

APPLICATIONS OF DEFINITE INTEGRALS

In this chapter applications volumes, lengths of plane curves, centers of mass, areas of surfaces of revolution, work, and fluid forces against planar walls.

6.1

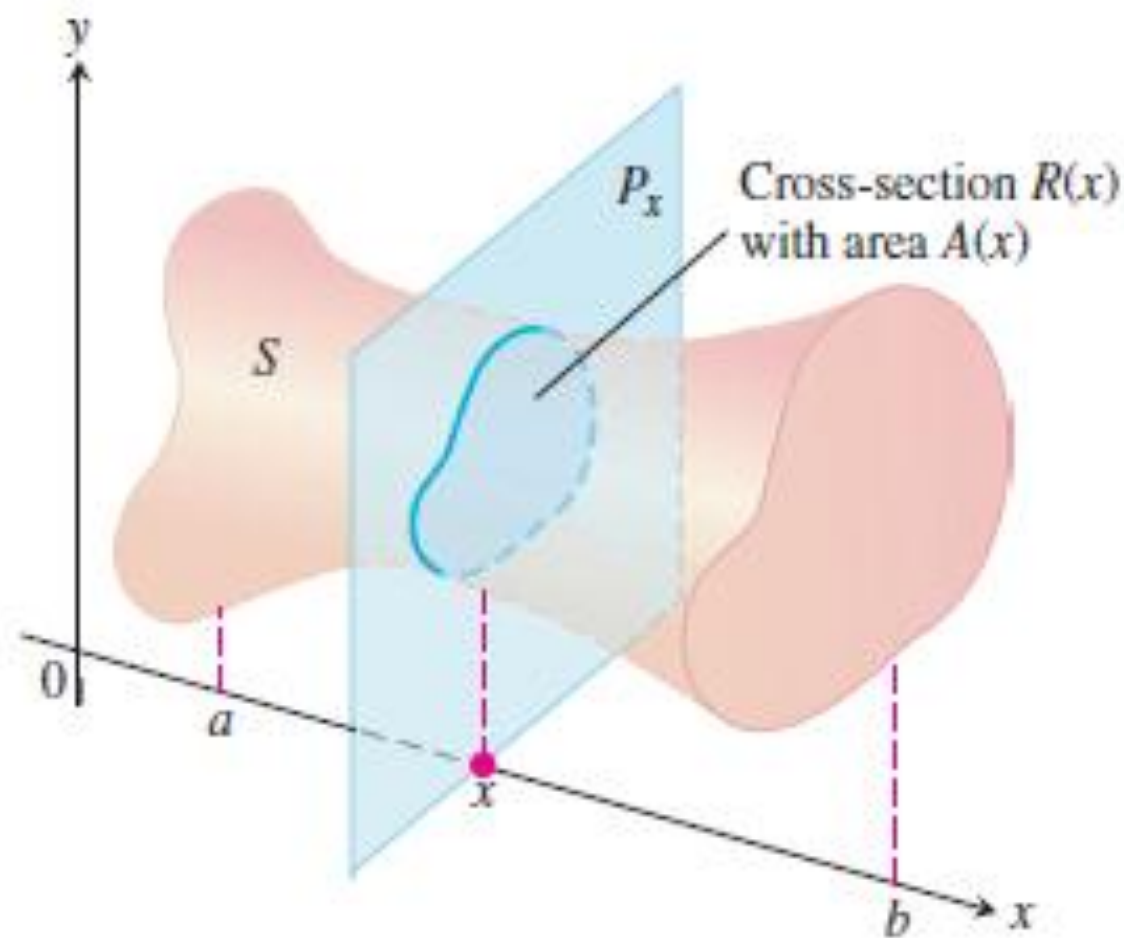
Volumes by Slicing and Rotation About an Axis

If the cylindrical solid has a known base area A and height h , then the volume of the cylindrical solid is

$$\text{Volume} = \text{area} \times \text{height} = A \cdot h.$$

This equation forms the basis for defining the volumes of many solids that are not cylindrical by the *method of slicing*.

If the cross-section of the solid S at each point x in the interval $[a, b]$ is a region $R(x)$ of area $A(x)$, and A is a continuous function of x , we can define and calculate the volume of the solid S as a definite integral in the following way.



A cross-section of the solid S formed by intersecting S with a plane P_x perpendicular to the x -axis through the point x in the interval $[a, b]$.

DEFINITION Volume

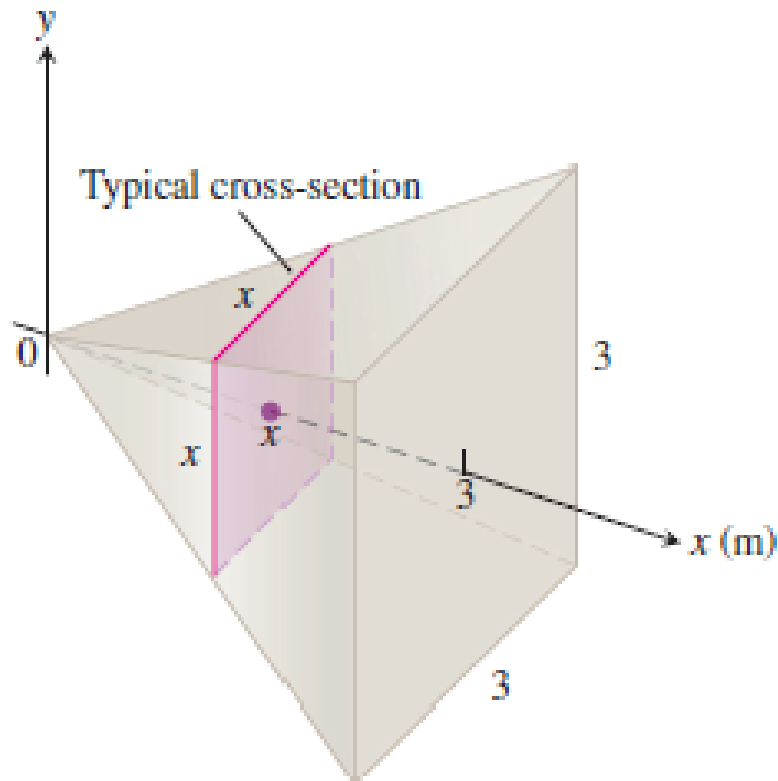
The volume of a solid of known integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b ,

$$V = \int_a^b A(x) dx.$$

Calculating the Volume of a Solid

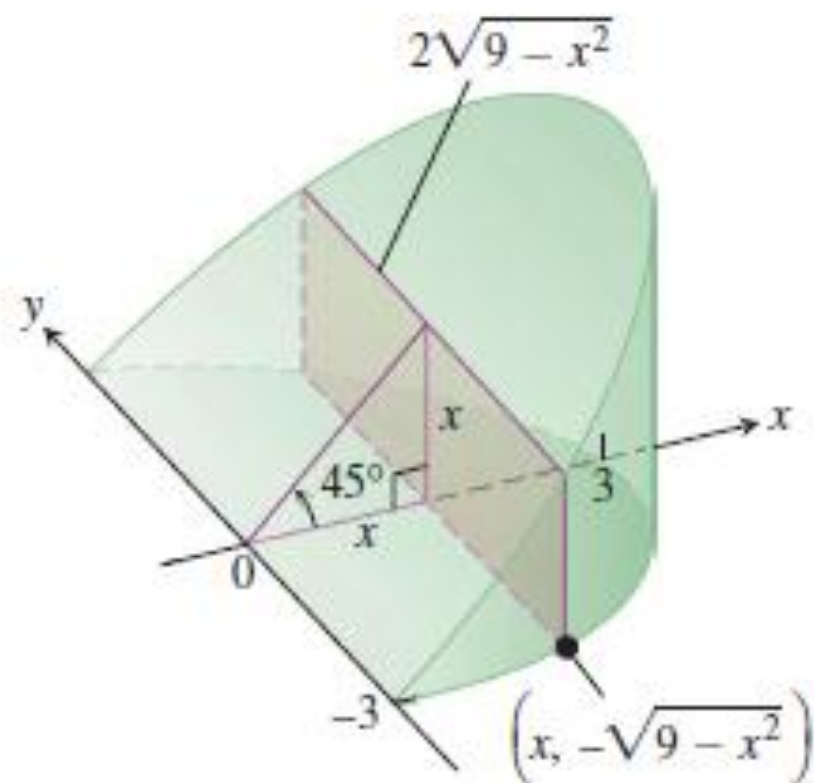
1. *Sketch the solid and a typical cross-section.*
2. *Find a formula for $A(x)$, the area of a typical cross-section.*
3. *Find the limits of integration.*
4. *Integrate $A(x)$ using the Fundamental Theorem.*

