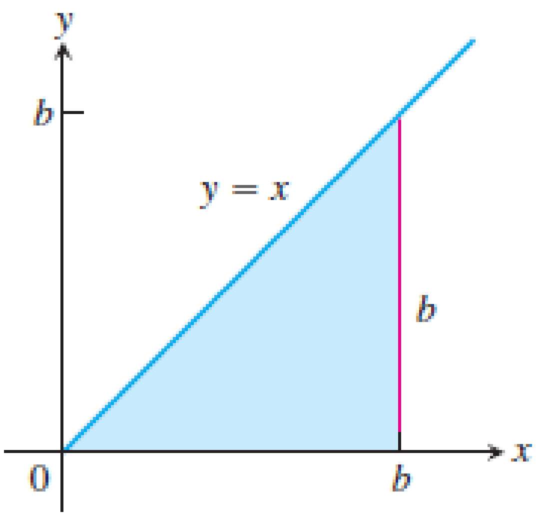
e. $\int_1^7 f(x)\,dx$

f. $\int_9^7 [h(x) - f(x)]\,dx$

DEFINITION Area Under a Curve as a Definite Integral

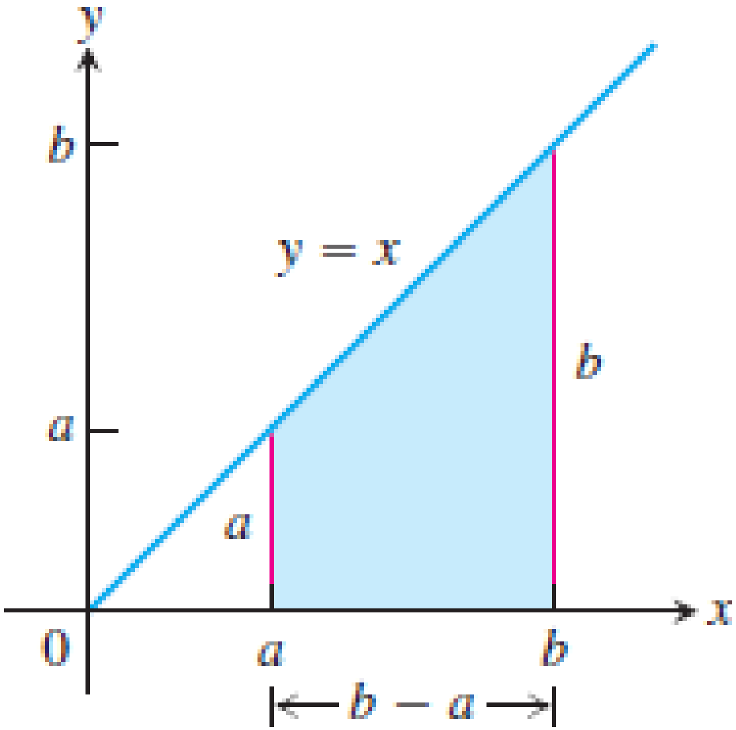
If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve** $y = f(x)$ **over** $[a, b]$ is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx.$$



$$\int_a^b x\,dx = \frac{b^2}{2} - \frac{a^2}{2}, \qquad a < b$$

Compute $\int_0^b x\,dx$ and find the area A under $y = x$ over the interval $[0, b]$, $b > 0$.



$$\int_a^b c\,dx = c(b-a), \qquad c \text{ any constant}$$

$$\int_a^b x^2\,dx = \frac{b^3}{3} - \frac{a^3}{3}, \qquad a < b$$

The area of this trapezoidal region is $A = (b^2 - a^2)/2$.

Using Area to Evaluate Definite Integrals

graph the integrands and use areas to evaluate the integrals.

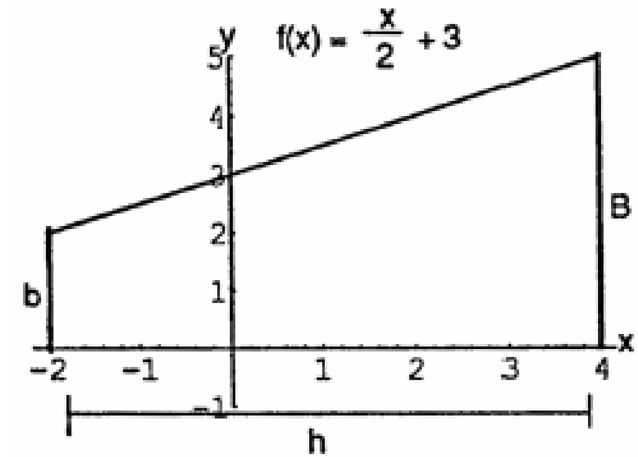
$$\int_{-2}^{4} \left(\frac{x}{2} + 3\right) dx$$

$$\int_{1/2}^{3/2} (-2x + 4)\, dx$$

. The area of the trapezoid is $A = \frac{1}{2}(B + b)h$

$= \frac{1}{2}(5 + 2)(6) = 21 \Rightarrow \int_{-2}^{4} \left(\frac{x}{2} + 3\right) dx$

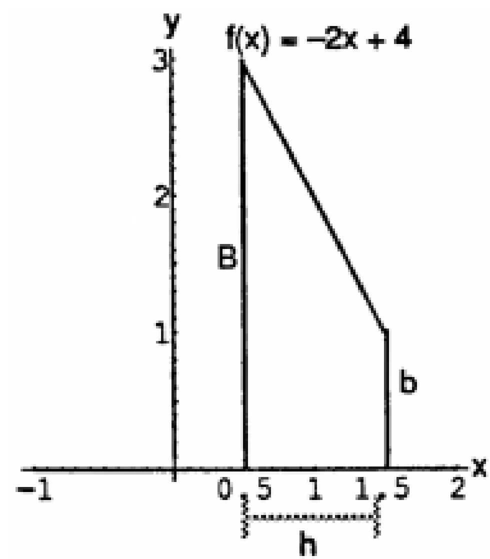
$= 21$ square units



. The area of the trapezoid is $A = \frac{1}{2}(B + b)h$

$= \frac{1}{2}(3 + 1)(1) = 2 \Rightarrow \int_{1/2}^{3/2} (-2x + 4)\, dx$

$= 2$ square units



$$\int_0^2 (3x^2 + x - 5)\,dx$$

$$\int_1^0 (3x^2 + x - 5)\,dx$$

$$\int_0^2 (3x^2 + x - 5)\,dx = 3\int_0^2 x^2\,dx + \int_0^2 x\,dx - \int_0^2 5\,dx = 3\left[\frac{2^3}{3} - \frac{0^3}{3}\right] + \left[\frac{2^2}{2} - \frac{0^2}{2}\right] - 5[2 - 0] =$$

$$\int_1^0 (3x^2 + x - 5)\,dx = -\int_0^1 (3x^2 + x - 5)\,dx = -\left[3\int_0^1 x^2\,dx + \int_0^1 x\,dx - \int_0^1 5\,dx\right]$$

$$= -\left[3\left(\frac{1^3}{3} - \frac{0^3}{3}\right) + \left(\frac{1^2}{2} - \frac{0^2}{2}\right) - 5(1 - 0)\right] = -\left(\frac{3}{2} - 5\right) = \frac{7}{2}$$

DEFINITION **The Average or Mean Value of a Function**

If f is integrable on $[a, b]$, then its **average value on** $[a, b]$, also called its **mean value**, is

$$\text{av}(f) = \frac{1}{b - a}\int_a^b f(x)\,dx.$$

Average Value

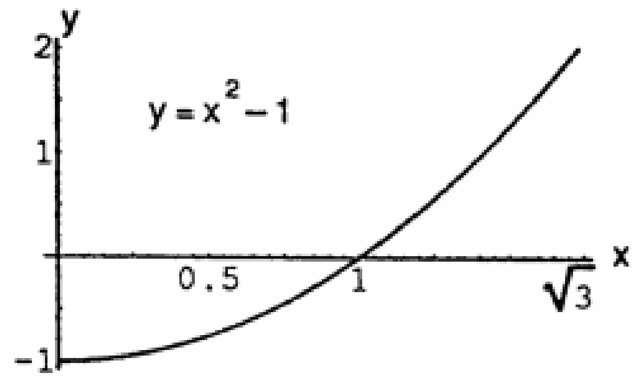
graph the function and find its average value over the given interval.

$$f(x) = x^2 - 1 \quad \text{on} \quad \left[0, \sqrt{3}\right]$$

$$\text{av}(f) = \left(\frac{1}{\sqrt{3}-0}\right)\int_0^{\sqrt{3}} (x^2 - 1)\,dx$$

$$= \frac{1}{\sqrt{3}}\int_0^{\sqrt{3}} x^2\,dx - \frac{1}{\sqrt{3}}\int_0^{\sqrt{3}} 1\,dx$$

$$= \frac{1}{\sqrt{3}}\left(\frac{\left(\sqrt{3}\right)^3}{3}\right) - \frac{1}{\sqrt{3}}\left(\sqrt{3} - 0\right) = 1 - 1 = 0.$$



If f is continuous at every point of $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Evaluate the integrals in Exercises 1–26.

