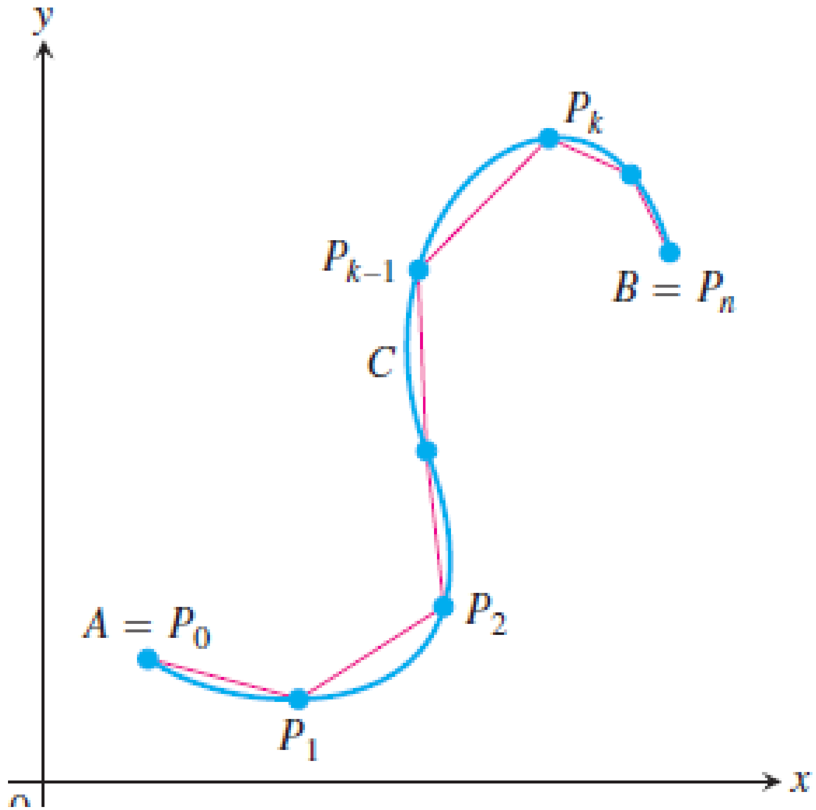
6.3 Lengths of Plane Curves

Length of a Parametrically Defined Curve

Let C be a curve given parametrically by the equations

$$x = f(t) \qquad \text{and} \qquad y = g(t), \qquad a \leq t \leq b.$$

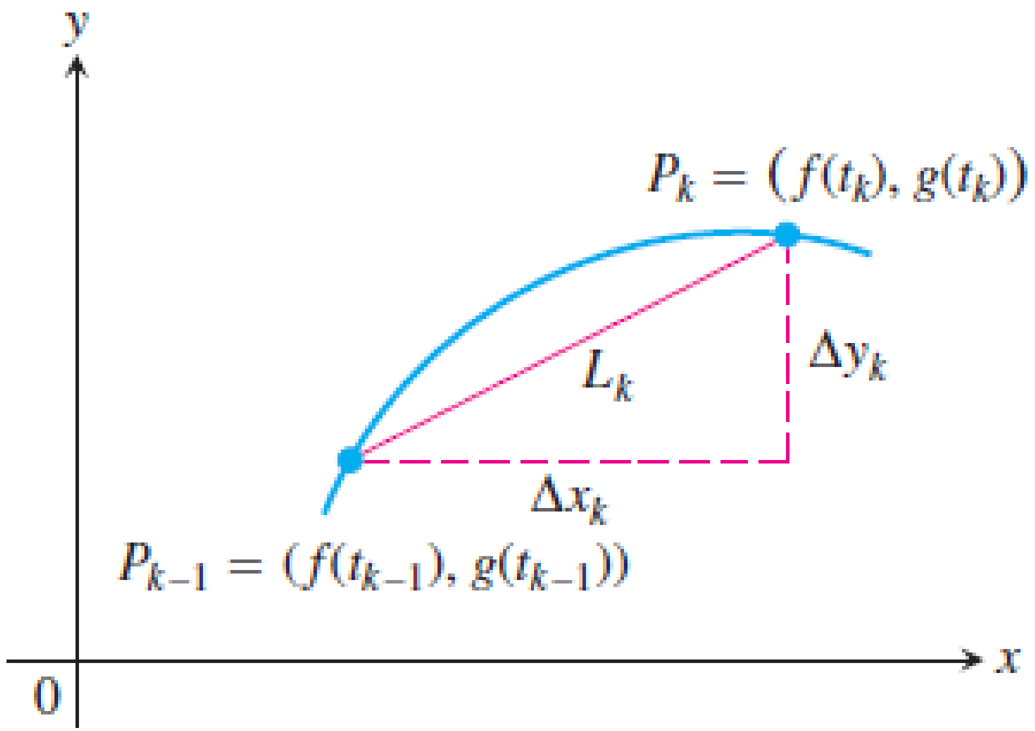
We assume the functions f and g have continuous derivatives on the interval $[a, b]$ that are not simultaneously zero. Such functions are said to be **continuously differentiable**, and the curve C defined by them is called a **smooth curve**. It may be helpful to imagine the curve as



DEFINITION **Length of a Parametric Curve**

If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \le t \le b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$, then **the length of C** is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$



$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

The Circumference of a Circle

Find the length of the circle of radius r defined parametrically by

$$x = r\cos t \qquad \text{and} \qquad y = r\sin t, \qquad 0 \le t \le 2\pi.$$

Solution As t varies from 0 to 2π, the circle is traversed exactly once,
ence is

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

We find

$$\frac{dx}{dt} = -r\sin t, \quad \frac{dy}{dt} = r\cos t$$

and

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = r^2(\sin^2 t + \cos^2 t) = r^2.$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = f'(t)$$

giving

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= 1 + [f'(t)]^2 \\ &= 1 + \left(\frac{dy}{dx}\right)^2 \\ &= 1 + [f'(x)]^2.\end{aligned}$$

Formula for the Length of $y = f(x)$, $\quad a \le x \le b$

If f is continuously differentiable on the closed interval $[a, b]$, the length of the curve (graph) $y = f(x)$ from $x = a$ to $x = b$ is

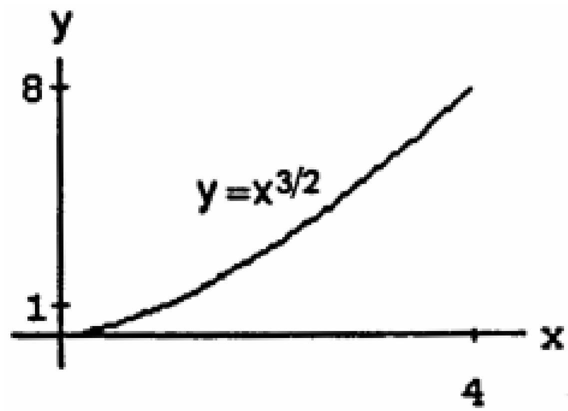
$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx. \tag{2}$$

Formula for the Length of $x = g(y)$, $\quad c \le y \le d$

If g is continuously differentiable on $[c, d]$, the length of the curve $x = g(y)$ from $y = c$ to $y = d$ is

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy. \tag{3}$$

$$y = x^{3/2} \quad \text{from} \quad x = 0 \text{ to } x = 4$$



$\frac{dy}{dx} = \frac{3}{2}\sqrt{x} \Rightarrow L = \int_0^4 \sqrt{1 + \frac{9}{4}x}\,dx$; $\left[u = 1 + \frac{9}{4}x\right.$
$\Rightarrow du = \frac{9}{4}\,dx \Rightarrow \frac{4}{9}\,du = dx$; $x = 0 \Rightarrow u = 1$; $x = 4$
$\left.\Rightarrow u = 10\right] \rightarrow L = \int_1^{10} u^{1/2}\left(\frac{4}{9}\,du\right) = \frac{4}{9}\left[\frac{2}{3}u^{3/2}\right]_1^{10}$
$= \frac{8}{27}\left(10\sqrt{10} - 1\right)$

6.4 Moments and Centers of Mass

Masses Along a Line

We develop our mathematical model in stages. The first stage is to imagine masses m_1, m_2, and m_3 on a rigid x-axis supported by a fulcrum at the origin.

The resulting system might balance, or it might not. It depends on how large the masses are and how they are arranged.

Each mass m_k exerts a downward force $m_k g$ (the weight of m_k) equal to the magnitude of the mass times the acceleration of gravity. Each of these forces has a tendency to turn the axis about the origin, the way you turn a seesaw. This turning effect, called a **torque**, is measured by multiplying the force $m_k g$ by the signed distance x_k from the point of application to the origin. Masses to the left of the origin exert negative (counterclockwise) torque. Masses to the right of the origin exert positive (clockwise) torque.

The sum of the torques measures the tendency of a system to rotate about the origin. This sum is called the **system torque**.

$$\text{System torque} = m_1 g x_1 + m_2 g x_2 + m_3 g x_3 \tag{1}$$

The system will balance if and only if its torque is zero.

If we factor out the g in Equation (1), we see that the system torque is

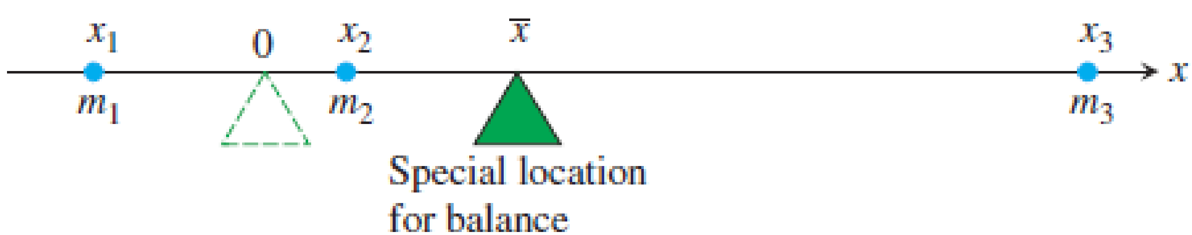
$$\underbrace{g}_{\text{a feature of the environment}} \cdot \underbrace{(m_1 x_1 + m_2 x_2 + m_3 x_3)}_{\text{a feature of the system}}$$

The number $(m_1x_1 + m_2x_2 + m_3x_3)$ is called the **moment of the system about the origin**. It is the sum of the **moments** m_1x_1, m_2x_2, m_3x_3 of the individual masses.

$$M_0 = \text{Moment of system about origin} = \sum m_k x_k.$$

(We shift to sigma notation here to allow for sums with more terms.)