A pyramid 3 m high has a square base that is 3 m on a side. The cross-section of the pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the volume of the pyramid.



$$V = \int_0^3 A(x) dx = \int_0^3 x^2 dx = \left. \frac{x^3}{3} \right]_0^3 = 9 \text{ m}^3$$

A curved wedge is cut from a cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.



$$\begin{aligned}
 V &= \int_a^b A(x) dx = \int_0^3 2x\sqrt{9-x^2} dx \\
 &= -\frac{2}{3}(9-x^2)^{3/2} \Big|_0^3 \\
 &= 0 + \frac{2}{3}(9)^{3/2} \\
 &= 18.
 \end{aligned}$$

Solids of Revolution: The Disk Method

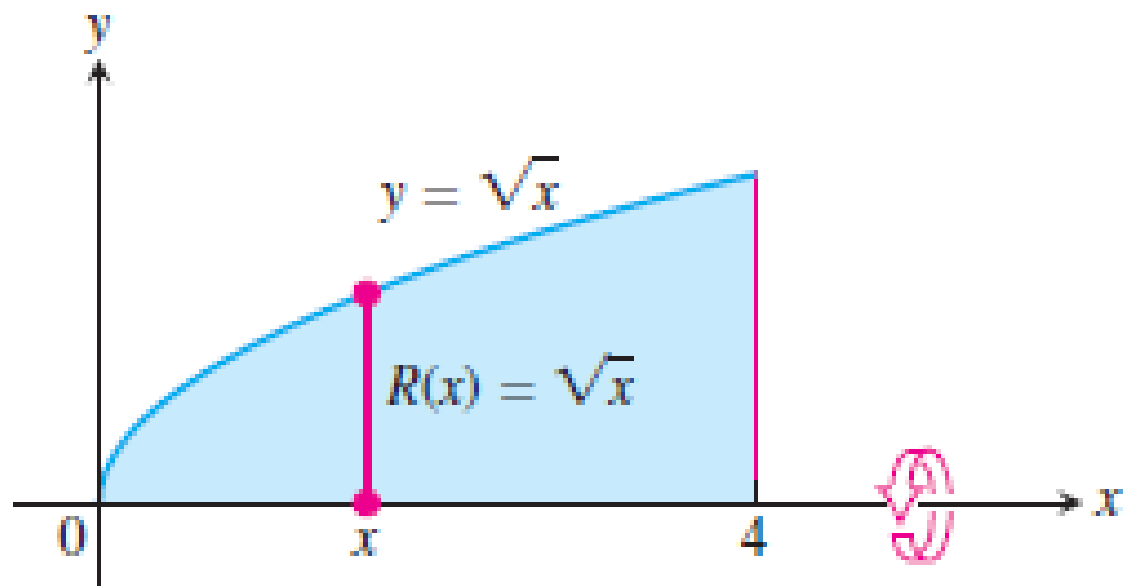
The solid generated by rotating a plane region about an axis in its plane is called a **solid of revolution**. To find the volume of a solid we need only observe that the cross-sectional area $A(x)$ is the area of a disk of radius $R(x)$, the distance of the planar region's boundary from the axis of revolution. The area is then

$$A(x) = \pi(\text{radius})^2 = \pi[R(x)]^2.$$

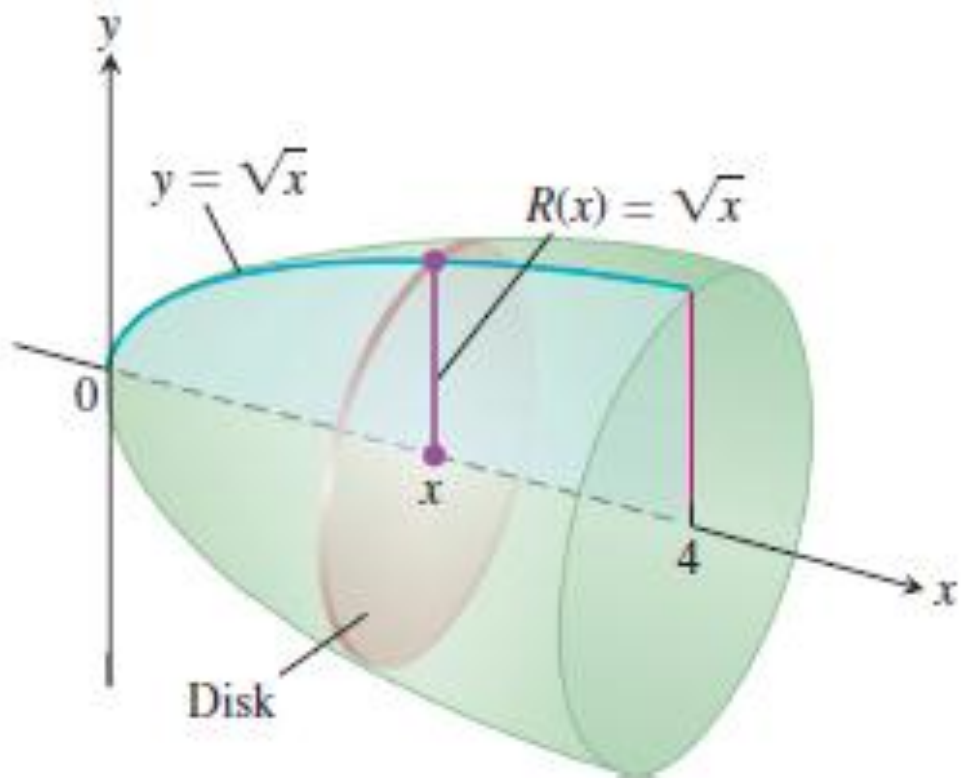
So the definition of volume gives

$$V = \int_a^b A(x) dx = \int_a^b \pi[R(x)]^2 dx.$$

The region between the curve $y = \sqrt{x}$, $0 \leq x \leq 4$, and the x -axis is revolved about the x -axis to generate a solid. Find its volume.