1. $\displaystyle\int_{-2}^{0} (2x + 5)\,dx$

2. $\displaystyle\int_{-3}^{4} \left(5 - \frac{x}{2}\right) dx$

3. $\displaystyle\int_{0}^{4} \left(3x - \frac{x^3}{4}\right) dx$

4. $\displaystyle\int_{-2}^{2} (x^3 - 2x + 3)\,dx$

5. $\displaystyle\int_{0}^{1} (x^2 + \sqrt{x})\,dx$

6. $\displaystyle\int_{0}^{5} x^{3/2}\,dx$

7. $\displaystyle\int_{1}^{32} x^{-6/5}\,dx$

8. $\displaystyle\int_{-2}^{-1} \frac{2}{x^2}\,dx$

9. $\displaystyle\int_{0}^{\pi} \sin x\,dx$

10. $\displaystyle\int_{0}^{\pi} (1 + \cos x)\,dx$

13. $\displaystyle\int_{\pi/4}^{3\pi/4} \csc\theta \cot\theta \, d\theta$

14. $\displaystyle\int_{0}^{\pi/3} 4 \sec u \tan u \, du$

15. $\displaystyle\int_{\pi/2}^{0} \frac{1 + \cos 2t}{2} \, dt$

16. $\displaystyle\int_{-\pi/3}^{\pi/3} \frac{1 - \cos 2t}{2} \, dt$

17. $\displaystyle\int_{-\pi/2}^{\pi/2} (8y^2 + \sin y) \, dy$

18. $\displaystyle\int_{-\pi/3}^{-\pi/4} \left(4 \sec^2 t + \frac{\pi}{t^2}\right) dt$

19. $\displaystyle\int_{1}^{-1} (r + 1)^2 \, dr$

20. $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} (t + 1)(t^2 + 4) \, dt$

21. $\displaystyle\int_{\sqrt{2}}^{1} \left(\frac{u^7}{2} - \frac{1}{u^5}\right) du$

22. $\displaystyle\int_{1/2}^{1} \left(\frac{1}{v^3} - \frac{1}{v^4}\right) dv$

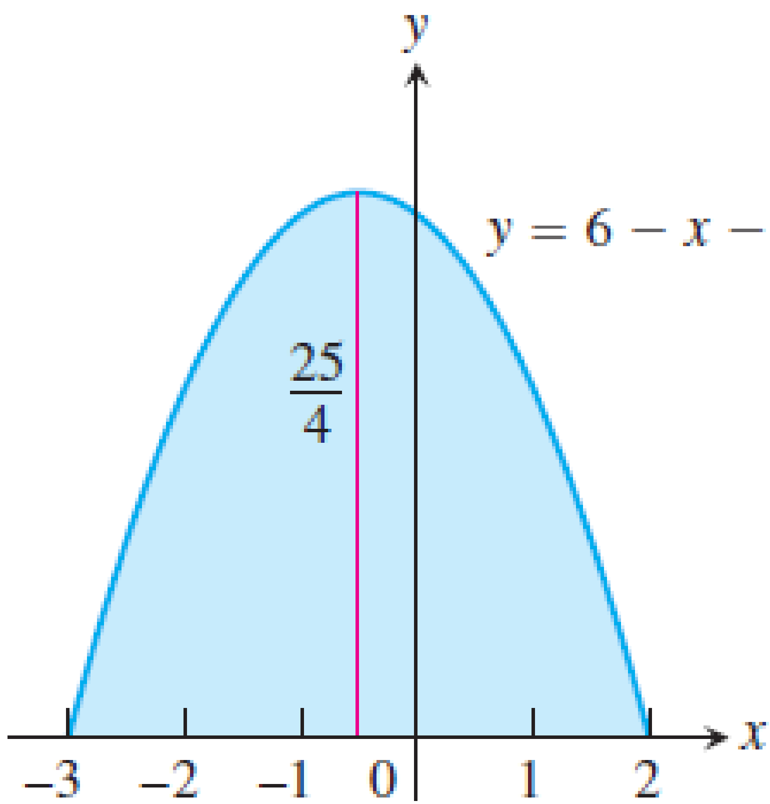
23. $\displaystyle\int_{1}^{\sqrt{2}} \frac{s^2 + \sqrt{s}}{s^2} \, ds$

24. $\displaystyle\int_{9}^{4} \frac{1 - \sqrt{u}}{\sqrt{u}} \, du$

25. $\displaystyle\int_{-4}^{4} |x| \, dx$

26. $\displaystyle\int_{0}^{\pi} \frac{1}{2}(\cos x + |\cos x|) \, dx$

Calculate the area bounded by the x-axis and the parabola $y = 6 - x - x^2$.



The area is

$$\int_{-3}^{2}(6 - x - x^2)\,dx = \left[6x - \frac{x^2}{2} - \frac{x^3}{3}\right]_{-3}^{2}$$

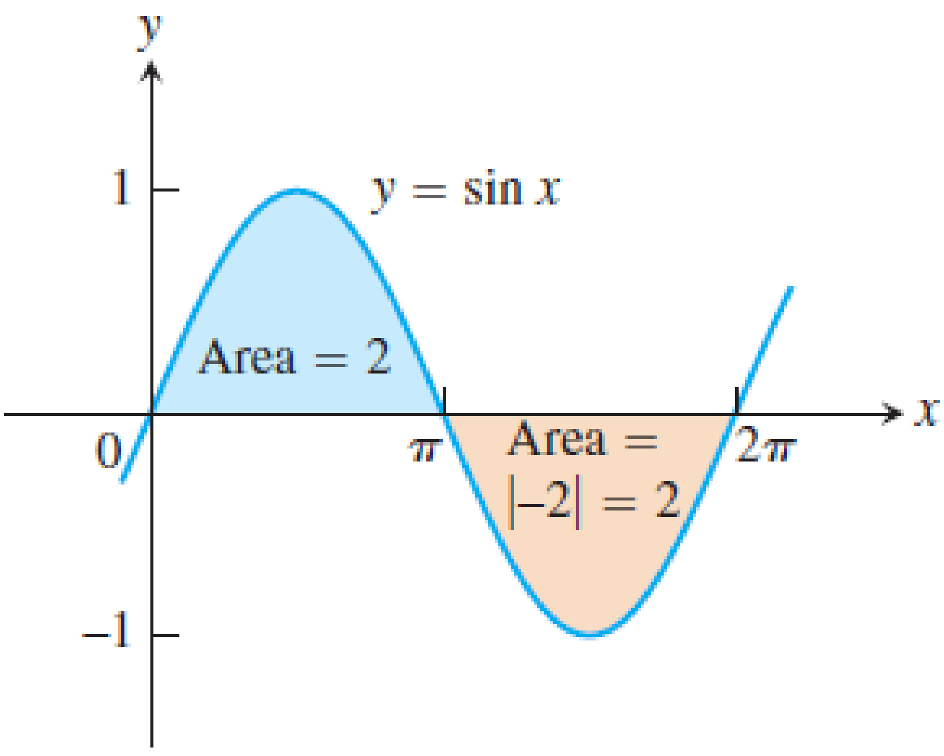
$$= \left(12 - 2 - \frac{8}{3}\right) - \left(-18 - \frac{9}{2} + \frac{27}{3}\right) = 20\tfrac{5}{6}.$$

Canceling Areas

shows the graph of the function $f(x) = \sin x$ between $x = 0$ and $x = 2\pi$. Compute

(a) the definite integral of $f(x)$ over $[0, 2\pi]$.

(b) the area between the graph of $f(x)$ and the x-axis over $[0, 2\pi]$.

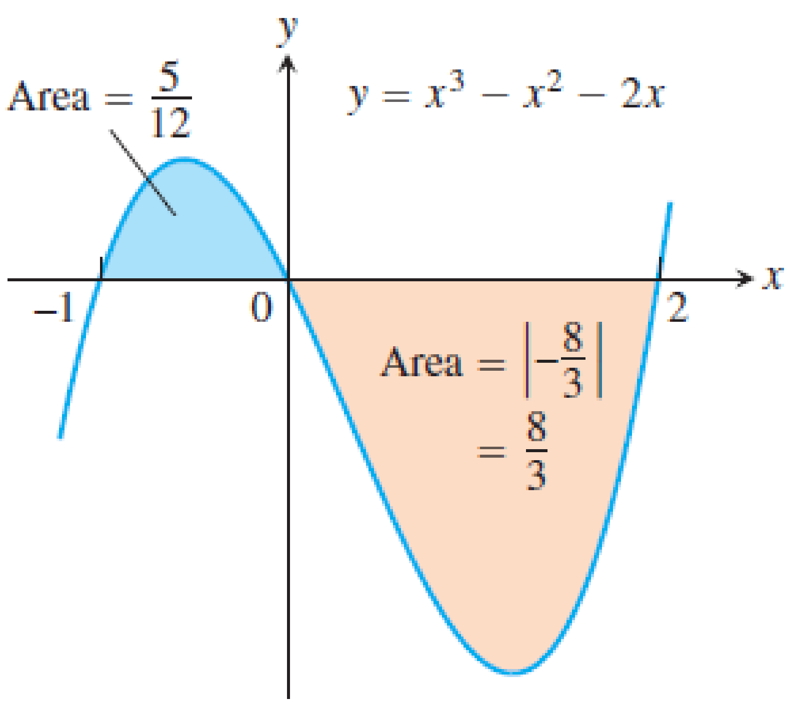


Summary:

To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$, do the following:

1. Subdivide $[a, b]$ at the zeros of f.
2. Integrate f over each subinterval.
3. Add the absolute values of the integrals.

Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \leq x \leq 2$.



Area

find the total area between the region and x-axis.

37. $y = -x^2 - 2x, \quad -3 \leq x \leq 2$

38. $y = 3x^2 - 3, \quad -2 \leq x \leq 2$

39. $y = x^3 - 3x^2 + 2x, \quad 0 \leq x \leq 2$

40. $y = x^3 - 4x, \quad -2 \leq x \leq 2$

41. $y = x^{1/3}, \quad -1 \leq x \leq 8$

42. $y = x^{1/3} - x, \quad -1 \leq x \leq 8$

5.5 Indefinite Integrals and the Substitution Rule

We must distinguish carefully between definite and indefinite integrals. A definite integral $\int_a^b f(x)\,dx$ is a *number*. An indefinite integral $\int f(x)\,dx$ is a *function* plus an arbitrary constant C.