We usually want to know where to place the fulcrum to make the system balance, that is, at what point $\bar{x}$ to place it to make the torques add to zero.



When we write the equation that says that the sum of these torques is zero, we get an equation we can solve for $\bar{x}$:

$\sum (x_k - \bar{x})m_k g = 0$	Sum of the torques equals zero
$g\sum (x_k - \bar{x})m_k = 0$	Constant Multiple Rule for Sums
$\sum (m_k x_k - \bar{x}m_k) = 0$	g divided out, m_k distributed
$\sum m_k x_k - \sum \bar{x}m_k = 0$	Difference Rule for Sums
$\sum m_k x_k = \bar{x}\sum m_k$	Rearranged, Constant Multiple Rule again
$\bar{x} = \dfrac{\sum m_k x_k}{\sum m_k}.$	Solved for $\bar{x}$

This last equation tells us to find $\bar{x}$ by dividing the system's moment about the origin by the system's total mass:

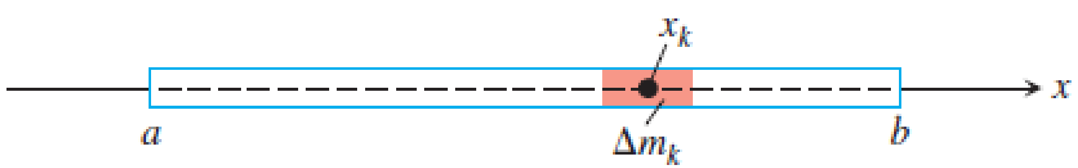
$$\bar{x} = \frac{\sum m_k x_k}{\sum m_k} = \frac{\text{system moment about origin}}{\text{system mass}}.$$

The point $\bar{x}$ is called the system's **center of mass.**

Wires and Thin Rods

In many applications, we want to know the center of mass of a rod or a thin strip of metal. In cases like these where we can model the distribution of mass with a continuous function, the summation signs in our formulas become integrals in a manner we now describe.

Imagine a long, thin strip lying along the x-axis from $x = a$ to $x = b$ and cut into small pieces of mass Δm_k by a partition of the interval $[a, b]$. Choose x_k to be any point in the kth subinterval of the partition.



The kth piece is Δx_k units long and lies approximately x_k units from the origin. Now observe three things.

First, the strip's center of mass $\bar{x}$ is nearly the same as that of the system of point masses we would get by attaching each mass Δm_k to the point x_k:

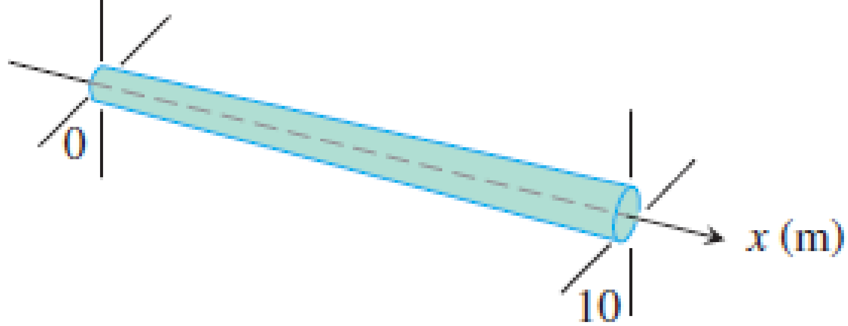
$$\bar{x} \approx \frac{\text{system moment}}{\text{system mass}}.$$

Moment, Mass, and Center of Mass of a Thin Rod or Strip Along the x-Axis with Density Function $\delta(x)$

Moment about the origin: $$M_0 = \int_a^b x\delta(x)\,dx \qquad (3a)$$

Mass: $$M = \int_a^b \delta(x)\,dx \qquad (3b)$$

Center of mass: $$\bar{x} = \frac{M_0}{M} \qquad (3c)$$



The 10-m-long rod in Figure   thickens from left to right so that its density, instead of being constant, is $\delta(x) = 1 + (x/10)$ kg/m. Find the rod's center of mass.

$$M_0 = \int_0^{10} x\delta(x)\,dx = \int_0^{10} x\left(1 + \frac{x}{10}\right)dx = \int_0^{10}\left(x + \frac{x^2}{10}\right)dx$$

$$= \left[\frac{x^2}{2} + \frac{x^3}{30}\right]_0^{10} = 50 + \frac{100}{3} = \frac{250}{3}\ \text{kg}\cdot\text{m}.$$

The units of a moment are mass × length.

The rod's mass (Equation 3b) is

$$M = \int_0^{10} \delta(x)\,dx = \int_0^{10}\left(1 + \frac{x}{10}\right)dx = \left[x + \frac{x^2}{20}\right]_0^{10} = 10 + 5 = 15\ \text{kg}.$$

The center of mass (Equation 3c) is located at the point

$$\bar{x} = \frac{M_0}{M} = \frac{250}{3}\cdot\frac{1}{15} = \frac{50}{9} \approx 5.56\ \text{m}.$$

Moments, Mass, and Center of Mass of a Thin Plate Covering a Region in the *xy*-Plane

Moment about the x-axis: $$M_x = \int \tilde{y}\, dm$$

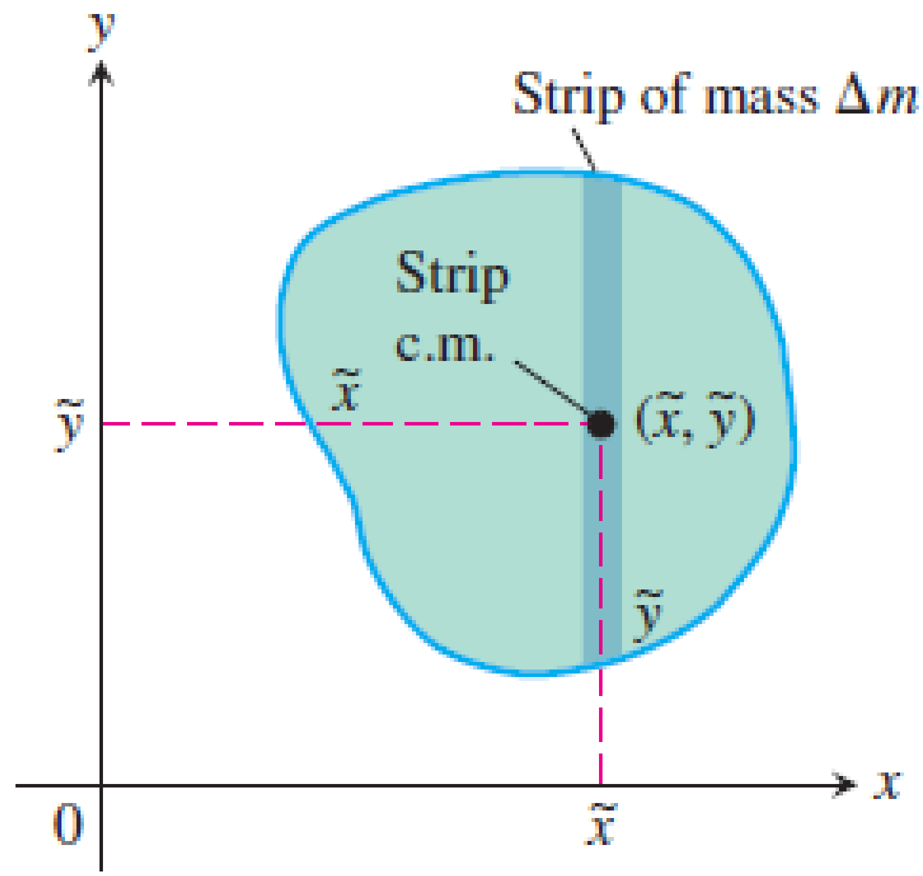
Moment about the y-axis: $$M_y = \int \tilde{x}\, dm$$

Mass: $$M = \int dm$$

Center of mass: $$\bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}$$

How to Find a Plate's Center of Mass

1. Picture the plate in the xy-plane.
2. Sketch a strip of mass parallel to one of the coordinate axes and find its dimensions.
3. Find the strip's mass dm and center of mass $(\tilde{x}, \tilde{y})$.
4. Integrate $\tilde{y}\,dm$, $\tilde{x}\,dm$, and dm to find M_x, M_y, and M.
5. Divide the moments by the mass to calculate $\bar{x}$ and $\bar{y}$.