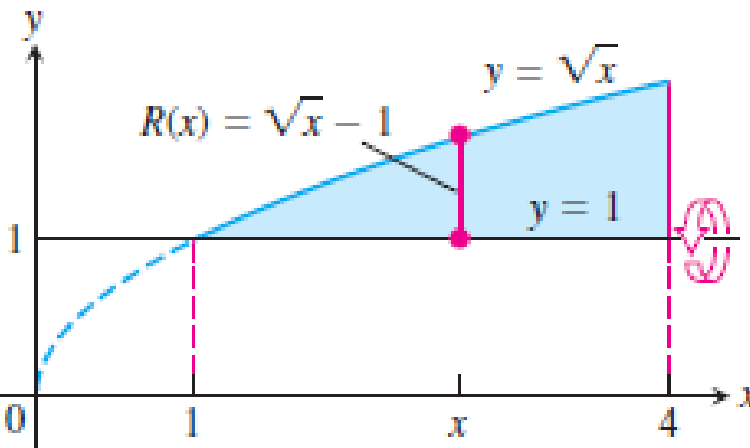


$$\begin{aligned} V &= \int_a^b \pi [R(x)]^2 dx \\ &= \int_0^4 \pi [\sqrt{x}]^2 dx \end{aligned}$$

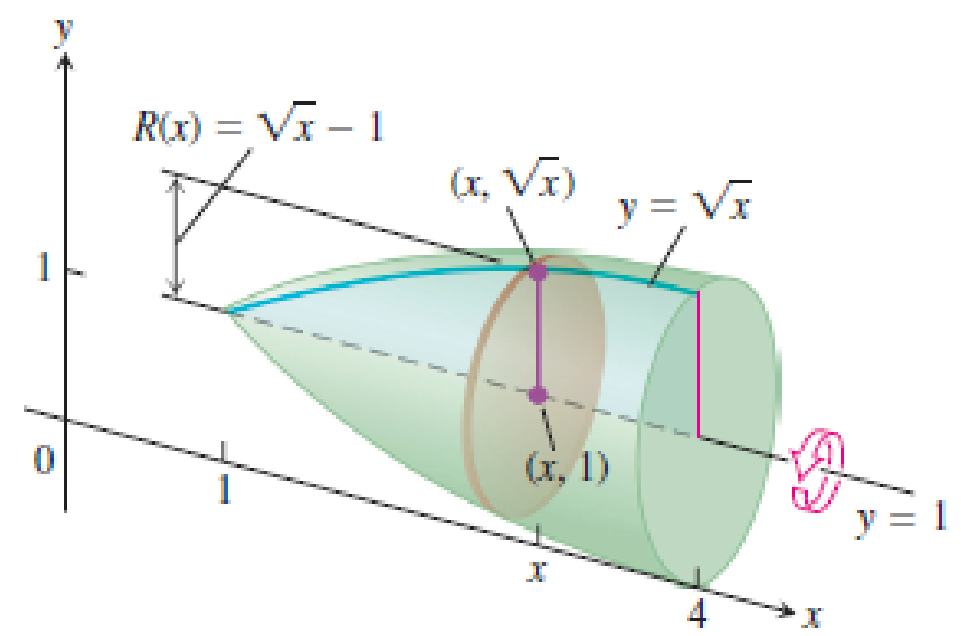


$$= \pi \int_0^4 x \, dx = \pi \left. \frac{x^2}{2} \right]_0^4 = \pi \frac{(4)^2}{2} = 8\pi.$$

Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1, x = 4$ about the line $y = 1$.



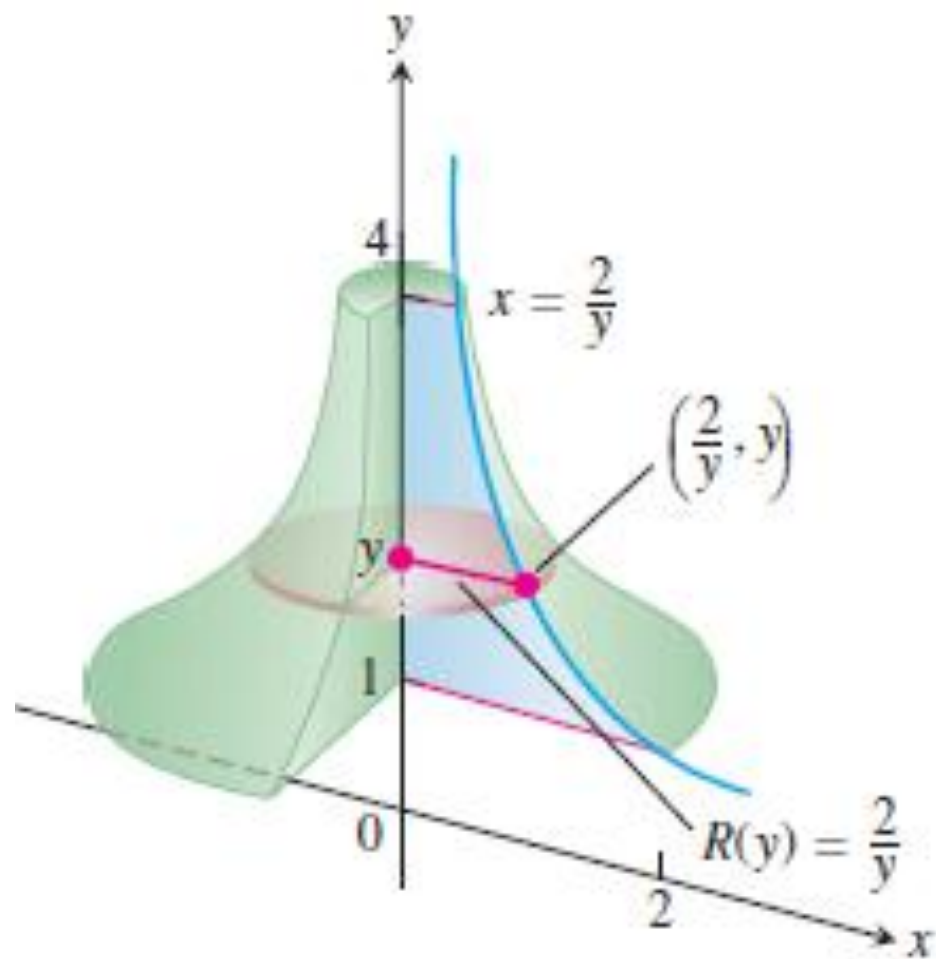
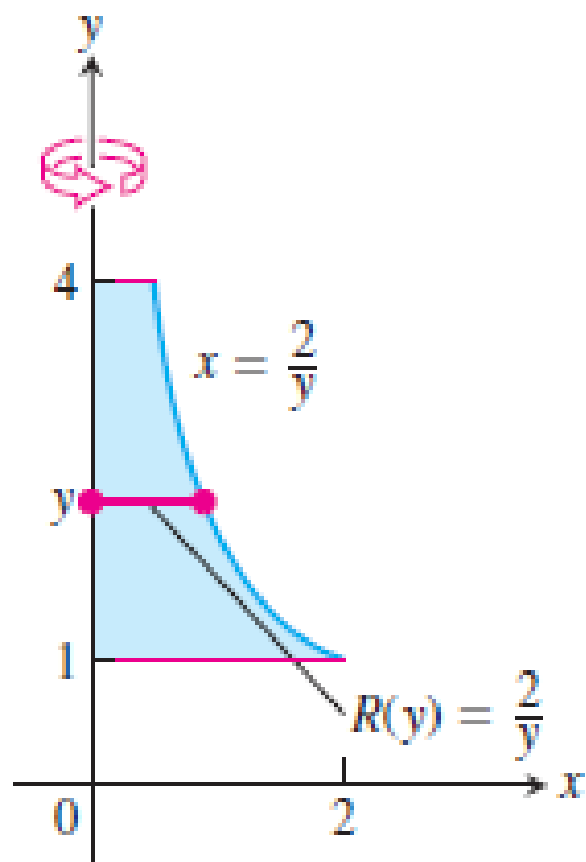
(a)



(b)