If u is any differentiable function, then

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C \qquad (n \neq -1,\ n \text{ rational}).$$

$$\int_1^{-1} (r+1)^2\,dr$$

$r + 1 = u$

$dr = du$

$$\int \sqrt{1+y^2} \cdot 2y\,dy = \int \sqrt{u} \cdot \left(\frac{du}{dy}\right) dy$$

Let $u = 1 + y^2$, $du/dy = 2y$

$$= \int u^{1/2}\,du$$

$$= \frac{u^{(1/2)+1}}{(1/2)+1} + C$$

Integrate, using Eq. (1) with $n = 1/2$.

$$= \frac{2}{3} u^{3/2} + C$$

Simpler form

$$= \frac{2}{3}(1+y^2)^{3/2} + C$$

Replace u by $1 + y^2$. ■

THEOREM 5 The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

$$\int x^2 \sin(x^3)\,dx = \int \sin(x^3) \cdot x^2\,dx$$

$$= \int \sin u \cdot \frac{1}{3}\,du$$

Let $u = x^3$,
$du = 3x^2\,dx$,
$(1/3)\,du = x^2\,dx$.

$$= \frac{1}{3}\int \sin u\,du$$

$$= \frac{1}{3}(-\cos u) + C$$

Integrate with respect to u.

$$= -\frac{1}{3}\cos(x^3) + C$$

Replace u by x^3.

$$\int \frac{2z\,dz}{\sqrt[3]{z^2+1}} = \int \frac{du}{u^{1/3}}$$ Let $u = z^2 + 1$, $du = 2z\,dz$.

$$= \int u^{-1/3}\,du$$ In the form $\int u^n\,du$

$$= \frac{u^{2/3}}{2/3} + C$$ Integrate with respect to u.

$$= \frac{3}{2}u^{2/3} + C$$

$$= \frac{3}{2}(z^2+1)^{2/3} + C$$ Replace u by $z^2 + 1$.

$$\int \sin^2 x\, dx = \int \frac{1 - \cos 2x}{2}\, dx \qquad \sin^2 x = \frac{1 - \cos 2x}{2}$$

$$= \frac{1}{2}\int (1 - \cos 2x)\, dx = \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\, dx$$

$$= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

$$\int \cos^2 x\, dx = \int \frac{1 + \cos 2x}{2}\, dx \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$= \frac{x}{2} + \frac{\sin 2x}{4} + C$$

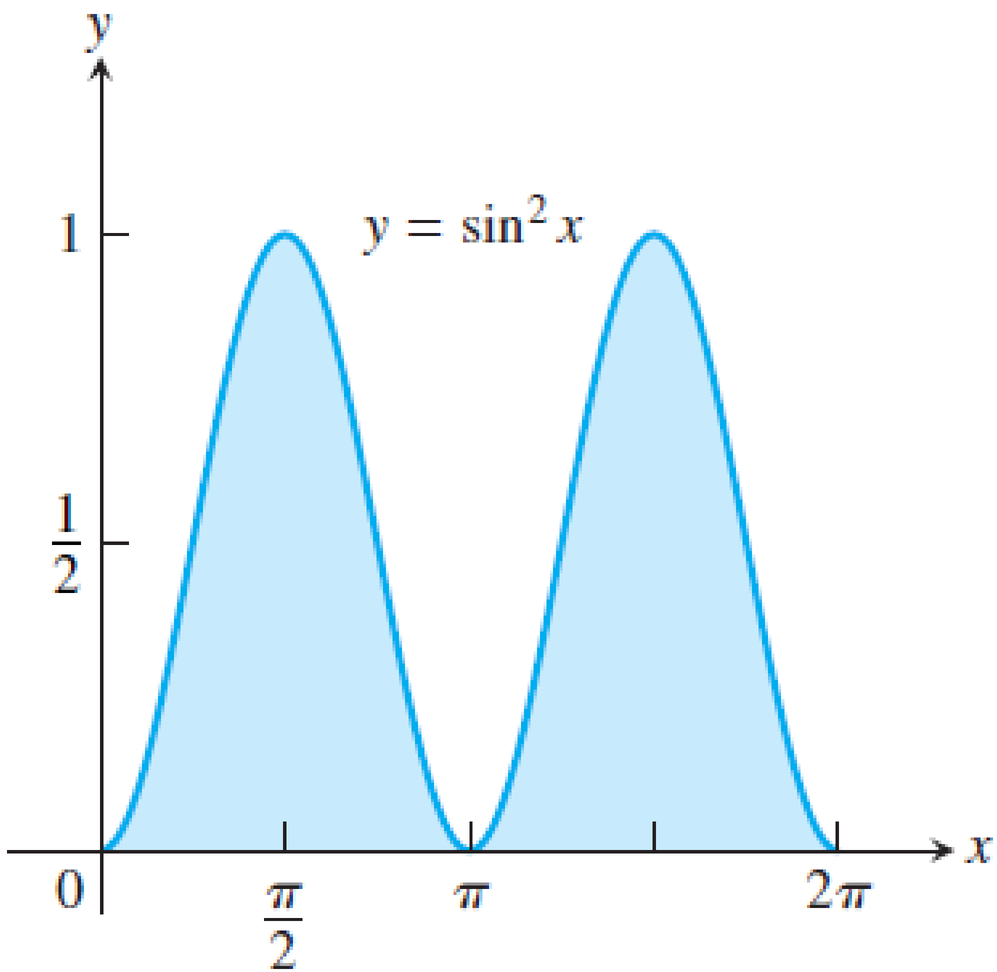
As in part (a), but with a sign change

Area Beneath the Curve $y = \sin^2 x$

Figure shows the graph of $g(x) = \sin^2 x$ over the interval $[0, 2\pi]$. Find

(a) the definite integral of $g(x)$ over $[0, 2\pi]$.

(b) the area between the graph of the function and the x-axis over $[0, 2\pi]$.



(a) From Example 7(a), the definite integral is

$$\int_0^{2\pi} \sin^2 x \, dx = \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{2\pi} = \left[\frac{2\pi}{2} - \frac{\sin 4\pi}{4}\right] - \left[\frac{0}{2} - \frac{\sin 0}{4}\right]$$

$$= [\pi - 0] - [0 - 0] = \pi.$$

(b) The function $\sin^2 x$ is nonnegative, so the area is equal to the definite integral, or π.

5. $\int 28(7x - 2)^{-5}\, dx, \quad u = 7x - 2$

6. $\int x^3(x^4 - 1)^2\, dx, \quad u = x^4 - 1$

7. $\int \frac{9r^2\, dr}{\sqrt{1 - r^3}}, \quad u = 1 - r^3$

8. $\int 12(y^4 + 4y^2 + 1)^2(y^3 + 2y)\, dy, \quad u = y^4 + 4y^2 + 1$

9. $\int \sqrt{x}\, \sin^2(x^{3/2} - 1)\, dx, \quad u = x^{3/2} - 1$

10. $\int \frac{1}{x^2} \cos^2\left(\frac{1}{x}\right) dx, \quad u = -\frac{1}{x}$

11. $\int \csc^2 2\theta \cot 2\theta\, d\theta$

a. Using $u = \cot 2\theta$

b. Using $u = \csc 2\theta$

Initial Value Problems

Solve the initial value problems in Exercises

53. $\dfrac{ds}{dt} = 12t\,(3t^2 - 1)^3, \quad s(1) = 3$

54. $\dfrac{dy}{dx} = 4x\,(x^2 + 8)^{-1/3}, \quad y(0) = 0$

55. $\dfrac{ds}{dt} = 8\sin^2\left(t + \dfrac{\pi}{12}\right), \quad s(0) = 8$

56. $\dfrac{dr}{d\theta} = 3\cos^2\left(\dfrac{\pi}{4} - \theta\right), \quad r(0) = \dfrac{\pi}{8}$

53. Let $u = 3t^2 - 1 \Rightarrow du = 6t\,dt \Rightarrow 2\,du = 12t\,dt$

$s = \int 12t\left(3t^2 - 1\right)^3 dt = \int u^3\,(2\,du) = 2\left(\frac{1}{4}u^4\right) + C = \frac{1}{2}u^4 + C = \frac{1}{2}\left(3t^2 - 1\right)^4 + C;$

$s = 3$ when $t = 1 \Rightarrow 3 = \frac{1}{2}(3-1)^4 + C \Rightarrow 3 = 8 + C \Rightarrow C = -5 \Rightarrow s = \frac{1}{2}\left(3t^2 - 1\right)^4 - 5$

54. Let $u = x^2 + 8 \Rightarrow du = 2x\,dx \Rightarrow 2\,du = 4x\,dx$

$y = \int 4x\left(x^2 + 8\right)^{-1/3} dx = \int u^{-1/3}\,(2\,du) = 2\left(\frac{3}{2}u^{2/3}\right) + C = 3u^{2/3} + C = 3\left(x^2 + 8\right)^{2/3} + C;$

$y = 0$ when $x = 0 \Rightarrow 0 = 3(8)^{2/3} + C \Rightarrow C = -12 \Rightarrow y = 3\left(x^2 + 8\right)^{2/3} - 12$

55. Let $u = t + \frac{\pi}{12} \Rightarrow du = dt$

$s = \int 8\sin^2\left(t + \frac{\pi}{12}\right) dt = \int 8\sin^2 u\,du = 8\left(\frac{u}{2} - \frac{1}{4}\sin 2u\right) + C = 4\left(t + \frac{\pi}{12}\right) - 2\sin\left(2t + \frac{\pi}{6}\right) + C;$

$s = 8$ when $t = 0 \Rightarrow 8 = 4\left(\frac{\pi}{12}\right) - 2\sin\left(\frac{\pi}{6}\right) + C \Rightarrow C = 8 - \frac{\pi}{3} + 1 = 9 - \frac{\pi}{3}$

$\Rightarrow s = 4\left(t + \frac{\pi}{12}\right) - 2\sin\left(2t + \frac{\pi}{6}\right) + 9 - \frac{\pi}{3} = 4t - 2\sin\left(2t + \frac{\pi}{6}\right) + 9$

5.6 Substitution and Area Between Curves

THEOREM 6 **Substitution in Definite Integrals**

If g' is continuous on the interval $[a, b]$ and f is continuous on the range of g, then

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

$$\begin{aligned}
\int_a^b f(g(x)) \cdot g'(x)\, dx &= F(g(x))\Big]_{x=a}^{x=b} \\
&= F(g(b)) - F(g(a)) \\
&= F(u)\Big]_{u=g(a)}^{u=g(b)} \\
&= \int_{g(a)}^{g(b)} f(u)\, du.
\end{aligned}$$

$\frac{d}{dx}F(g(x))$
$= F'(g(x))g'(x)$
$= f(g(x))g'(x)$

Fundamental Theorem, Part 2

`Method 1:` Transform the integral and evaluate the transformed integral with the transformed limits given in Theorem 6.