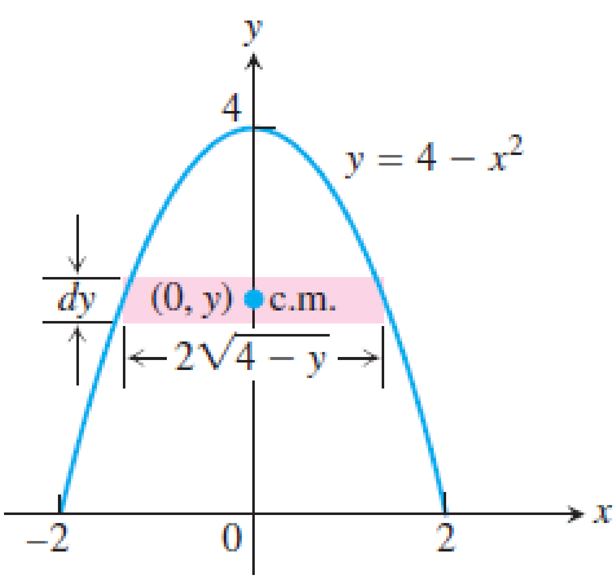


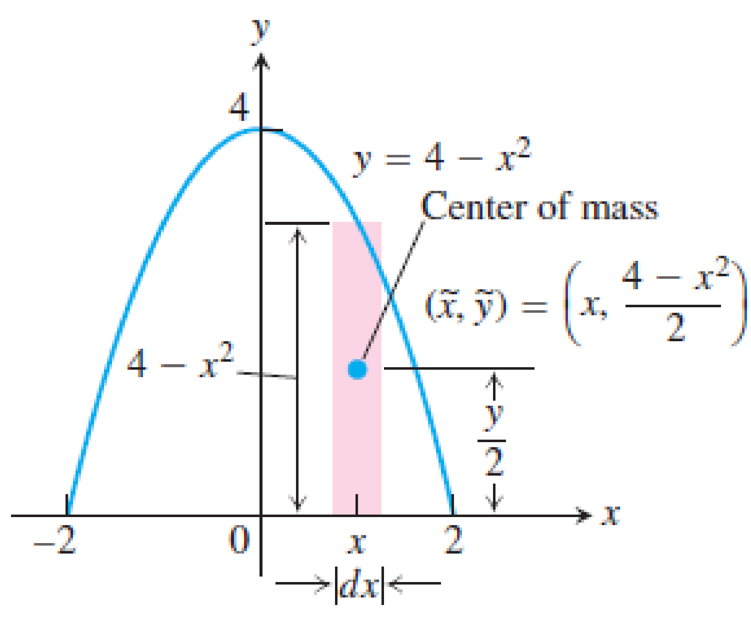
Constant-Density Plate

Find the center of mass of a thin plate of constant density δ covering the region bounded above by the parabola $y = 4 - x^2$ and below by the x-axis

Solution Since the plate is symmetric about the y-axis and its density is constant, the distribution of mass is symmetric about the y-axis and the center of mass lies on the y-axis. Thus, $\bar{x} = 0$. It remains to find $\bar{y} = M_x/M$.

Modeling the plate in horizontal strips leads to an inconvenient integration, so we model vertical strips instead





The typical vertical strip has

center of mass (c.m.): $(\tilde{x}, \tilde{y}) = \left(x, \frac{4 - x^2}{2}\right)$

length: $4 - x^2$

width: dx

area: $dA = (4 - x^2)\, dx$

mass: $dm = \delta\, dA = \delta(4 - x^2)\, dx$

distance from c.m. to x-axis: $\tilde{y} = \frac{4 - x^2}{2}.$

The moment of the plate about the x-axis is

$$M_x = \int \tilde{y}\, dm = \int_{-2}^{2} \frac{\delta}{2}(4 - x^2)^2\, dx$$

$$= \frac{\delta}{2}\int_{-2}^{2} (16 - 8x^2 + x^4)\, dx = \frac{256}{15}\delta.$$

The mass of the plate is

$$M = \int dm = \int_{-2}^{2} \delta(4 - x^2)\, dx = \frac{32}{3}\delta.$$

Therefore,

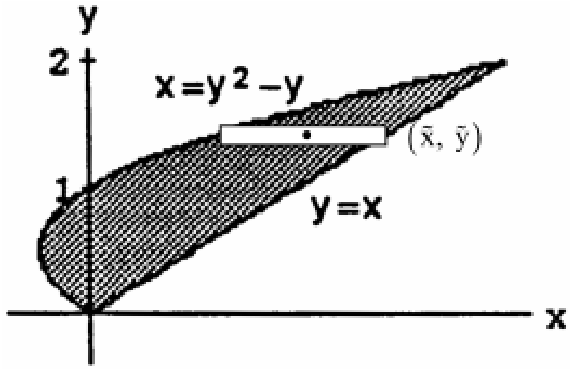
$$\bar{y} = \frac{M_x}{M} = \frac{(256/15)\,\delta}{(32/3)\,\delta} = \frac{8}{5}.$$

The plate's center of mass is the point

$$(\bar{x}, \bar{y}) = \left(0, \frac{8}{5}\right).$$

18. The region bounded by the parabola $x = y^2 - y$ and the line $y = x$

18. Intersection points: $y = y^2 - y \Rightarrow y^2 - 2y = 0$ $\Rightarrow y(y-2) = 0 \Rightarrow y = 0$ or $y = 2$. The typical *horizontal* strip has center of mass: $(\tilde{x}, \tilde{y}) = \left(\frac{(y^2-y)+y}{2}, y\right) = \left(\frac{y^2}{2}, y\right)$, length: $y - (y^2 - y) = 2y - y^2$, width: dy, area: $dA = (2y - y^2)\,dy$, mass: $dm = \delta\,dA = \delta\,(2y - y^2)\,dy$. The moment about the y-axis is $\tilde{x}\,dm = \frac{\delta}{2} \cdot y^2 (2y - y^2)\,dy$ $= \frac{\delta}{2}(2y^3 - y^4)\,dy$; the moment about the x-axis is $\tilde{y}\,dm = \delta y\,(2y - y^2)\,dy = \delta\,(2y^2 - y^3)\,dy$. Thus, $M_x = \int \tilde{y}\,dm = \int_0^2 \delta\,(2y^2 - y^3)\,dy = \delta\left[\frac{2y^3}{3} - \frac{y^4}{4}\right]_0^2 = \delta\left(\frac{16}{3} - \frac{16}{4}\right) = \frac{16\delta}{12}(4-3) = \frac{4\delta}{3}$; $M_y = \int \tilde{x}\,dm$ $= \int_0^2 \frac{\delta}{2}(2y^3 - y^4)\,dy = \frac{\delta}{2}\left[\frac{y^4}{2} - \frac{y^5}{5}\right]_0^2 = \frac{\delta}{2}\left(8 - \frac{32}{5}\right) = \frac{\delta}{2}\left(\frac{40-32}{5}\right) = \frac{4\delta}{5}$; $M = \int dm = \int_0^2 \delta\,(2y - y^2)\,dy$ $= \delta\left[y^2 - \frac{y^3}{3}\right]_0^2 = \delta\left(4 - \frac{8}{3}\right) = \frac{4\delta}{3}$. Therefore, $\bar{x} = \frac{M_y}{M} = \left(\frac{4\delta}{5}\right)\left(\frac{3}{4\delta}\right) = \frac{3}{5}$ and $\bar{y} = \frac{M_x}{M} = \left(\frac{4\delta}{3}\right)\left(\frac{3}{4\delta}\right) = 1$ $\Rightarrow (\bar{x}, \bar{y}) = \left(\frac{3}{5}, 1\right)$ is the center of mass.



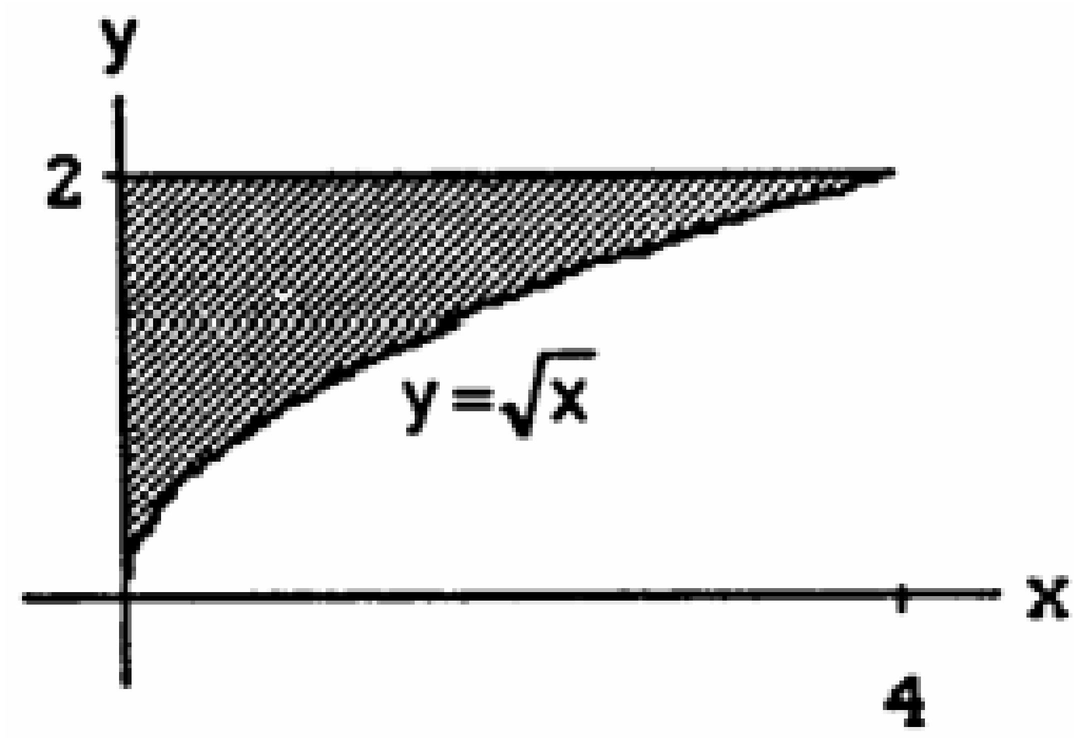
The region bounded by $y = \sqrt{x}$, $y = 2$, $x = 0$ about

a. the x-axis

b. the y-axis

c. the line $x = 4$

d. the line $y = 2$



$$\text{(a)}\quad V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}}\, dy = \int_0^2 2\pi y(y^2 - 0)\, dy$$