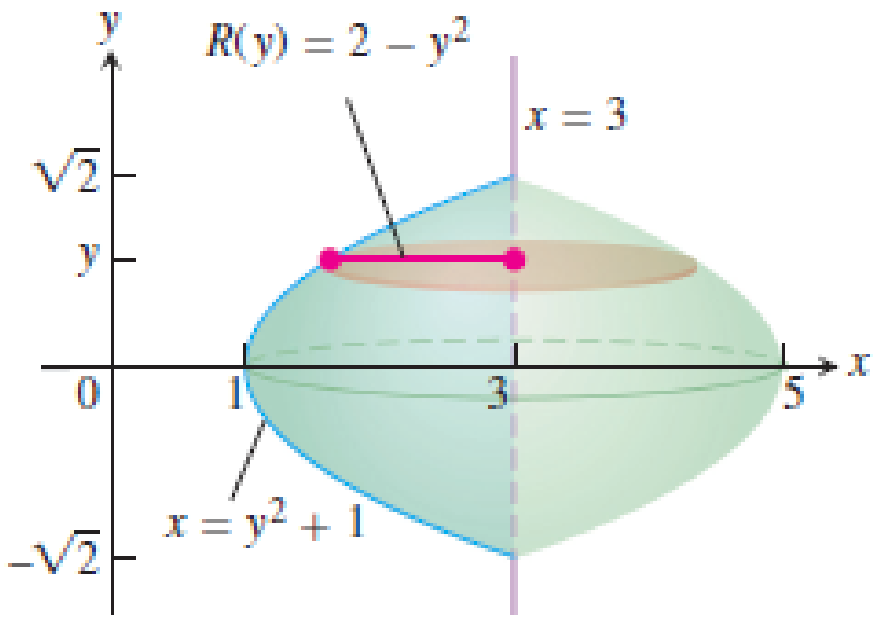
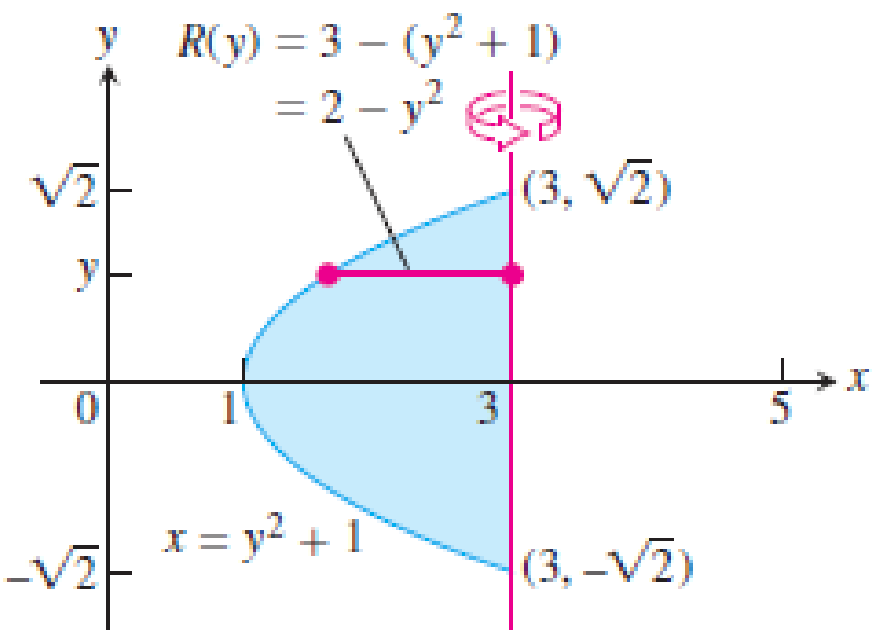
$$\begin{aligned} V &= \int_1^4 \pi [R(x)]^2 dx \\ &= \int_1^4 \pi [\sqrt{x} - 1]^2 dx \\ &= \pi \int_1^4 [x - 2\sqrt{x} + 1] dx \\ &= \pi \left[\frac{x^2}{2} - 2 \cdot \frac{2}{3} x^{3/2} + x \right]_1^4 = \frac{7\pi}{6}. \end{aligned}$$

Find the volume of the solid generated by revolving the region between the y -axis and the curve $x = 2/y$, $1 \leq y \leq 4$, about the y -axis.



$$\begin{aligned} V &= \int_1^4 \pi [R(y)]^2 dy \\ &= \int_1^4 \pi \left(\frac{2}{y}\right)^2 dy \\ &= \pi \int_1^4 \frac{4}{y^2} dy = 4\pi \left[-\frac{1}{y}\right]_1^4 = 4\pi \left[\frac{3}{4}\right] \\ &= 3\pi. \end{aligned}$$

Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.



$$\begin{aligned} V &= \int_{-\sqrt{2}}^{\sqrt{2}} \pi [R(y)]^2 dy \\ &= \int_{-\sqrt{2}}^{\sqrt{2}} \pi [2 - y^2]^2 dy \\ &= \pi \int_{-\sqrt{2}}^{\sqrt{2}} [4 - 4y^2 + y^4] dy \\ &= \pi \left[4y - \frac{4}{3}y^3 + \frac{y^5}{5} \right]_{-\sqrt{2}}^{\sqrt{2}} \\ &= \frac{64\pi\sqrt{2}}{15}. \end{aligned}$$

Solids of Revolution: The Washer Method

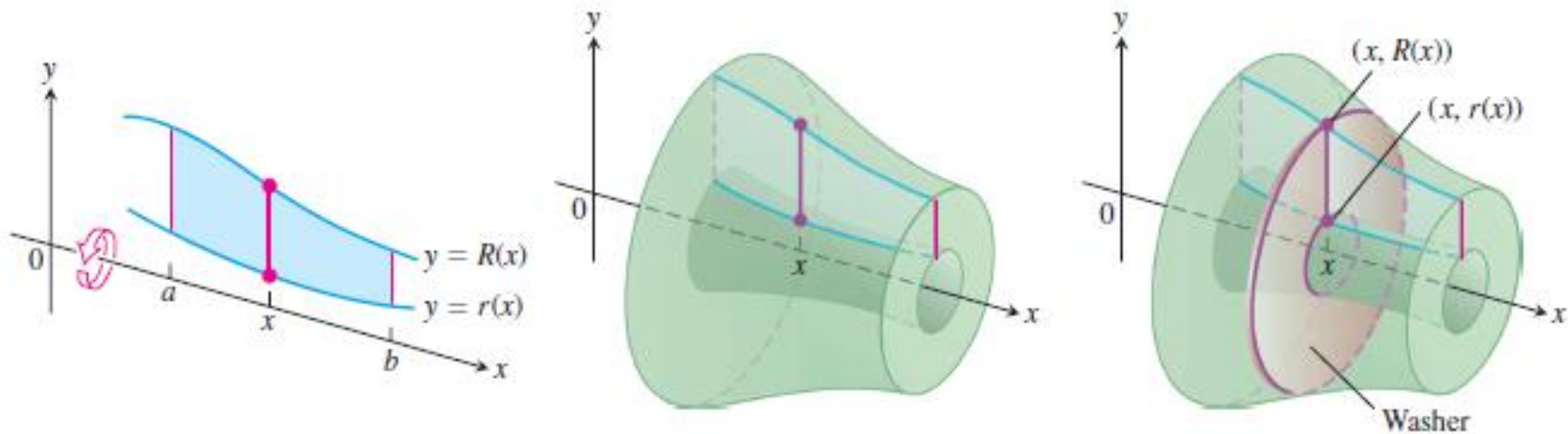
If the region we revolve to generate a solid does not border on or cross the axis of revolution, the solid has a hole in it (Figure 6.13). The cross-sections perpendicular to the axis of revolution are washers (the purplish circular surface in Figure 6.13) instead of disks. The dimensions of a typical washer are