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$$

$$= \int_{0}^{2} \sqrt{u}\,du$$

Let $u = x^3 + 1$, $du = 3x^2\,dx$.
When $x = -1$, $u = (-1)^3 + 1 = 0$.
When $x = 1$, $u = (1)^3 + 1 = 2$.

$$= \frac{2}{3}u^{3/2}\Big]_0^2$$

Evaluate the new definite integral.

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

Method 2: Transform the integral as an indefinite integral, integrate, change back to x, and use the original x-limits.

$$\int 3x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\,du \qquad \text{Let } u = x^3+1,\ du = 3x^2\,dx.$$

$$= \frac{2}{3}u^{3/2} + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{2}{3}(x^3+1)^{3/2} + C \qquad \text{Replace } u \text{ by } x^3+1.$$

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \frac{2}{3}(x^3+1)^{3/2}\Big]_{-1}^{1} \qquad \text{Use the integral just found, with limits of integration for } x.$$

$$= \frac{2}{3}\left[((1)^3+1)^{3/2} - ((-1)^3+1)^{3/2}\right]$$

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

■

Theorem 7

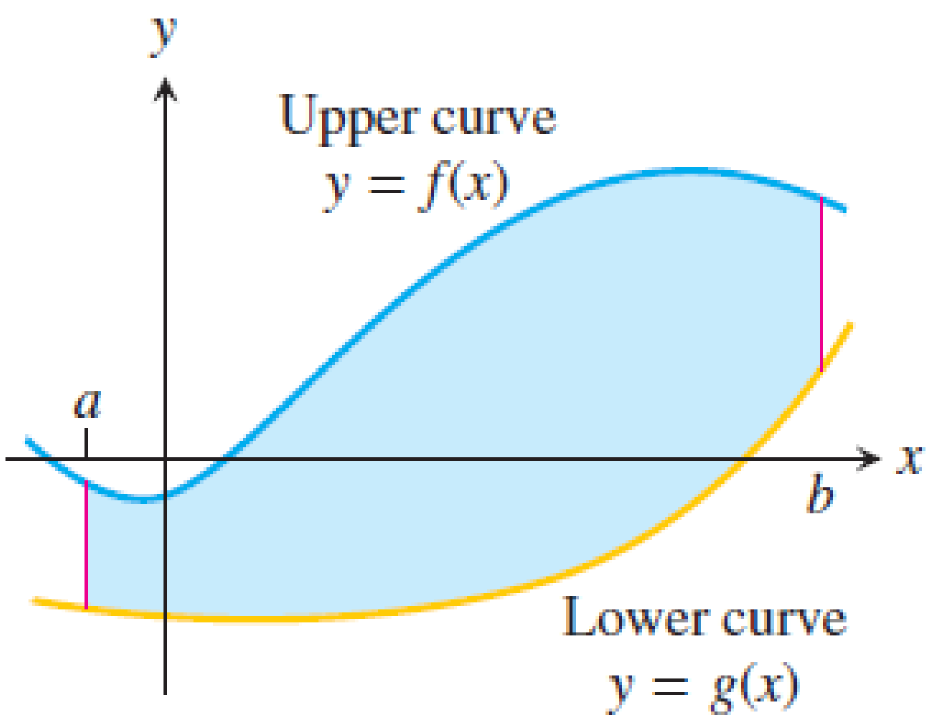
Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

Areas Between Curves

Suppose we want to find the area of a region that is bounded above by the curve $y = f(x)$, below by the curve $y = g(x)$, and on the left and right by the lines $x = a$ and $x = b$

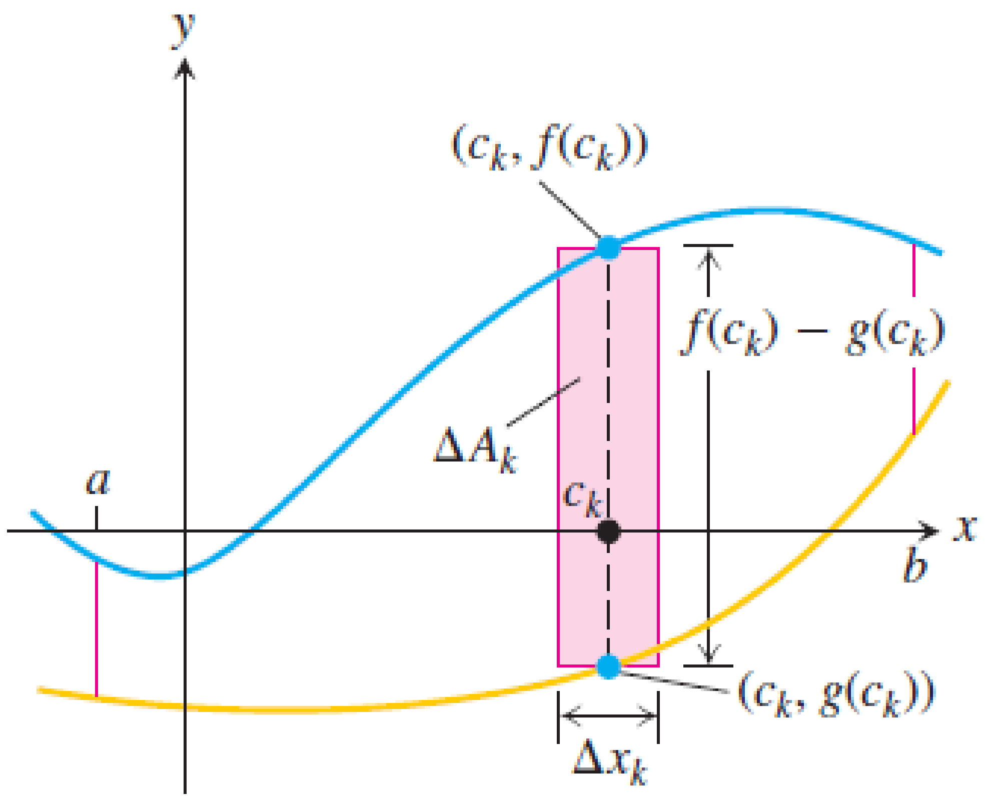


The region between the curves $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$.

DEFINITION **Area Between Curves**

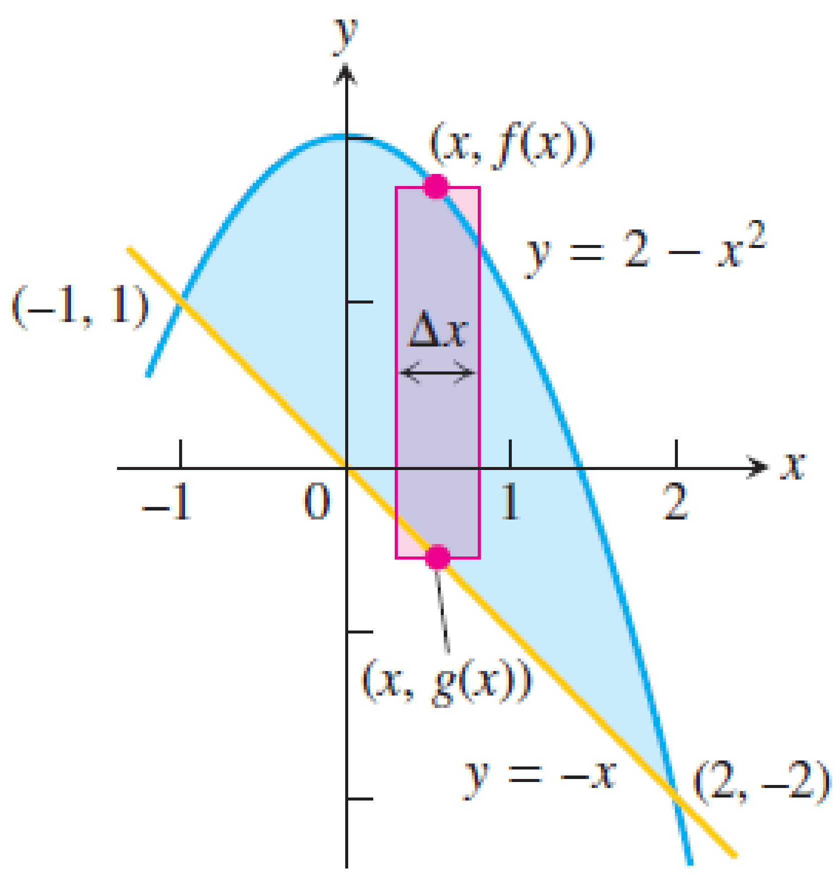
If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\, dx.$$