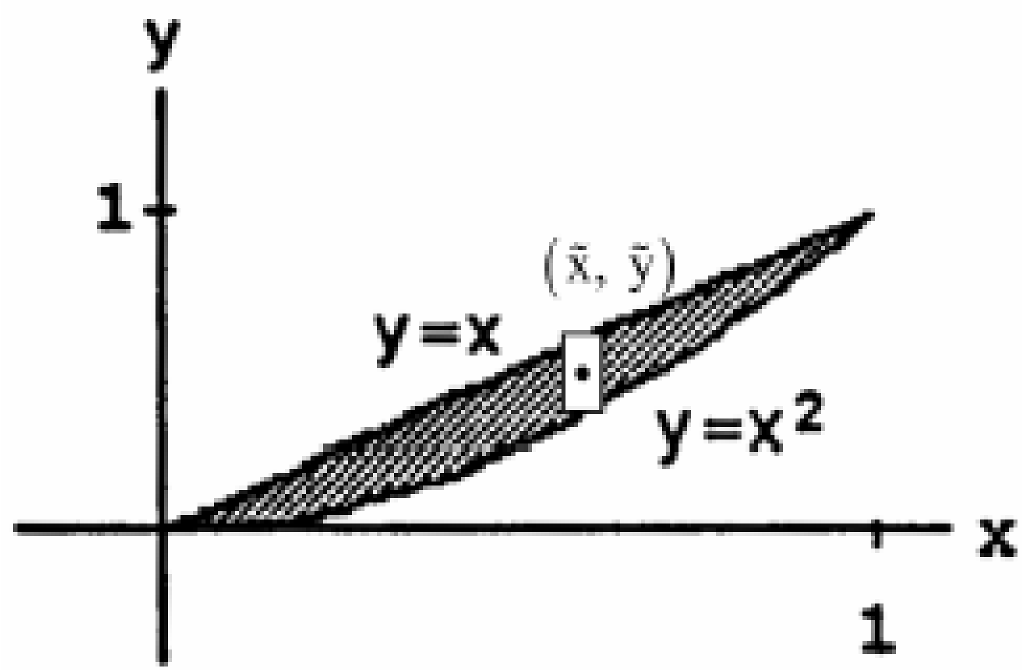
$$\text{(b)}\quad V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}}\, dx$$

$$= \int_0^4 2\pi x\left(2 - \sqrt{x}\right) dx = 2\pi \int_0^4 \left(2x - x^{3/2}\right) dx$$

$$\text{(c)}\quad V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}}\, dx = \int_0^4 2\pi(4 - x)\left(2 - \sqrt{x}\right) dx$$

$$\text{(d)}\quad V = \int_c^d 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}}\, dy = \int_0^2 2\pi(2 - y)\left(y^2\right) dy$$

Find the center of mass of a thin plate covering the region bounded below by the parabola $y = x^2$ and above by the line $y = x$ if the plate's density at the point (x, y) is $\delta(x) = 12x$.



$$M_x = \int \tilde{y}\, dm = \int_0^1 \frac{(x+x^2)}{2}\,(x - x^2) \cdot \delta\, dx$$

$$= \frac{1}{2}\int_0^1 (x^2 - x^4) \cdot 12x\, dx$$

$$= 6\int_0^1 (x^3 - x^5)\, dx = 6\left[\frac{x^4}{4} - \frac{x^6}{6}\right]_0^1$$

$$= 6\left(\frac{1}{4} - \frac{1}{6}\right) = \frac{6}{4} - 1 = \frac{1}{2}\,;$$

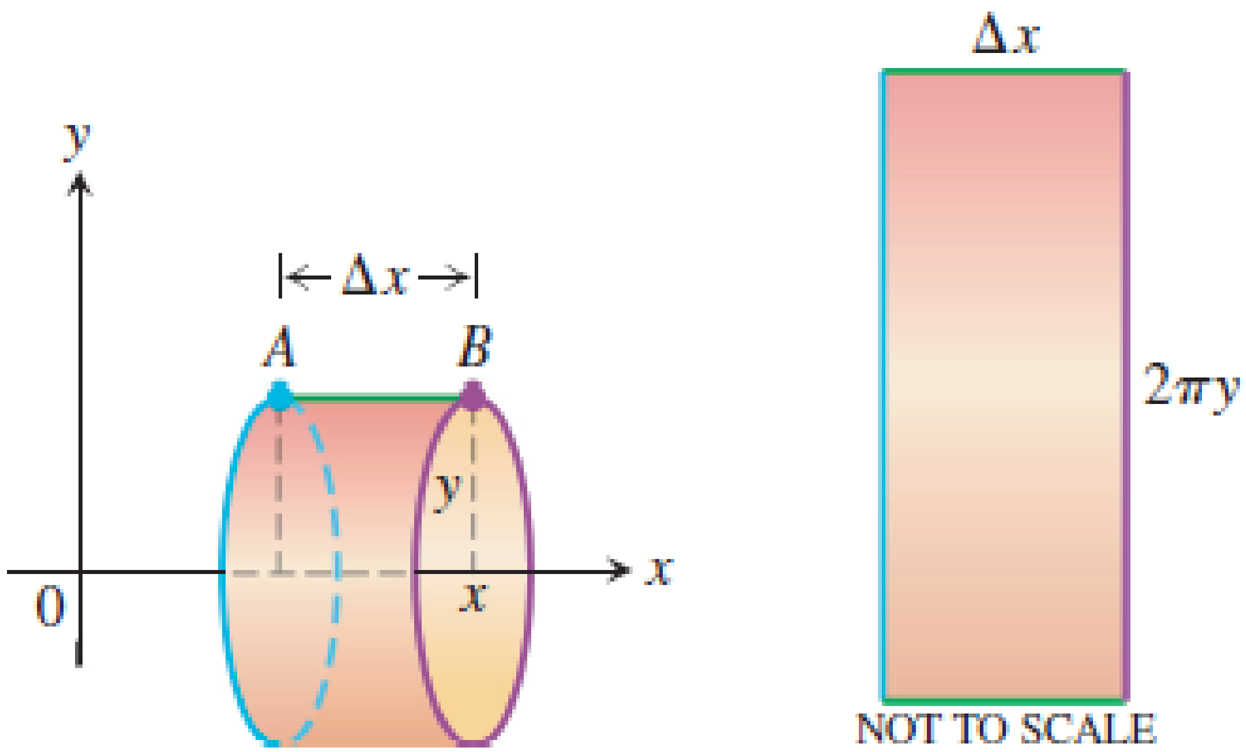
$$M_y = \int \tilde{x}\, dm = \int_0^1 x\,(x - x^2) \cdot \delta\, dx = \int_0^1 (x^2 - x^3) \cdot 12x\, dx$$

$$= \frac{12}{20} = \frac{3}{5}\,;\ M = \int dm = \int_0^1 (x - x^2) \cdot \delta\, dx = 12\int_0^1 (x^2 - x^3)\, dx$$

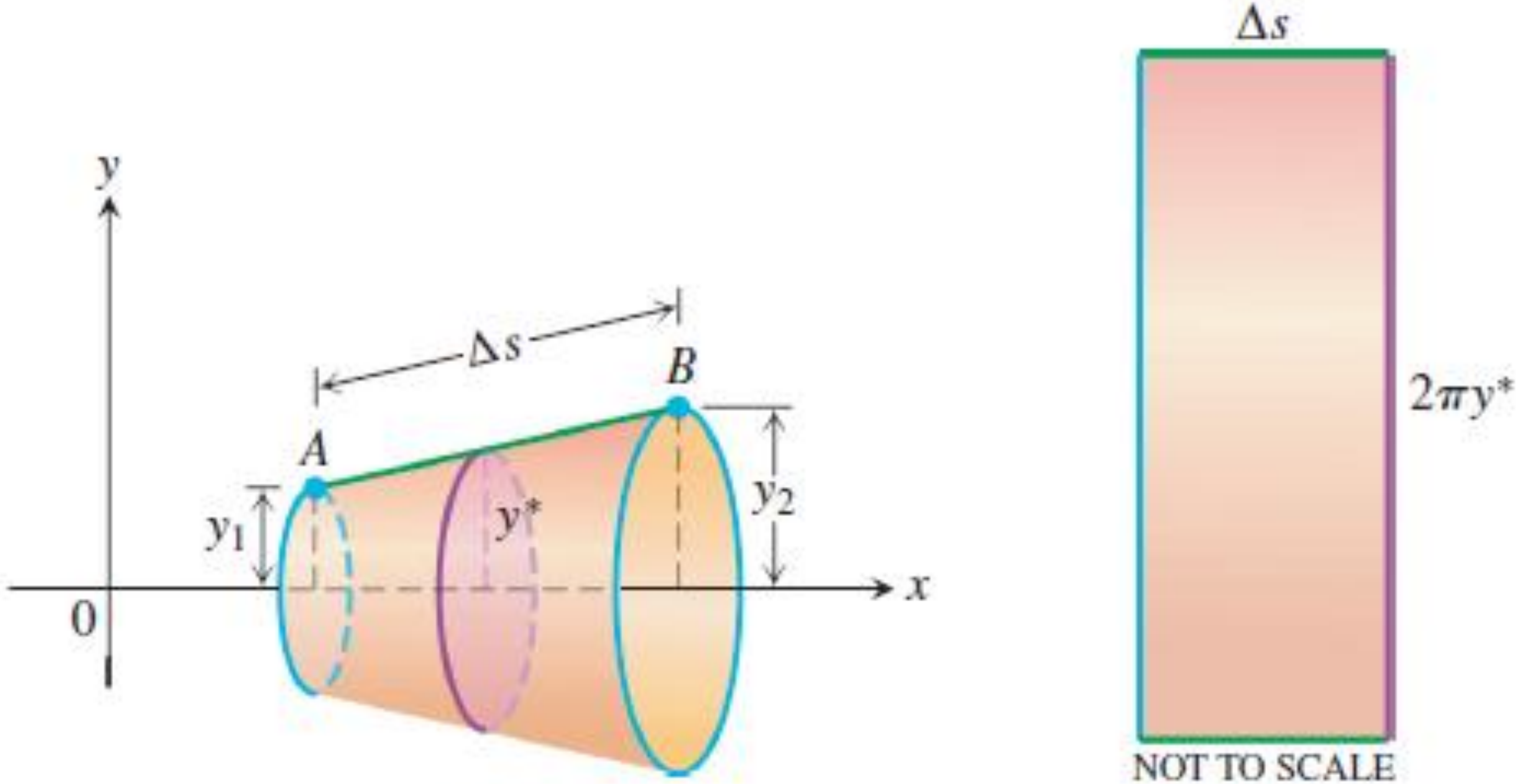
$$12\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 12\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{12}{12} = 1.$$

$$\bar{x} = \frac{M_y}{M} = \frac{3}{5} \text{ and } \bar{y} = \frac{M_x}{M} = \frac{1}{2} \Rightarrow \left(\frac{3}{5}, \frac{1}{2}\right)$$