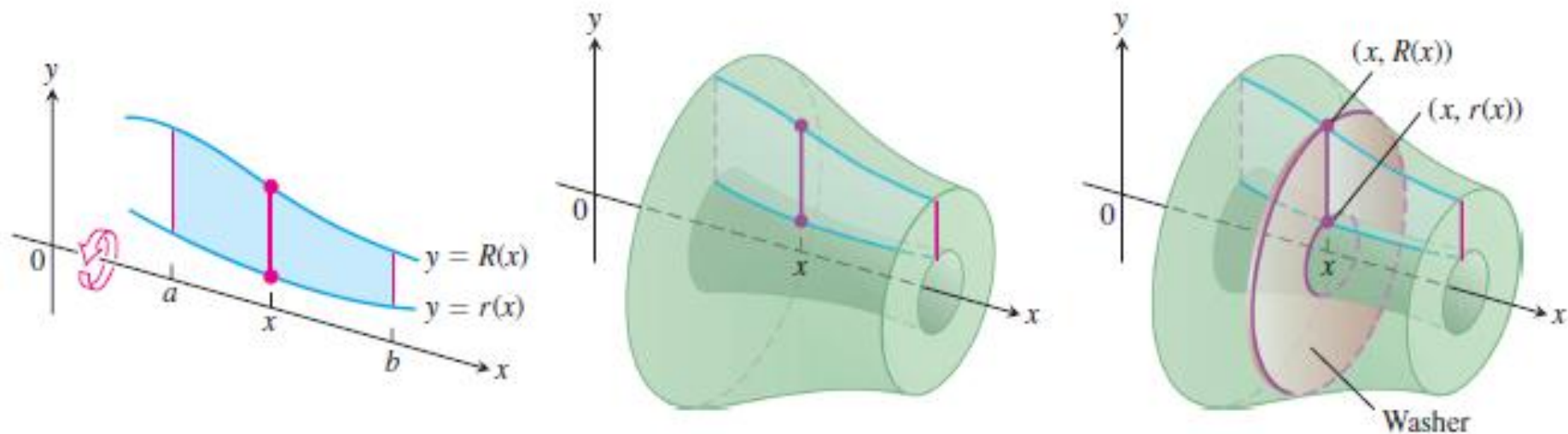
Outer radius: $R(x)$

Inner radius: $r(x)$

The washer's area is

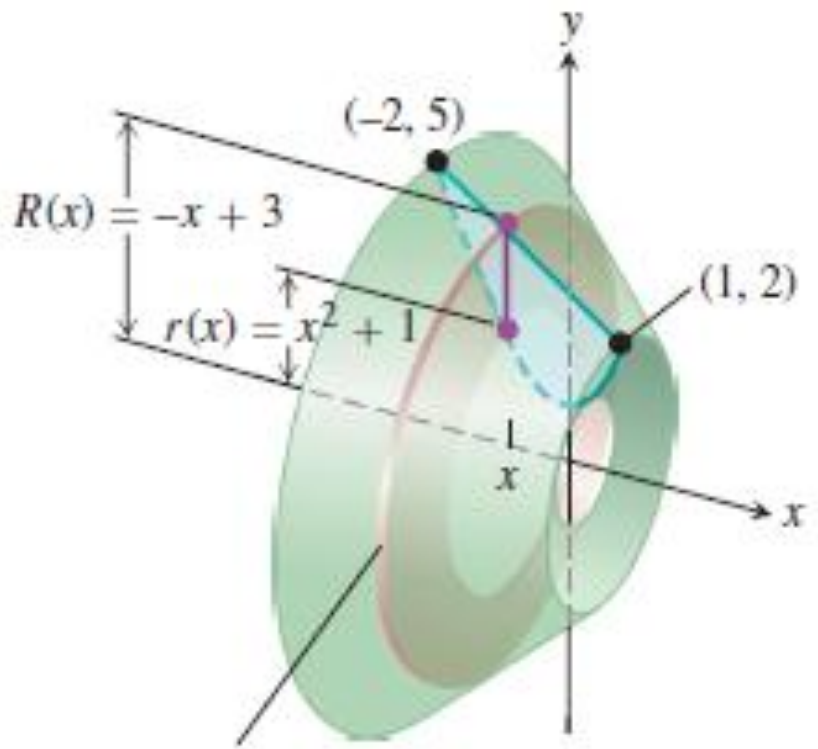
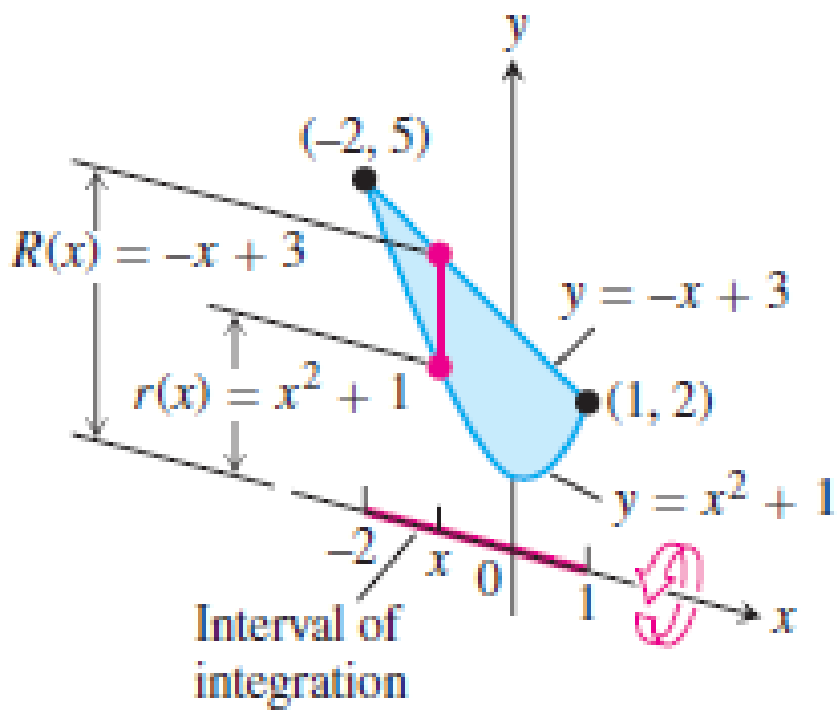
$$A(x) = \pi[R(x)]^2 - \pi[r(x)]^2 = \pi([R(x)]^2 - [r(x)]^2).$$





$$V = \int_a^b A(x) dx = \int_a^b \pi([R(x)]^2 - [r(x)]^2) dx.$$

The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x -axis to generate a solid. Find the volume of the solid.



$$x^2 + 1 = -x + 3$$

$$x^2 + x - 2 = 0$$

$$(x + 2)(x - 1) = 0$$

$$x = -2, \quad x = 1$$

Evaluate the volume integral.