Area Between Intersecting Curves

Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

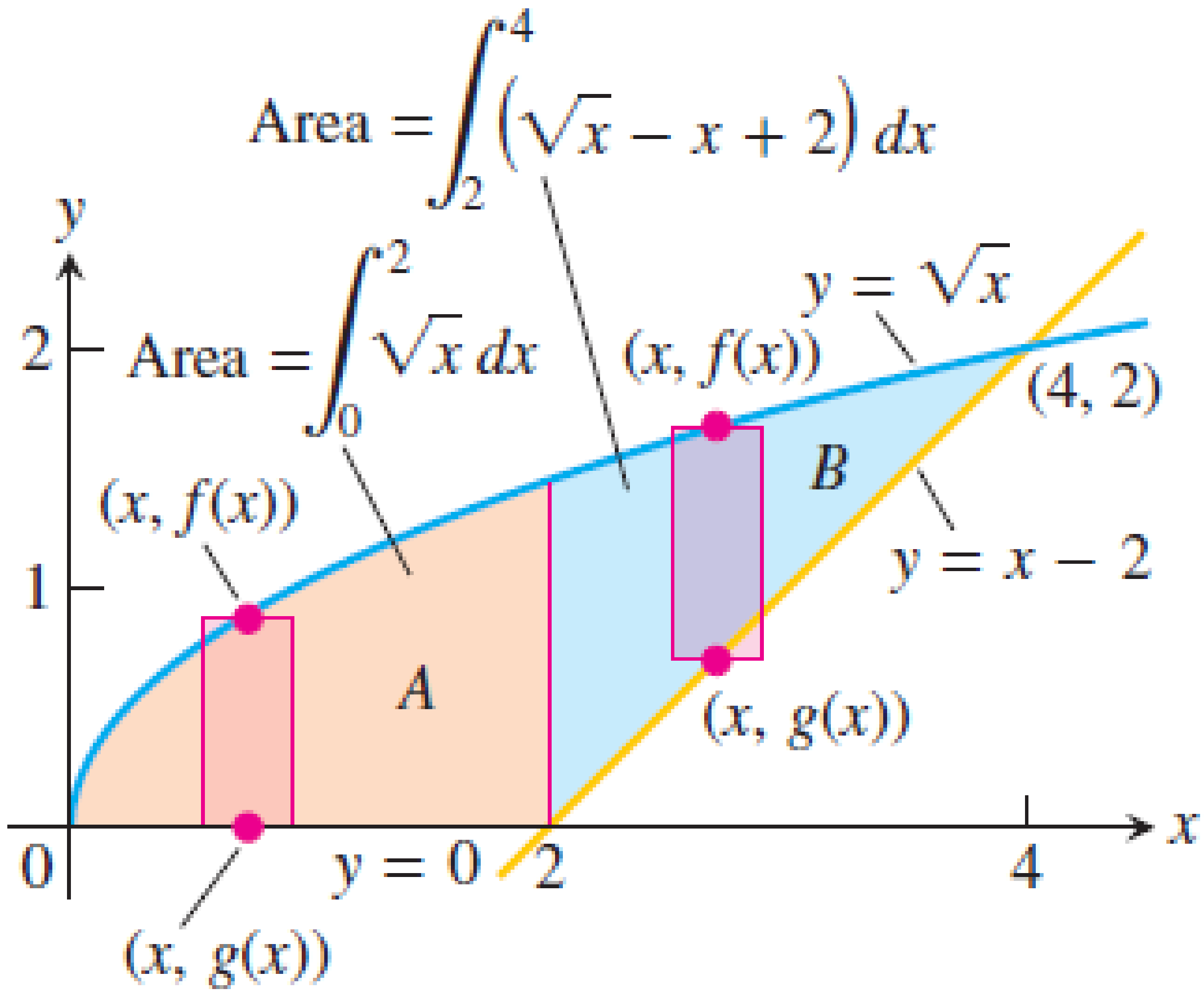


The region runs from $x = -1$ to $x = 2$. The limits of integration are $a = -1$, $b = 2$.
The area between the curves is

$$A = \int_a^b [f(x) - g(x)]\,dx = \int_{-1}^{2} [(2 - x^2) - (-x)]\,dx$$
$$= \int_{-1}^{2} (2 + x - x^2)\,dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2}$$
$$= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2}$$

Changing the Integral to Match a Boundary Change

Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.



$$\text{Total area} = \underbrace{\int_0^2 \sqrt{x}\,dx}_{\text{area of } A} + \underbrace{\int_2^4 (\sqrt{x} - x + 2)\,dx}_{\text{area of } B}$$

$$= \left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4$$

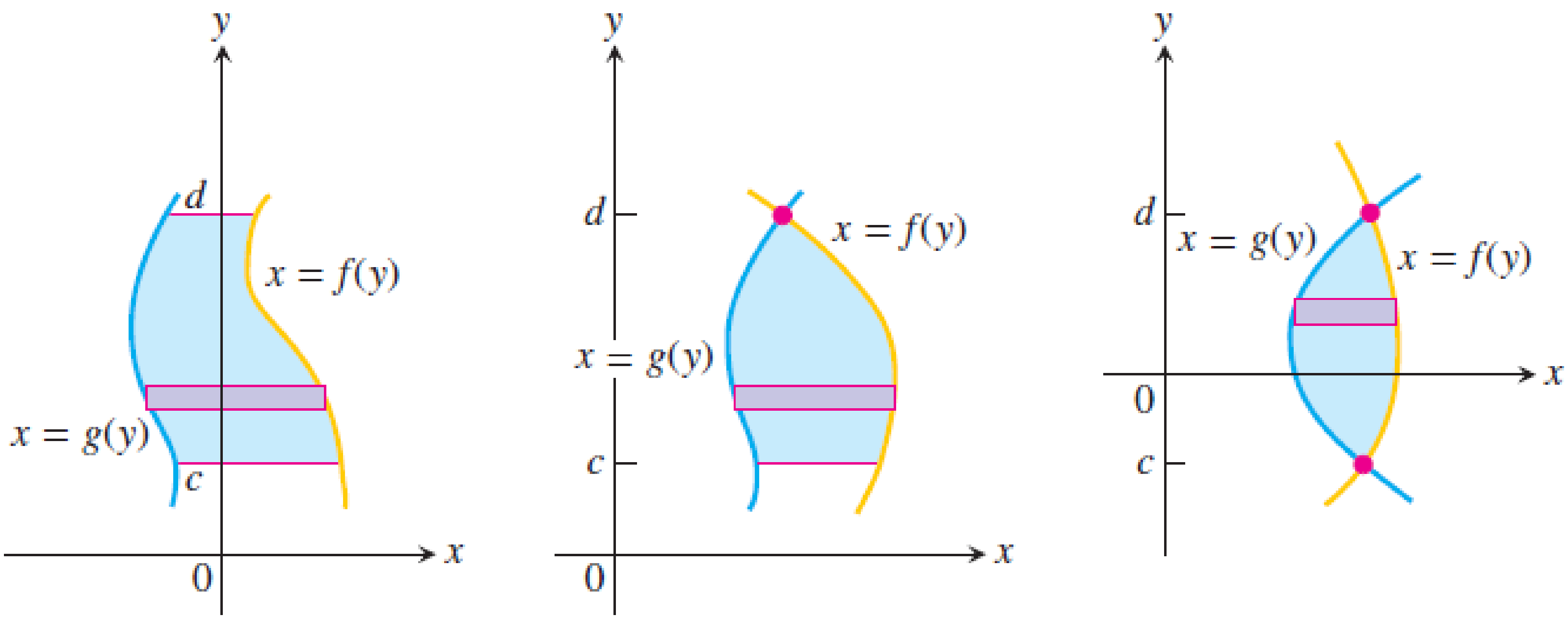
$$= \frac{2}{3}(2)^{3/2} - 0 + \left(\frac{2}{3}(4)^{3/2} - 8 + 8\right) - \left(\frac{2}{3}(2)^{3/2} - 2 + 4\right)$$