6.5 Areas of Surfaces of Revolution and the Theorems of Pappus



(a) (b)



(a) (b)

DEFINITION Surface Area for Revolution About the x-Axis

If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the **area** of the surface generated by revolving the curve $y = f(x)$ about the x-axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx. \qquad (3)$$

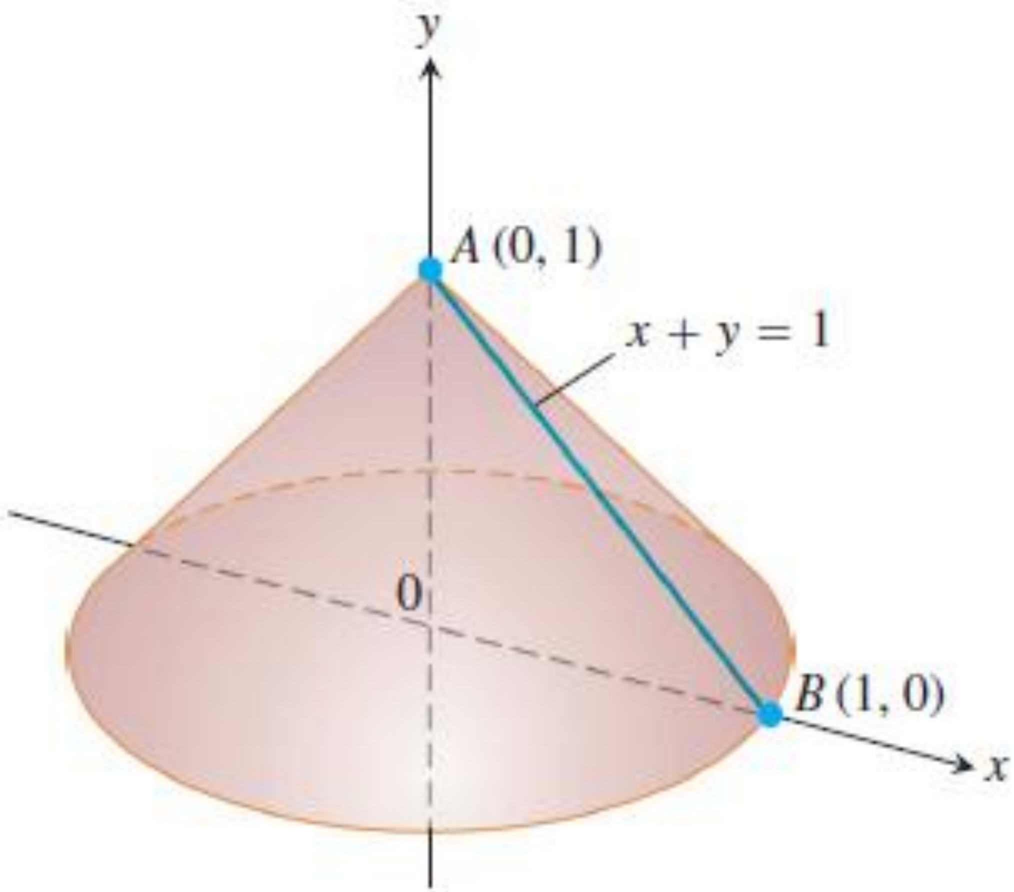
Surface Area for Revolution About the y-Axis

If $x = g(y) \geq 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the curve $x = g(y)$ about the y-axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\, dy. \qquad (4)$$

Finding Area for Revolution about the y-Axis

The line segment $x = 1 - y, 0 \leq y \leq 1$, is revolved about the y-axis to generate the cone Find its lateral surface area (which excludes the base area).



$$
\begin{aligned}
S &= \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 2\pi(1 - y)\sqrt{2}\, dy \\
&= 2\pi\sqrt{2}\left[y - \frac{y^2}{2}\right]_0^1 = 2\pi\sqrt{2}\left(1 - \frac{1}{2}\right) \\
&= \pi\sqrt{2}.
\end{aligned}
$$

Surface Area of Revolution for Parametrized Curves

If a smooth curve $x = f(t), y = g(t), a \leq t \leq b$, is traversed exactly once as t increases from a to b, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

1. Revolution about the x-axis $(y \geq 0)$:

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{5}$$

2. Revolution about the y-axis $(x \geq 0)$:

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{6}$$

6.6 Work

Work Done by a Constant Force

When a body moves a distance d along a straight line as a result of being acted on by a force of constant magnitude F in the direction of motion, we define the **work** W done by the force on the body with the formula

$$W = Fd \qquad \text{(Constant-force formula for work).} \tag{1}$$

Work Done by a Variable Force Along a Line

If the force you apply varies along the way, as it will if you are compressing a spring, the formula $W = Fd$ has to be replaced by an integral formula that takes the variation in F into account.

DEFINITION Work

The **work** done by a variable force $F(x)$ directed along the x-axis from $x = a$ to $x = b$ is

$$W = \int_a^b F(x)\,dx. \qquad (2)$$

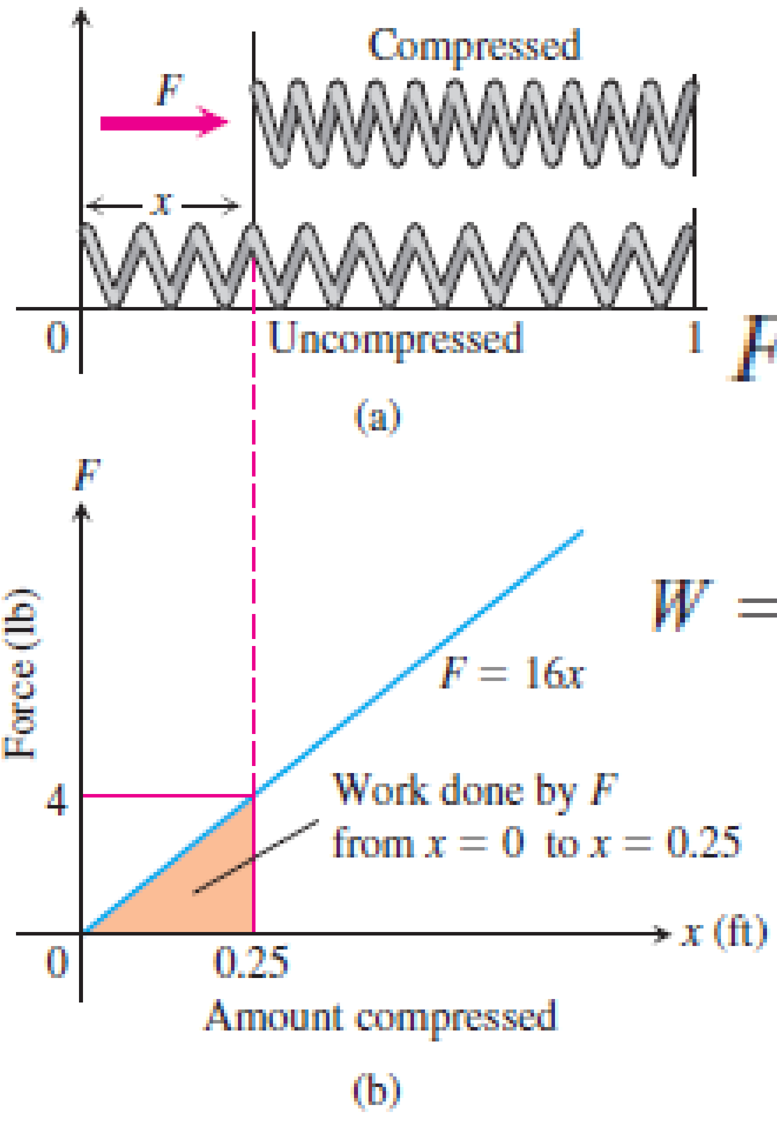
Hooke's Law for Springs: $F = kx$

Hooke's Law says that the force it takes to stretch or compress a spring x length units from its natural (unstressed) length is proportional to x. In symbols,

$$F = kx. \qquad (3)$$

The constant k, measured in force units per unit length, is a characteristic of the spring, called the **force constant** (or **spring constant**) of the spring. Hooke's Law, Equation (3), gives good results as long as the force doesn't distort the metal in the spring. We assume that the forces in this section are too small to do that.

Find the work required to compress a spring from its natural length of 1 ft to a length of 0.75 ft if the force constant is $k = 16$ lb/ft.



$$F(0) = 16 \cdot 0 = 0 \text{ lb}$$

$$F(0.25) = 16 \cdot 0.25 = 4 \text{ lb}.$$

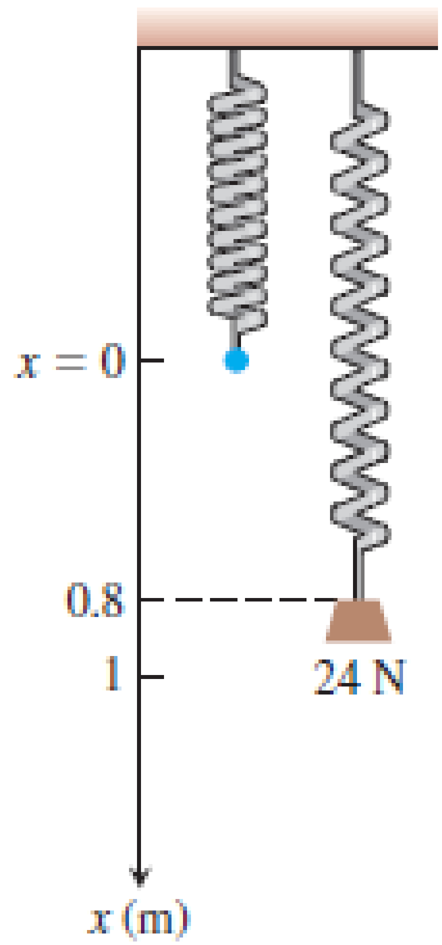
$$W = \int_0^{0.25} 16x\, dx = 8x^2 \Big]_0^{0.25} = 0.5 \text{ ft-lb}$$

A spring has a natural length of 1 m. A force of 24 N stretches the spring to a length of 1.8 m.