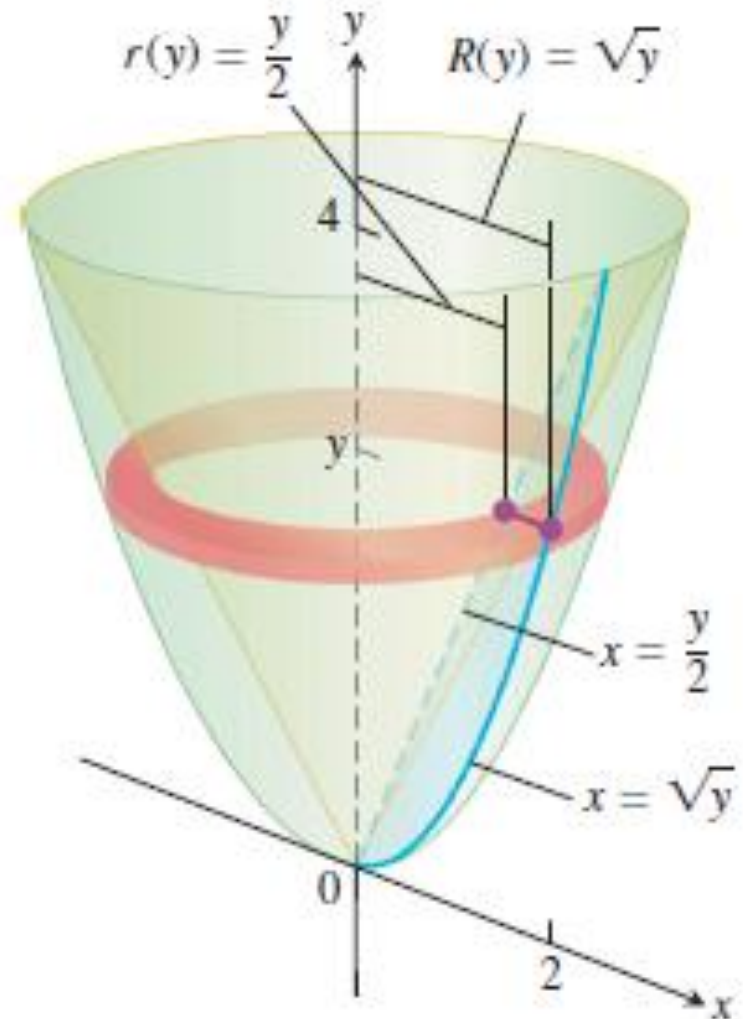
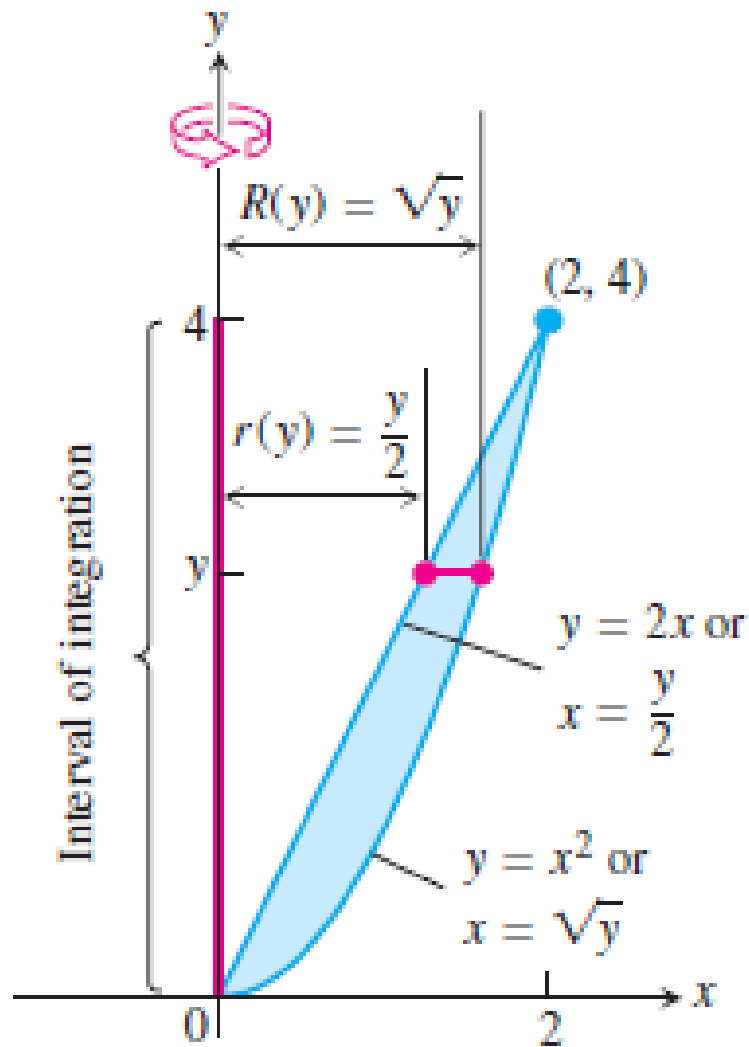
$$V = \int_a^b \pi([R(x)]^2 - [r(x)]^2) dx$$

$$= \int_{-2}^1 \pi((-x + 3)^2 - (x^2 + 1)^2) dx$$

$$= \int_{-2}^1 \pi(8 - 6x - x^2 - x^4) dx$$

$$= \pi \left[8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5} \right]_{-2}^1 = \frac{117\pi}{5}$$

The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y -axis to generate a solid. Find the volume of the solid.

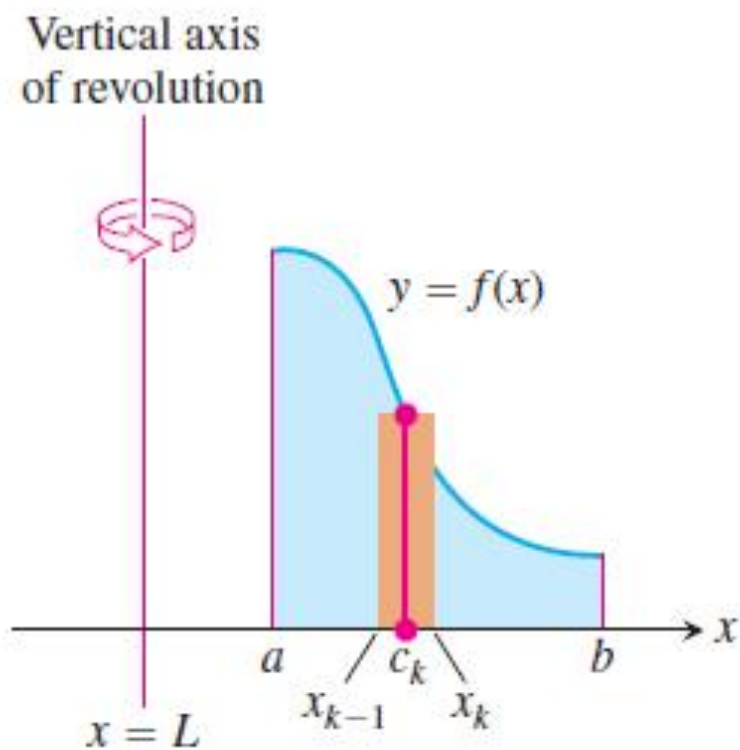


$$\begin{aligned} V &= \int_c^d \pi([R(y)]^2 - [r(y)]^2) dy \\ &= \int_0^4 \pi\left(\left[\sqrt{y}\right]^2 - \left[\frac{y}{2}\right]^2\right) dy \\ &= \pi \int_0^4 \left(y - \frac{y^2}{4}\right) dy = \pi \left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 = \frac{8}{3} \pi. \end{aligned}$$