$$= \frac{2}{3}(8) - 2 = \frac{10}{3}.$$

Integration with Respect to y

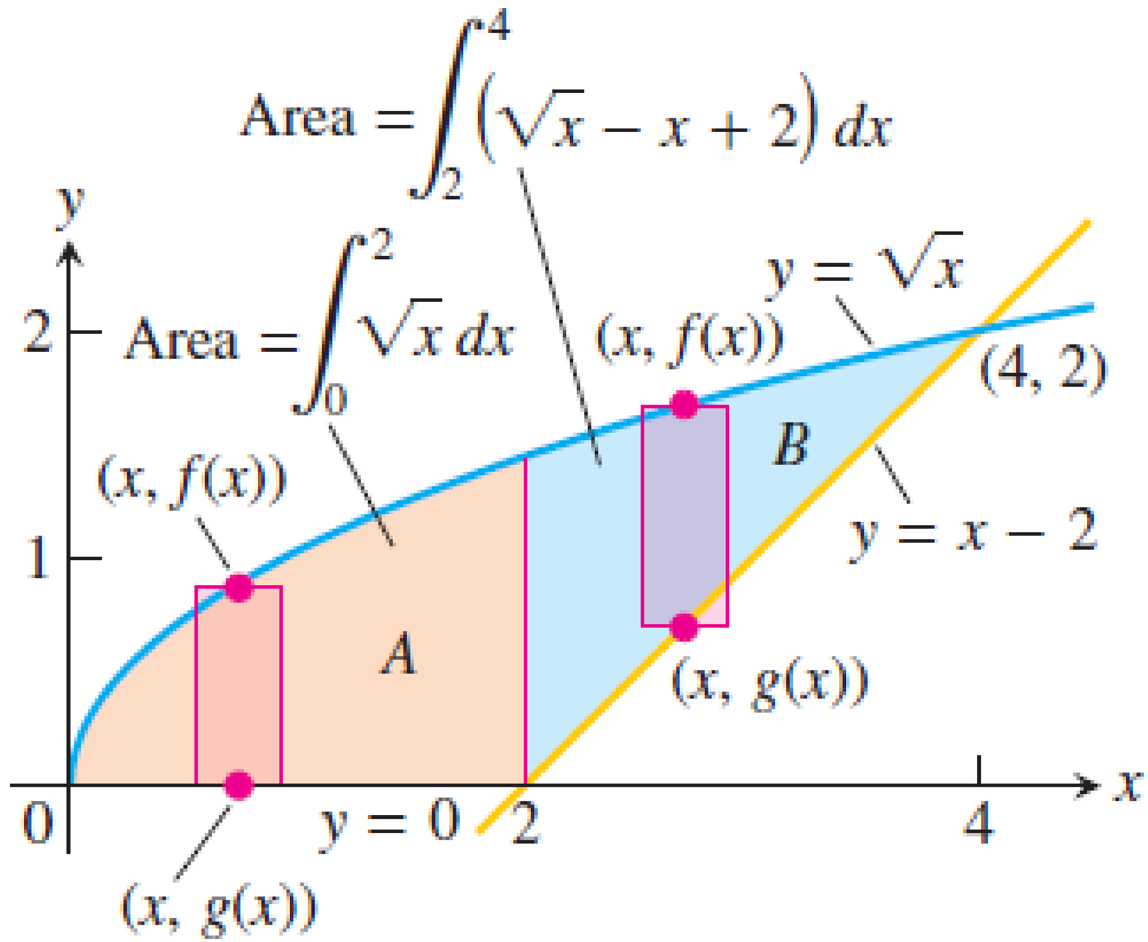
If a region's bounding curves are described by functions of y, the approximating rectangles are horizontal instead of vertical and the basic formula has y in place of x.

For regions like these

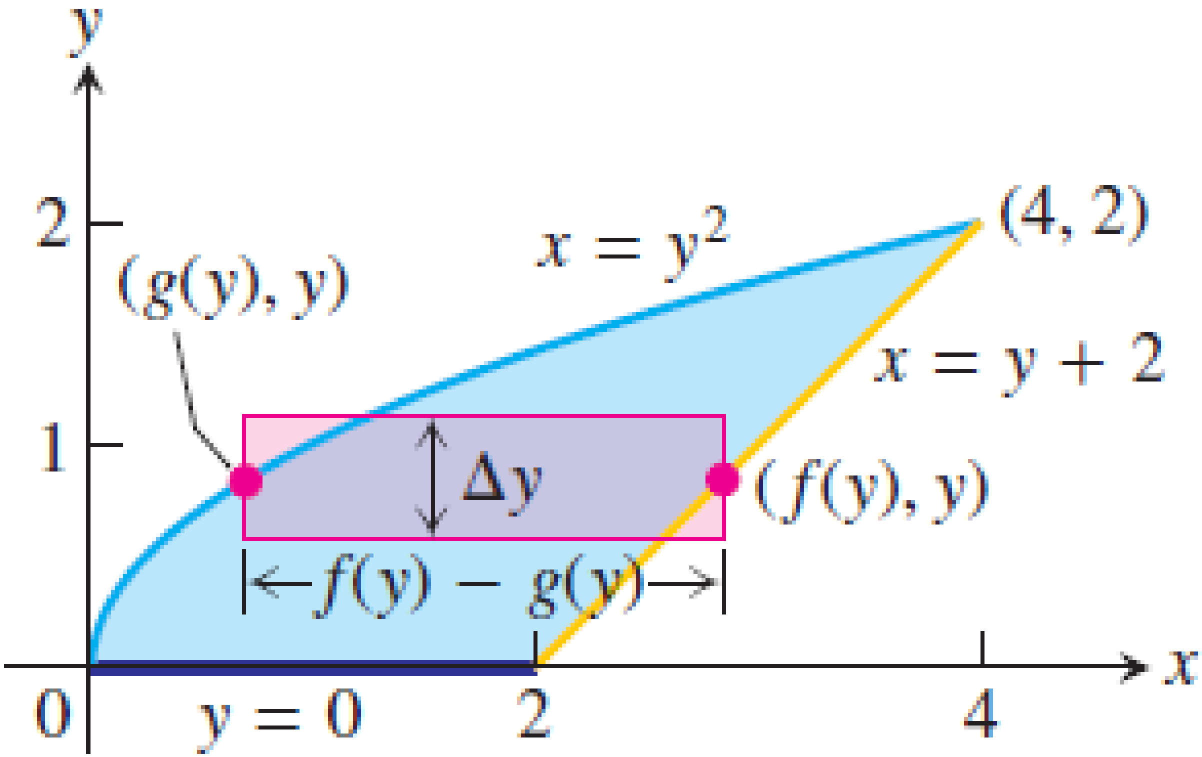


use the formula

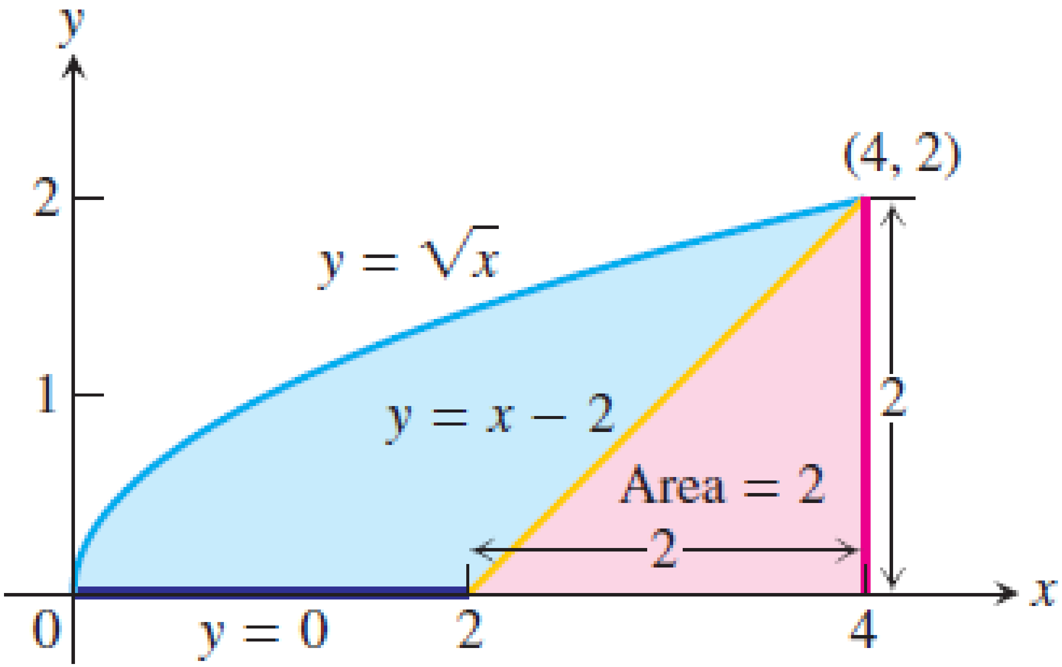
$$A = \int_c^d [f(y) - g(y)]\, dy.$$



$$\text{Total area} = \underbrace{\int_0^2 \sqrt{x}\,dx}_{\text{area of } A} + \underbrace{\int_2^4 (\sqrt{x} - x + 2)\,dx}_{\text{area of } B}$$