(a) Find the force constant k.

(b) How much work will it take to stretch the spring 2 m beyond its natural length?

(c) How far will a 45-N force stretch the spring?



(a) *The force constant.* We find the force constant from stretches the spring 0.8 m, so

$$24 = k(0.8)$$
$$k = 24/0.8 = 30\ \text{N/m}.$$

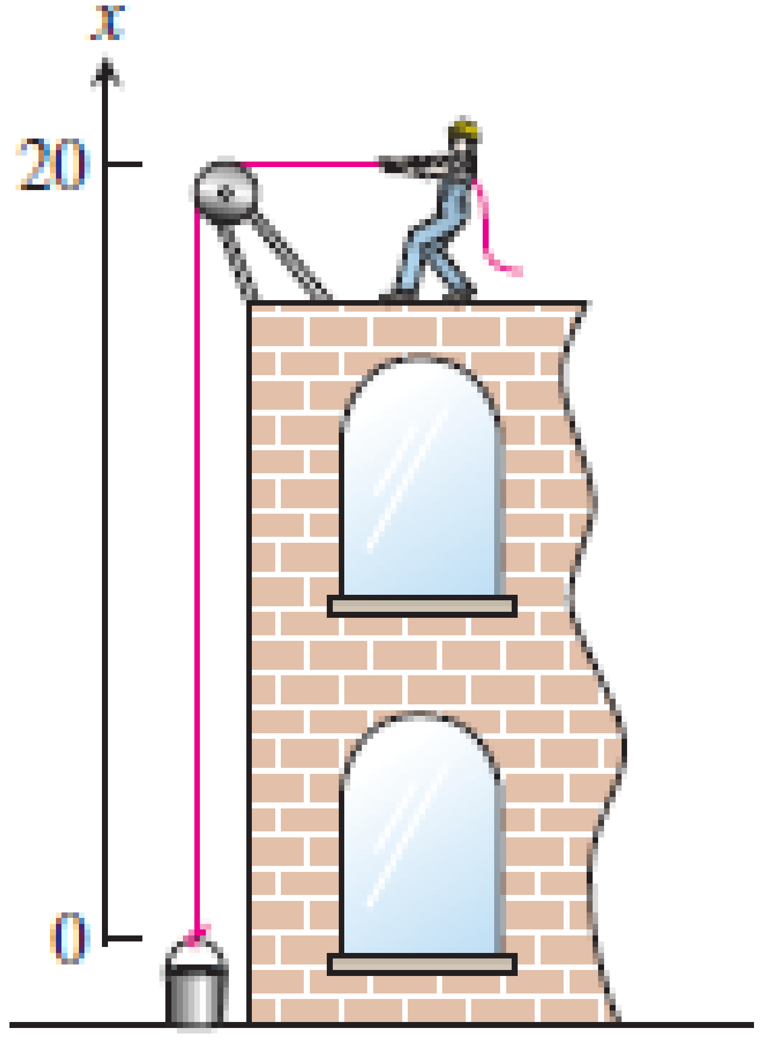
(b) The work done by F on the spring from $x = 0$ m to $x = 2$ m is

$$W = \int_0^2 30x\, dx = 15x^2\Big]_0^2 = 60\ \text{J}.$$

(c)

$$45 = 30x, \qquad \text{or} \qquad x = 1.5\ \text{m}.$$

A 5-lb bucket is lifted from the ground into the air by pulling in 20 ft of rope at a constant speed The rope weighs 0.08 lb/ft. How much work was spent lifting the bucket and rope?



The bucket has constant weight so the work done lifting it alone is

W = F . d   W = 5 . 20 = 100 ft.lb

$$\text{Work on rope} = \int_0^{20} (0.08)(20 - x)\,dx = \int_0^{20} (1.6 - 0.08x)\,dx$$

$$= \left[1.6x - 0.04x^2\right]_0^{20} = 32 - 16 = 16 \text{ ft-lb}.$$

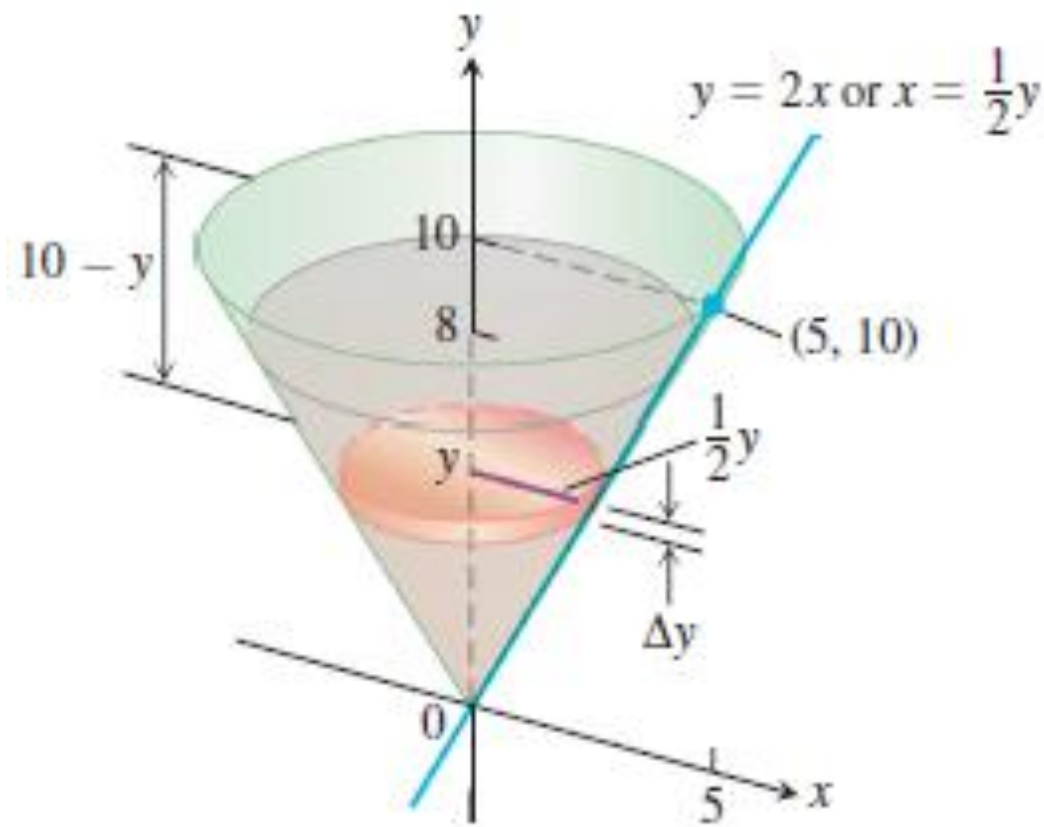
total work for the bucket and rope combined is

$$100 + 16 = 116 \text{ ft-lb}.$$

Pumping Liquids from Containers

How much work does it take to pump all or part of the liquid from a container? To find out, we imagine lifting the liquid out one thin horizontal slab at a time and applying the Equation ***W=F.d*** to each slab. We then evaluate the integral this leads to as the slabs become thinner and more numerous.

The conical tank in is filled to within 2 ft of the top with olive oil weighing 57 lb/ft^3. How much work does it take to pump the oil to the rim of the tank?



$$\Delta V = \pi(\text{radius})^2(\text{thickness}) = \pi\left(\frac{1}{2}y\right)^2 \Delta y = \frac{\pi}{4}y^2\,\Delta y \text{ ft}^3.$$

The force $F(y)$ required to lift this slab is equal to its weight,

$$F(y) = 57\,\Delta V = \frac{57\pi}{4}y^2\,\Delta y \text{ lb.}$$

Weight = weight per unit volume × volume

The distance through which $F(y)$ must act to lift this slab to the level of the rim of the cone is about $(10 - y)$ ft, so the work done lifting the slab is about

$$\Delta W = \frac{57\pi}{4}(10 - y)y^2\,\Delta y \text{ ft-lb.}$$

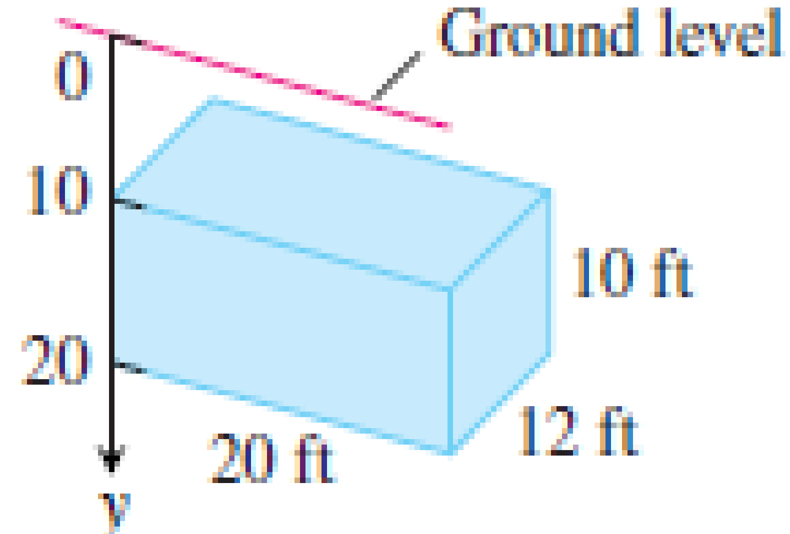
$$\begin{aligned} W &= \int_0^8 \frac{57\pi}{4}(10 - y)y^2 \, dy \\ &= \frac{57\pi}{4} \int_0^8 (10y^2 - y^3) \, dy \\ &= \frac{57\pi}{4} \left[\frac{10y^3}{3} - \frac{y^4}{4} \right]_0^8 \approx 30{,}561 \text{ ft-lb}. \end{aligned}$$

. **Emptying a cistern** The rectangular cistern (storage tank for rainwater) shown below has its top 10 ft below ground level. The cistern, currently full, is to be emptied for inspection by pumping its contents to ground level.