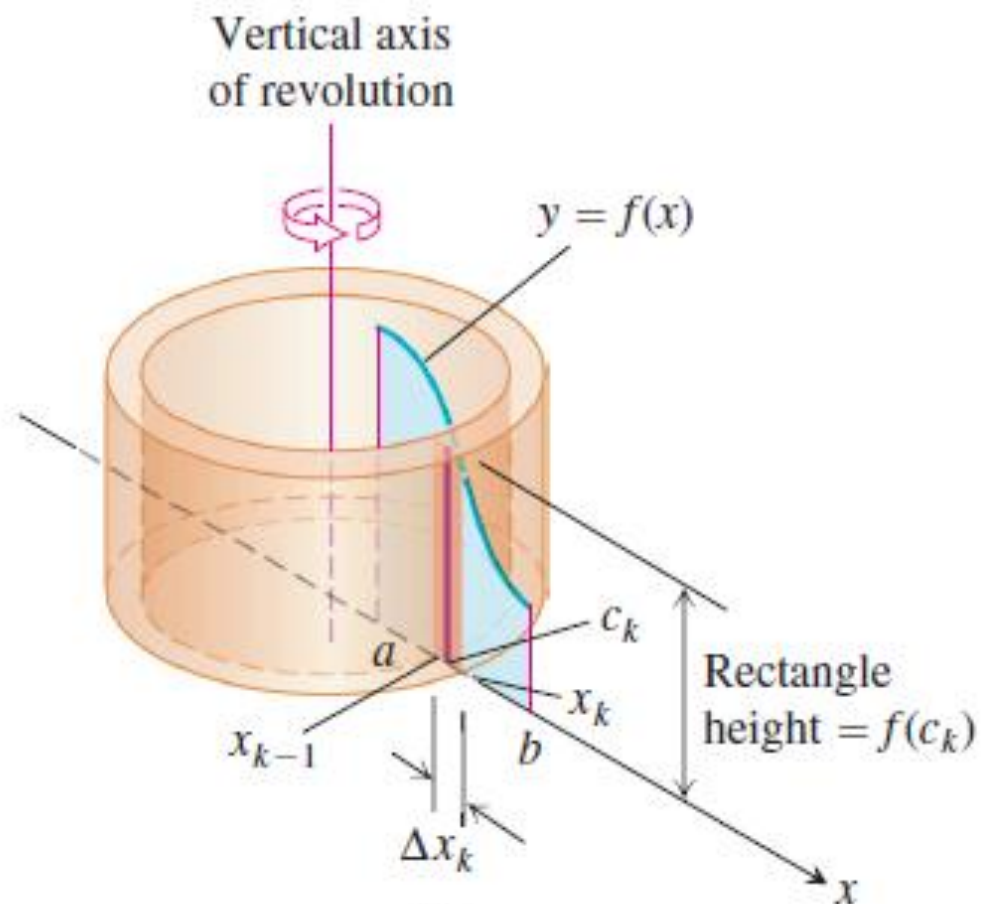
45. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 2$ and $x = 0$ about
- the x -axis.
 - the y -axis.
 - the line $y = 2$.
 - the line $x = 4$.
46. Find the volume of the solid generated by revolving the triangular region bounded by the lines $y = 2x$, $y = 0$, and $x = 1$ about
- the line $x = 1$.
 - the line $x = 2$.
47. Find the volume of the solid generated by revolving the region bounded by the parabola $y = x^2$ and the line $y = 1$ about
- the line $y = 1$.
 - the line $y = 2$.
 - the line $y = -1$.
48. By integration, find the volume of the solid generated by revolving the triangular region with vertices $(0, 0)$, $(b, 0)$, $(0, h)$ about
- the x -axis.
 - the y -axis.

6.2

Volumes by Cylindrical Shells



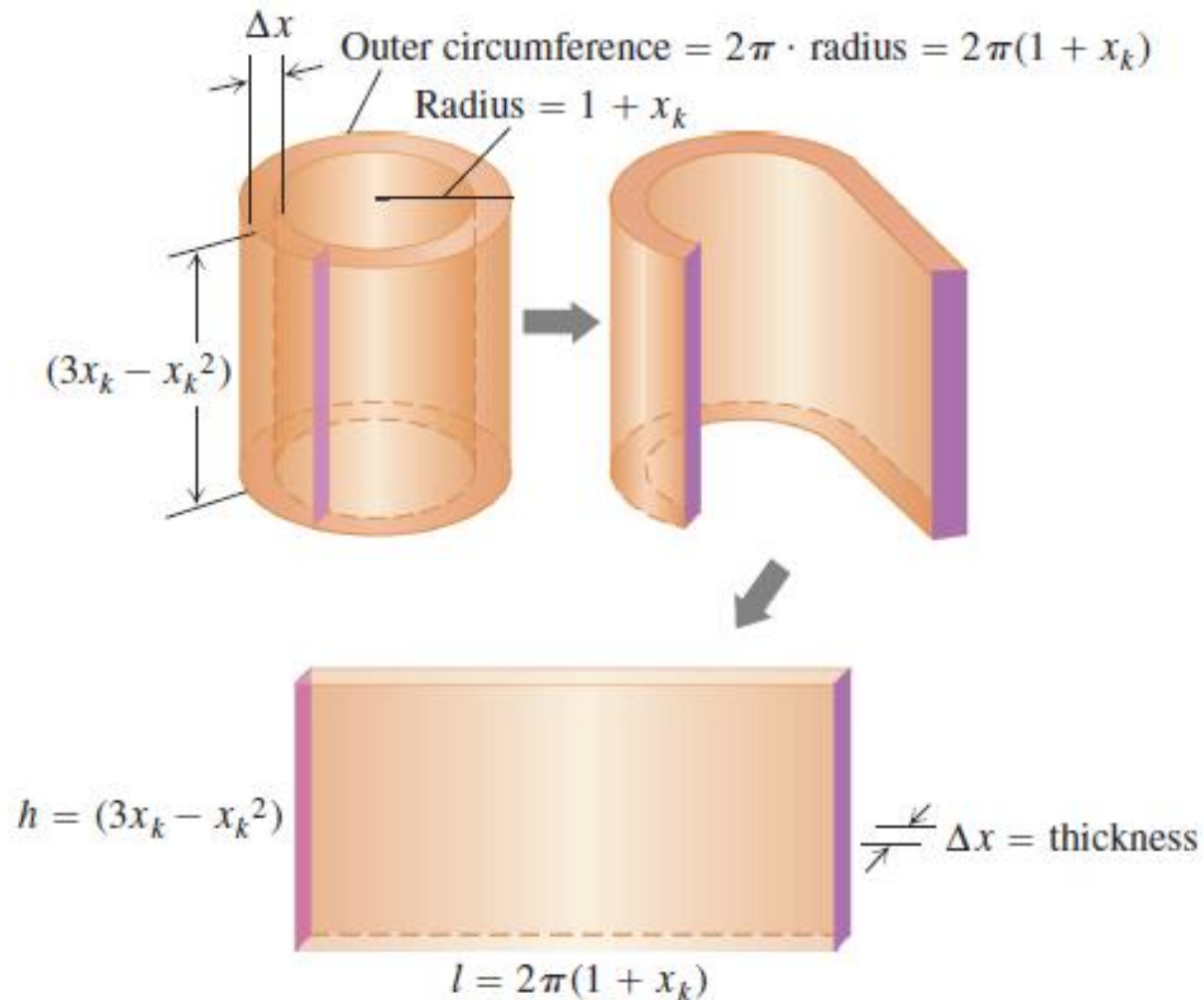
(a)



(b)

$$\Delta V_k = \text{circumference} \times \text{height} \times \text{thickness}$$

$$= 2\pi(1 + x_k) \cdot (3x_k - x_k^2) \cdot \Delta x.$$



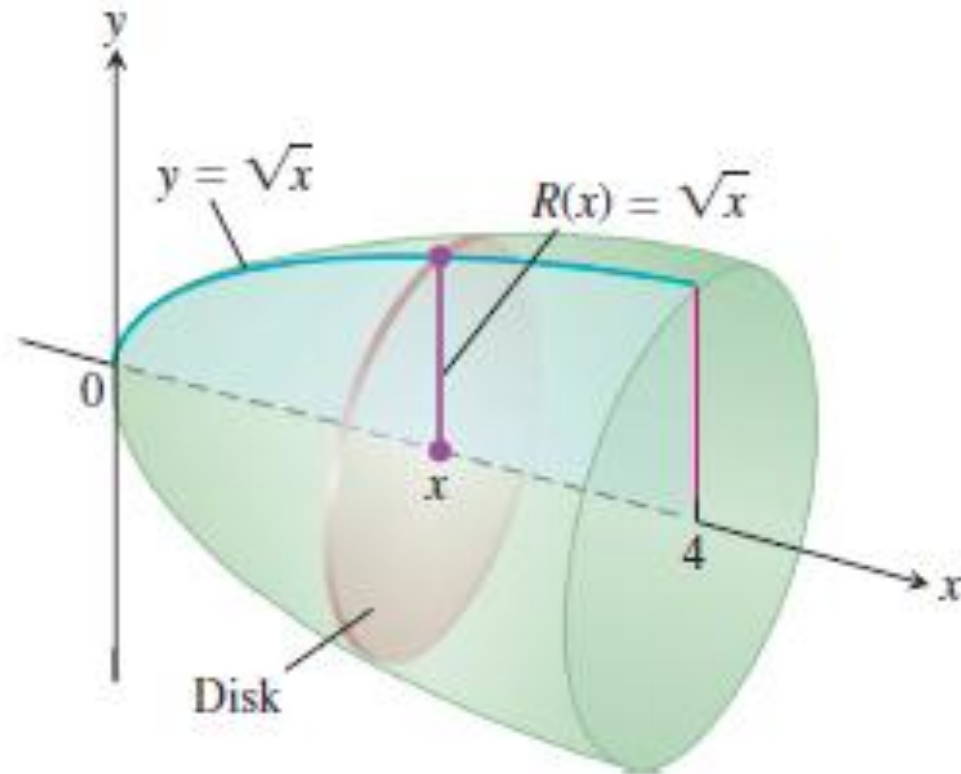
Shell Formula for Revolution About a Vertical Line