$$A = \int_a^b [f(y) - g(y)]\,dy = \int_0^2 [y + 2 - y^2]\,dy$$

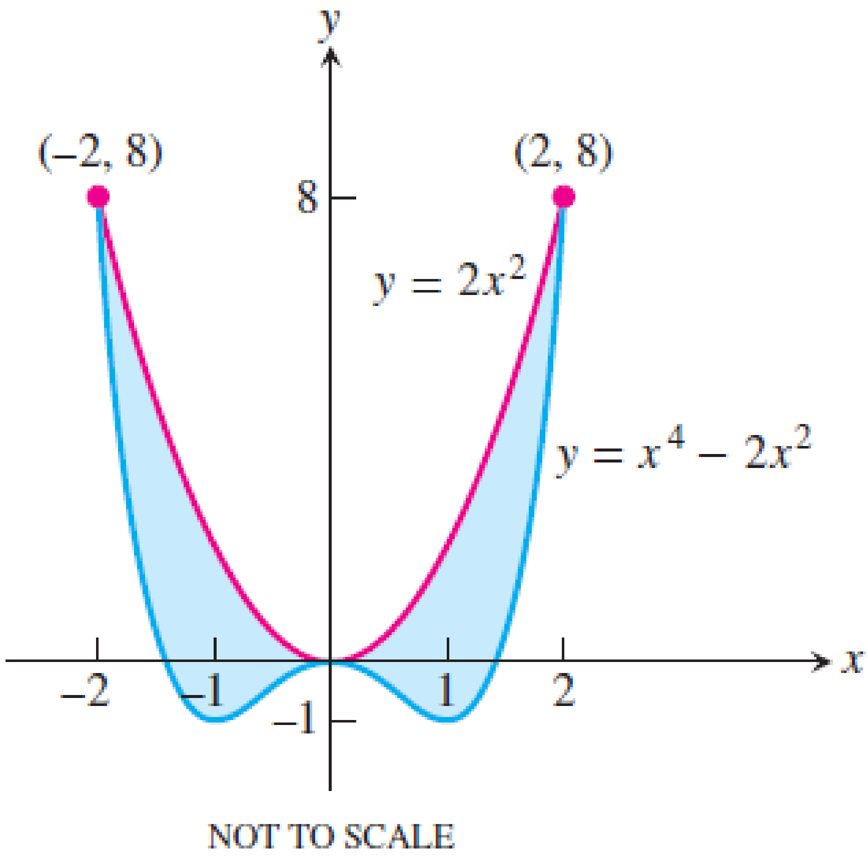


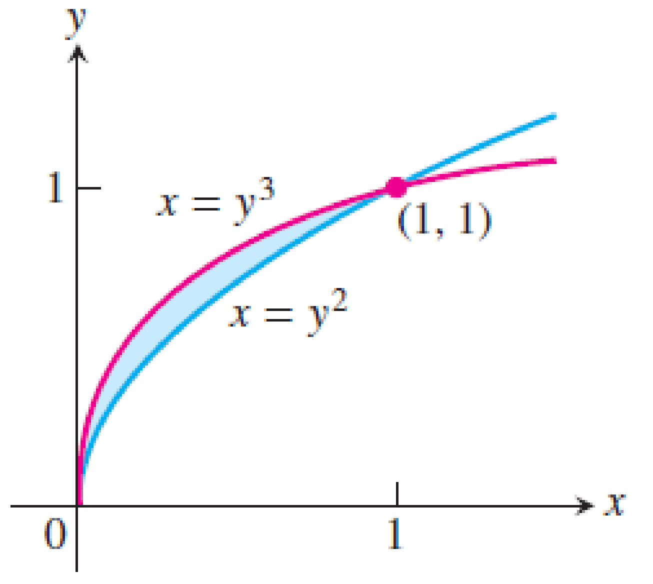
$$\text{Area} = \int_0^4 \sqrt{x}\,dx - \frac{1}{2}(2)(2)$$

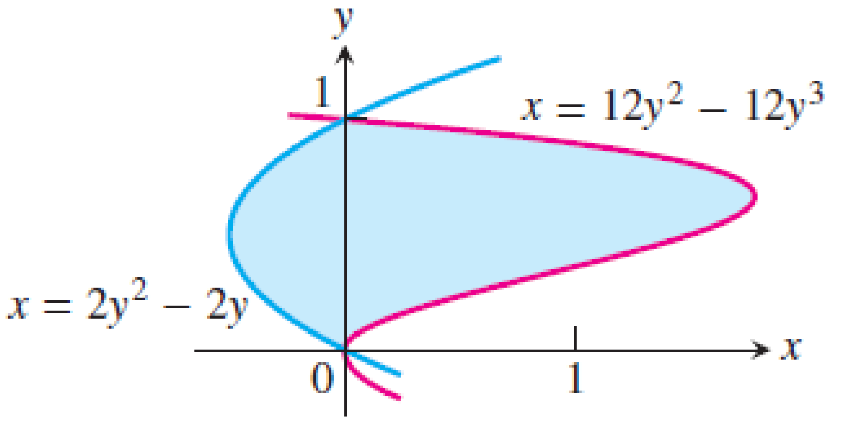
$$= \frac{2}{3}x^{3/2}\Big]_0^4 - 2$$

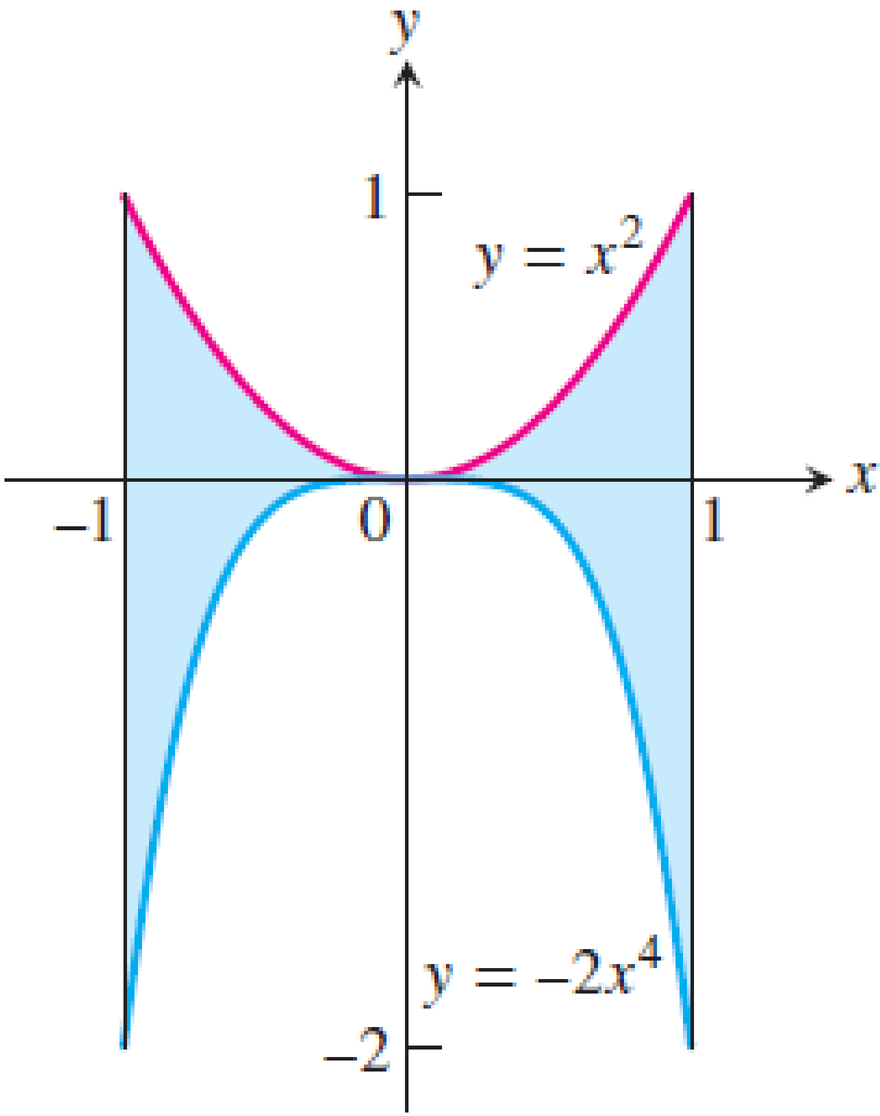
$$= \frac{2}{3}(8) - 0 - 2 = \frac{10}{3}.$$

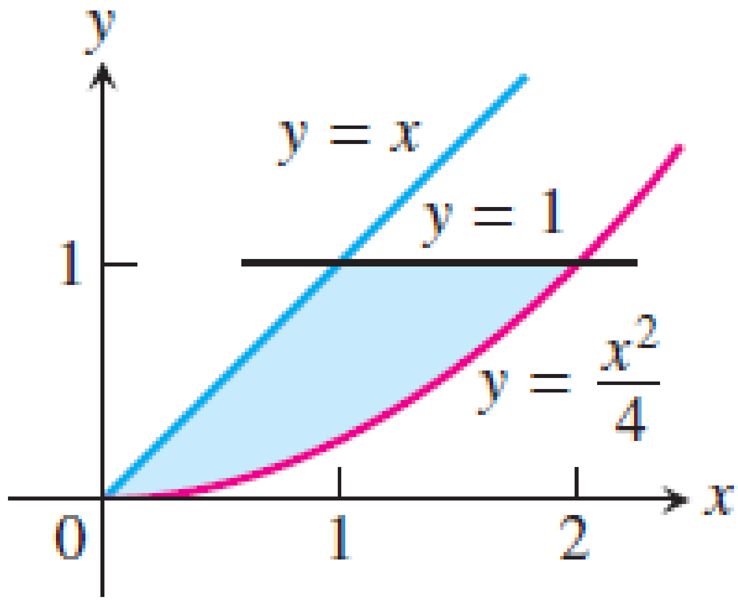
Find the total areas of the shaded regions