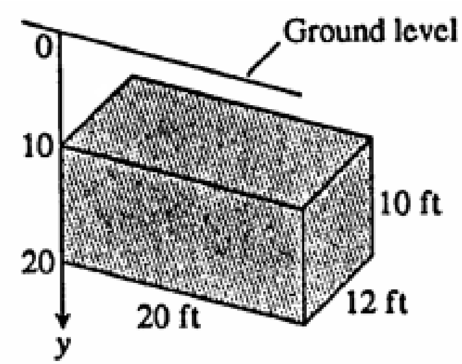
a. How much work will it take to empty the cistern?

b. How long will it take a 1/2 hp pump, rated at 275 ft-lb/sec, to pump the tank dry?

c. How long will it take the pump in part (b) to empty the tank halfway? (It will be less than half the time required to empty the tank completely.)



We will use the coordinate system given.



(a) The typical slab between the planes at y and y + Δy has a volume of $\Delta V = (20)(12)\,\Delta y = 240\,\Delta y$ ft^3. The force F required to lift the slab is equal to its weight: $F = 62.4\,\Delta V = 62.4 \cdot 240\,\Delta y$ lb. The distance through which F must act is about y ft, so the work done lifting the slab is about ΔW = force $\times$ distance $= 62.4 \cdot 240 \cdot y \cdot \Delta y$ ft $\cdot$ lb. The work it takes to lift all the water is approximately $W \approx \sum_{10}^{20} \Delta W = \sum_{10}^{20} 62.4 \cdot 240y \cdot \Delta y$ ft $\cdot$ lb. This is a Riemann sum for the function $62.4 \cdot 240y$ over the interval $10 \le y \le 20$. The work it takes to empty the cistern is the limit of these sums: $W = \int_{10}^{20} 62.4 \cdot 240y\,dy$

$$= (62.4)(240)\left[\frac{y^2}{2}\right]_{10}^{20} = (62.4)(240)(200 - 50) = (62.4)(240)(150) = 2{,}246{,}400 \text{ ft} \cdot \text{lb}$$

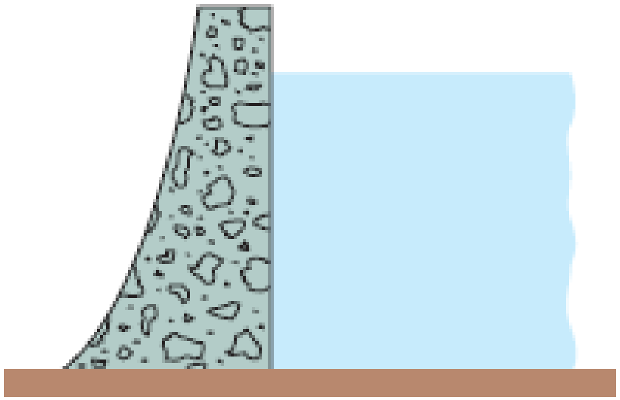
(b) $t = \frac{W}{275\,\frac{\text{ft}\cdot\text{lb}}{\text{sec}}} = \frac{2{,}246{,}400 \text{ ft-lb}}{275} \approx 8168.73$ sec ≈ 2.27 hours $\approx$ 2 hr and 16.1 min

(c) Following all the steps of part (a), we find that the work it takes to empty the tank halfway is

$$W = \int_{10}^{15} 62.4 \cdot 240y\,dy = (62.4)(240)\left[\frac{y^2}{2}\right]_{10}^{15} = (62.4)(240)\left(\frac{225}{2} - \frac{100}{2}\right) = (62.4)(240)\left(\frac{125}{2}\right) = 936{,}000 \text{ ft.}$$

Then the time is $t = \frac{W}{275\,\frac{\text{ft}\cdot\text{lb}}{\text{sec}}} = \frac{936{,}000}{275} \approx 3403.64$ sec ≈ 56.7 min

6.7 Fluid Pressures and Forces



To withstand the increasing pressure, dams are built thicker as they go down.

The Pressure-Depth Equation

In a fluid that is standing still, the pressure p at depth h is the fluid's weight-density w times h:

$$p = wh. \tag{1}$$

Fluid Force on a Constant-Depth Surface

$$F = pA = whA \tag{2}$$

Weight-density

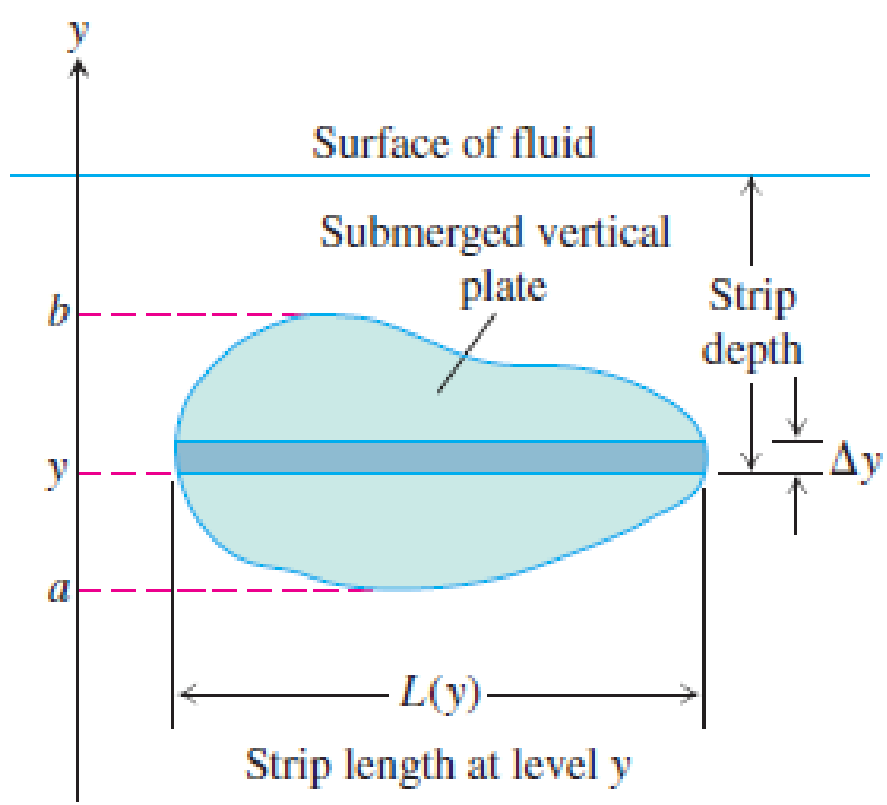
A fluid's weight-density is its weight per unit volume. Typical values (lb/ft^3) are

Gasoline	42
Mercury	849
Milk	64.5
Molasses	100
Olive oil	57
Seawater	64
Water	62.4

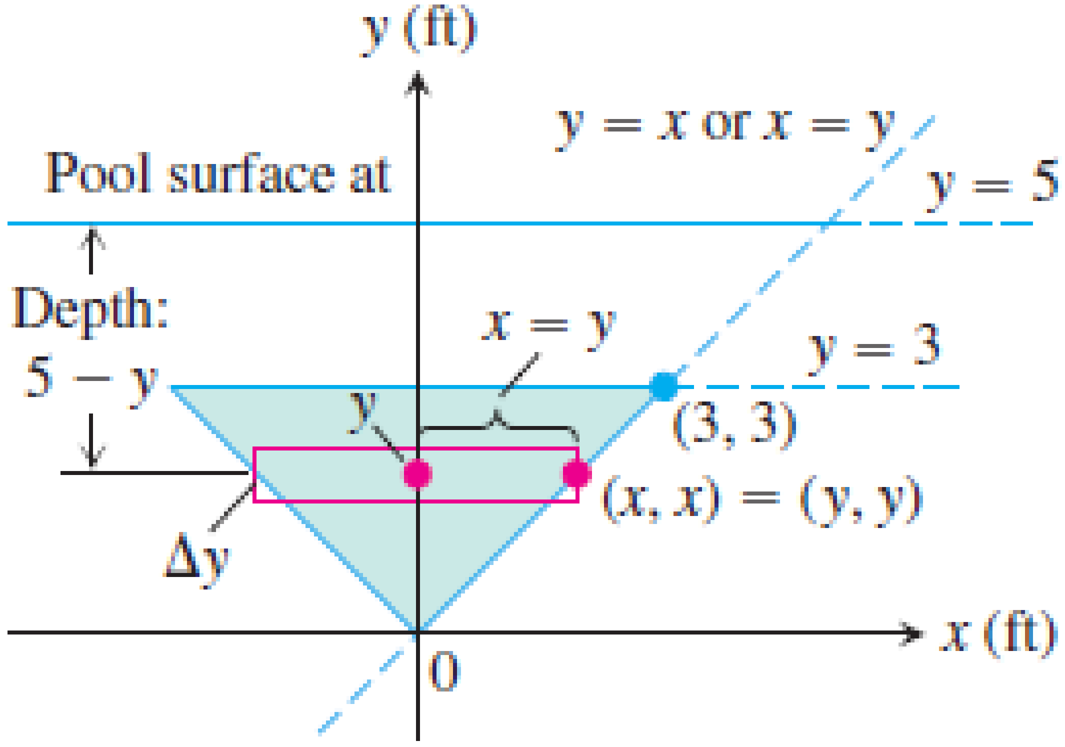
The Integral for Fluid Force Against a Vertical Flat Plate

Suppose that a plate submerged vertically in fluid of weight-density w runs from $y = a$ to $y = b$ on the y-axis. Let $L(y)$ be the length of the horizontal strip measured from left to right along the surface of the plate at level y. Then the force exerted by the fluid against one side of the plate is

$$F = \int_a^b w \cdot (\text{strip depth}) \cdot L(y)\, dy. \tag{4}$$



A flat isosceles right triangular plate with base 6 ft and height 3 ft is submerged vertically, base up, 2 ft below the surface of a swimming pool. Find the force exerted by the water against one side of the plate.