The volume of the solid generated by revolving the region between the x -axis and the graph of a continuous function $y = f(x) \geq 0$, $L \leq a \leq x \leq b$, about a vertical line $x = L$ is

$$V = \int_a^b 2\pi \left(\begin{array}{c} \text{shell} \\ \text{radius} \end{array} \right) \left(\begin{array}{c} \text{shell} \\ \text{height} \end{array} \right) dx.$$

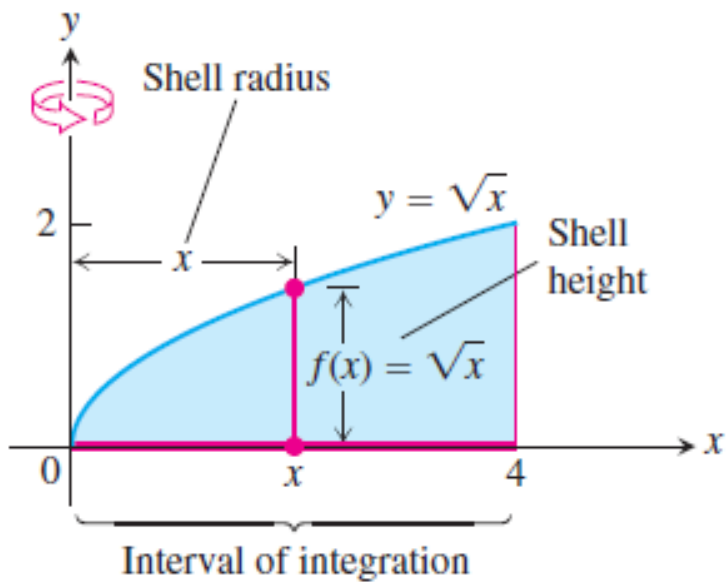
The region bounded by the curve $y = \sqrt{x}$, the x -axis, and the line $x = 4$ is revolved about the y -axis to generate a solid. Find the volume of the solid.

Disk Method

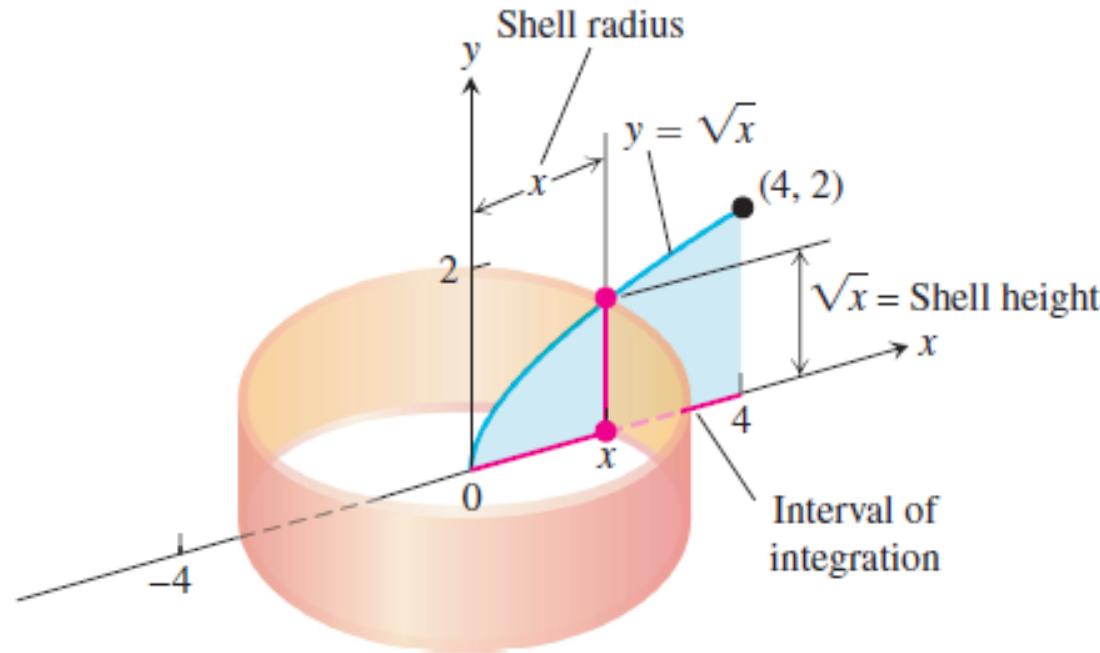


$$= \pi \int_0^4 x \, dx = \pi \left. \frac{x^2}{2} \right|_0^4 = \pi \frac{(4)^2}{2} = 8\pi.$$

Shell Method



(a)



(b)

The shell thickness variable is x , so the limits of integration for the shell formula are $a = 0$ and $b = 4$ (Figure 6.20). The volume is then

$$\begin{aligned} V &= \int_a^b 2\pi \left(\begin{array}{c} \text{shell} \\ \text{radius} \end{array} \right) \left(\begin{array}{c} \text{shell} \\ \text{height} \end{array} \right) dx \\ &= \int_0^4 2\pi(x)(\sqrt{x}) dx \\ &= 2\pi \int_0^4 x^{3/2} dx = 2\pi \left[\frac{2}{5} x^{5/2} \right]_0^4 = \frac{128\pi}{5}. \end{aligned}$$

So far, we have used vertical axes of revolution. For horizontal axes, we replace the x 's with y 's. ■

Summary of the Shell Method

Regardless of the position of the axis of revolution (horizontal or vertical), the steps for implementing the shell method are these.

1. *Draw the region and sketch a line segment across it parallel to the axis of revolution. Label the segment's height or length (shell height) and distance from the axis of revolution (shell radius).*
2. *Find the limits of integration for the thickness variable.*
3. *Integrate the product 2π (shell radius) (shell height) with respect to the thickness variable (x or y) to find the volume.*

Choosing Shells or Washers

In Exercises 27–32, find the volumes of the solids generated by revolving the regions about the given axes. If you think it would be better to use washers in any given instance, feel free to do so.

27. The triangle with vertices $(1, 1)$, $(1, 2)$, and $(2, 2)$ about
- a. the x -axis
 - b. the y -axis
 - c. the line $x = 10/3$
 - d. the line $y = 1$
28. The region bounded by $y = \sqrt{x}$, $y = 2$, $x = 0$ about
- a. the x -axis
 - b. the y -axis
 - c. the line $x = 4$
 - d. the line $y = 2$
29. The region in the first quadrant bounded by the curve $x = y - y^3$ and the y -axis about
- a. the x -axis
 - b. the line $y = 1$
30. The region in the first quadrant bounded by $x = y - y^3$, $x = 1$, and $y = 1$ about
- a. the x -axis
 - b. the y -axis
 - c. the line $x = 1$
 - d. the line $y = 1$
31. The region bounded by $y = \sqrt{x}$ and $y = x^2/8$ about
- a. the x -axis
 - b. the y -axis
32. The region bounded by $y = 2x - x^2$ and $y = x$ about
- a. the y -axis
 - b. the line $x = 1$