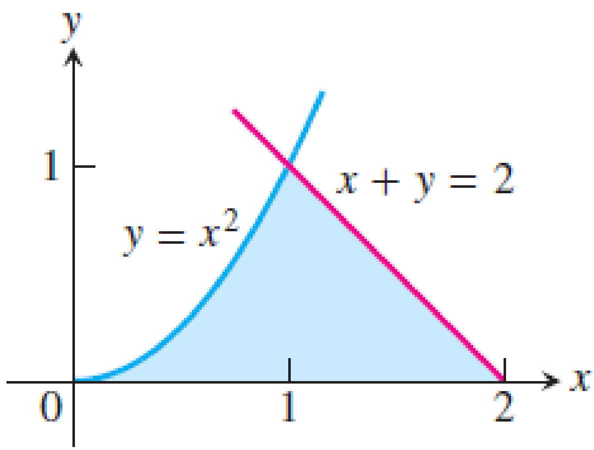




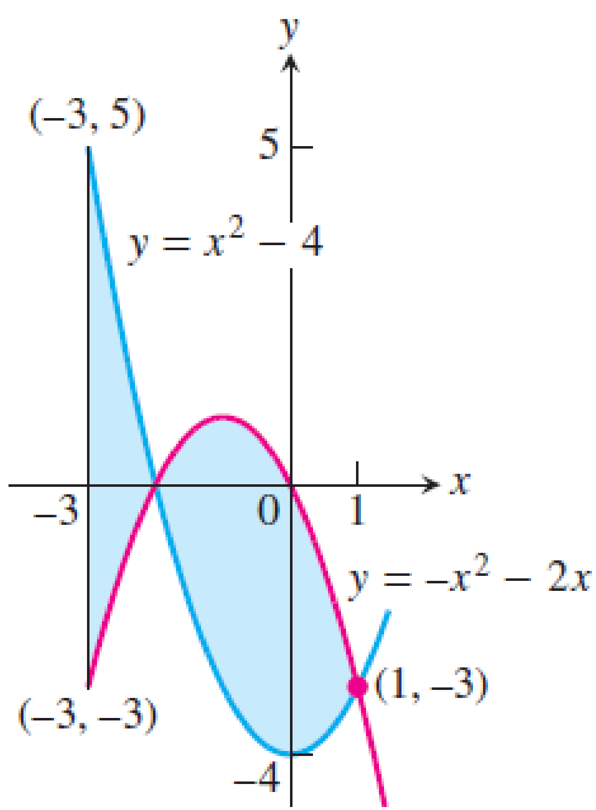

y
1
$y = x^2$
x
−1
0
1
$y = -2x^4$
−2


y
$y = x$
$y = 1$
1
$y = \frac{x^2}{4}$
x
0
1
2

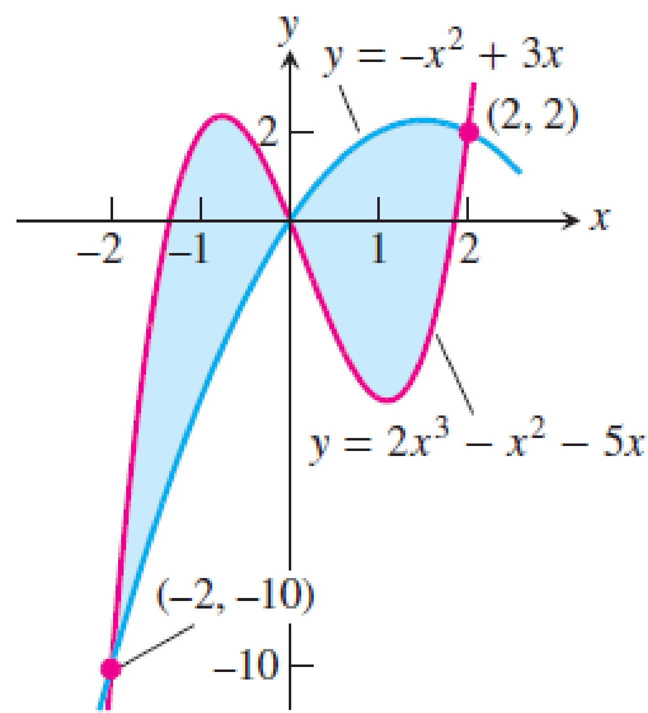
36.


y
1
x + y = 2
y = x²
0
1
2
x

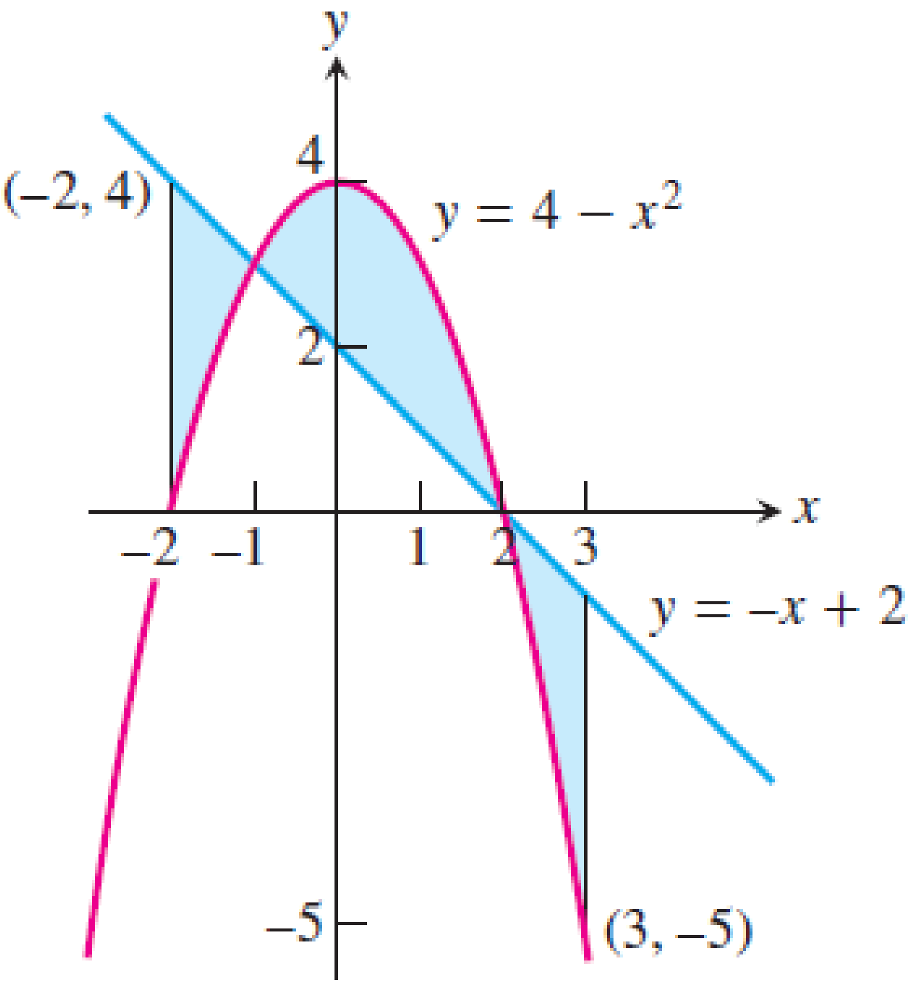
37.


y
(−3, 5)
5
y = x² − 4
−3
0
1
x
y = −x² − 2x
(1, −3)
(−3, −3)
−4

38.

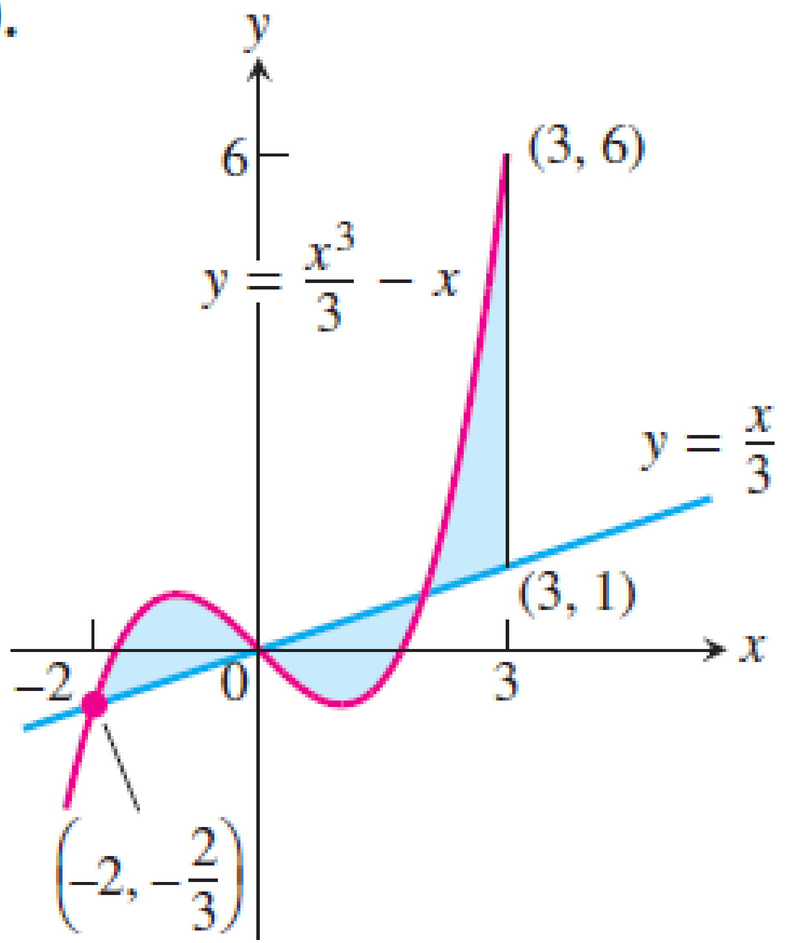

y
y = −x² + 3x
(2, 2)
2
−2
−1
1
2
x
y = 2x³ − x² − 5x
(−2, −10)
−10

39.



y
4
2
–5
(–2, 4)
$y = 4 - x^2$
–2
–1
1
2
3
x
$y = -x + 2$
(3, –5)

40.



y
6
(3, 6)
$y = \frac{x^3}{3} - x$
$y = \frac{x}{3}$
(3, 1)
–2
0
3
x
$\left(-2, -\frac{2}{3}\right)$

Find the areas of the regions enclosed by the lines and curves

41. $y = x^2 - 2 \quad \text{and} \quad y = 2$

42. $y = 2x - x^2 \quad \text{and} \quad y = -3$

43. $y = x^4 \quad \text{and} \quad y = 8x$

44. $y = x^2 - 2x \quad \text{and} \quad y = x$

45. $y = x^2 \quad \text{and} \quad y = -x^2 + 4x$

46. $y = 7 - 2x^2 \quad \text{and} \quad y = x^2 + 4$

47. $y = x^4 - 4x^2 + 4 \quad \text{and} \quad y = x^2$

Find the areas of the regions enclosed by the curves in Exercises

59. $4x^2 + y = 4$ and $x^4 - y = 1$

60. $x^3 - y = 0$ and $3x^2 - y = 4$

61. $x + 4y^2 = 4$ and $x + y^4 = 1$, for $x \geq 0$

62. $x + y^2 = 3$ and $4x + y^2 = 0$

77. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the line $y = x/4$, above left by the curve $y = 1 + \sqrt{x}$, and above right by the curve $y = 2/\sqrt{x}$.

78. Find the area of the region in the first quadrant bounded on the left by the y-axis, below by the curve $x = 2\sqrt{y}$, above left by the curve $x = (y - 1)^2$, and above right by the line $x = 3 - y$.