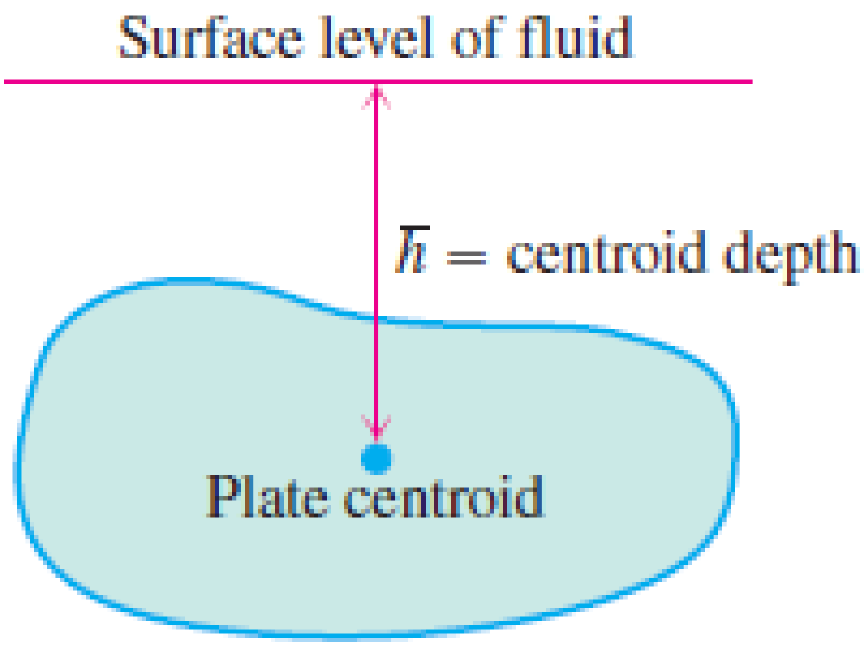


$$
\begin{aligned}
F &= \int_a^b w \cdot \begin{pmatrix}\text{strip}\\ \text{depth}\end{pmatrix} \cdot L(y)\, dy \qquad \text{Eq. (4)}\\
&= \int_0^3 62.4(5 - y)2y\, dy\\
&= 124.8\int_0^3 (5y - y^2)\, dy\\
&= 124.8\left[\frac{5}{2}y^2 - \frac{y^3}{3}\right]_0^3 = 1684.8 \text{ lb}.
\end{aligned}
$$

Fluid Forces and Centroids

The force of a fluid of weight-density w against one side of a submerged flat vertical plate is the product of w, the distance $\bar{h}$ from the plate's centroid to the fluid surface, and the plate's area:

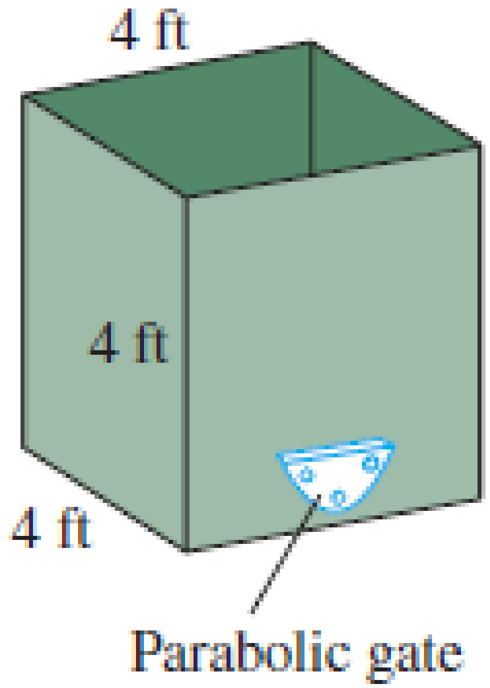
$$F = w\bar{h}A. \tag{5}$$

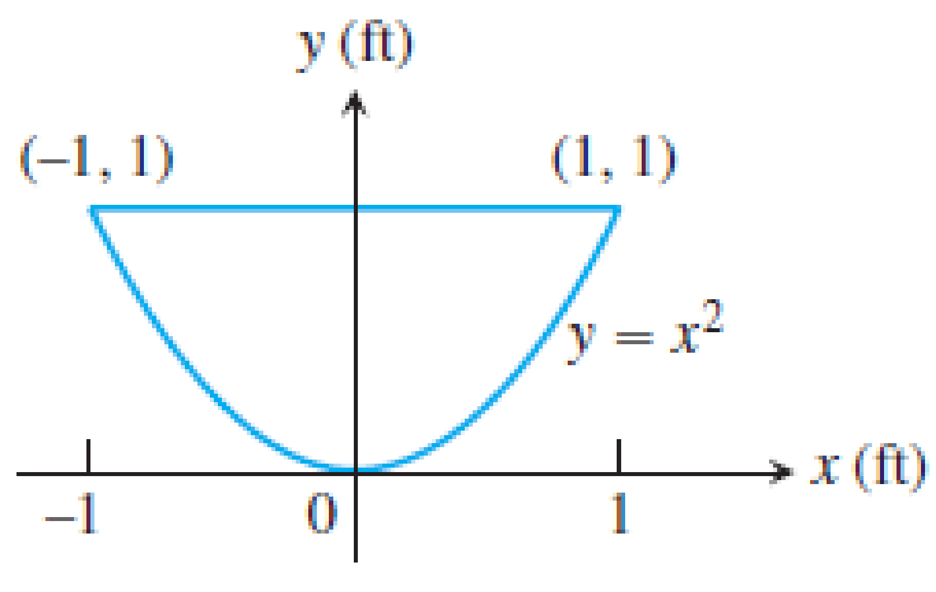


The cubical metal tank shown here has a parabolic gate, held in place by bolts and designed to withstand a fluid force of 160 lb without rupturing. The liquid you plan to store has a weight-density of 50 lb/ft^3.

a. What is the fluid force on the gate when the liquid is 2 ft deep?

b. What is the maximum height to which the container can be filled without exceeding its design limitation?





Enlarged view of parabolic gate

The coordinate system is given in the text. The right-hand edge is $x = \sqrt{y}$ and the total width is $L(y) = 2x = 2\sqrt{y}$.

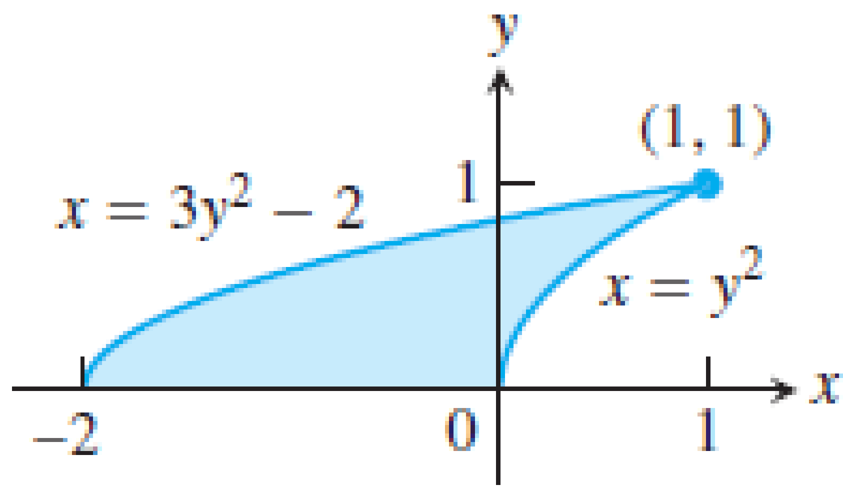
(a) The depth of the strip is $(2 - y)$ so the force exerted by the liquid on the gate is $F = \int_0^1 w(2-y)L(y)\,dy$

$$= \int_0^1 50(2-y)\cdot 2\sqrt{y}\,dy = 100\int_0^1 (2-y)\sqrt{y}\,dy = 100\int_0^1 \left(2y^{1/2} - y^{3/2}\right) dy = 100\left[\tfrac{4}{3}y^{3/2} - \tfrac{2}{5}y^{5/2}\right]_0^1$$

$$= 100\left(\tfrac{4}{3} - \tfrac{2}{5}\right) = \left(\tfrac{100}{15}\right)(20-6) = 93.33 \text{ lb}$$

(b) We need to solve $160 = \int_0^1 w(H-y)\cdot 2\sqrt{y}\,dy$ for h. $160 = 100\left(\frac{2H}{3} - \frac{2}{5}\right) \Rightarrow H = 3$ ft.

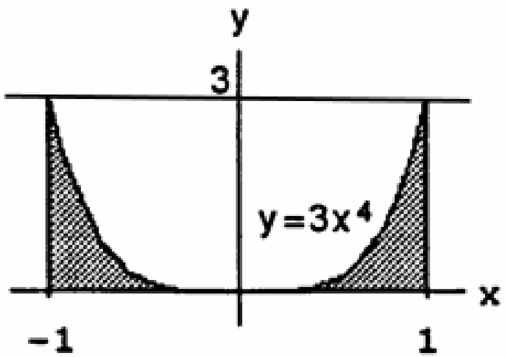
The region shown here is to be revolved about the x-axis to generate a solid. Which of the methods (disk, washer, shell) could you use to find the volume of the solid? How many integrals would be required in each case? Explain.



Find the volume of the solid generated by revolving the region bounded by the x-axis, the curve $y = 3x^4$, and the lines $x = 1$ and $x = -1$ about (a) the x-axis; (b) the y-axis; (c) the line $x = 1$; (d) the line $y = 3$.

(a) *disk method*:

$$V = \int_a^b \pi R^2(x)\,dx = \int_{-1}^{1} \pi\,(3x^4)^2\,dx = \pi \int_{-1}^{1} 9x^8\,dx$$
$$= \pi\,[x^9]_{-1}^{1} = 2\pi$$



(b) *shell method*:

$$V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dx = \int_0^1 2\pi x\,(3x^4)\,dx = 2\pi \cdot 3 \int_0^1 x^5\,dx = 2\pi \cdot 3 \left[\tfrac{x^6}{6}\right]_0^1 = \pi$$

Note: The lower limit of integration is 0 rather than -1.

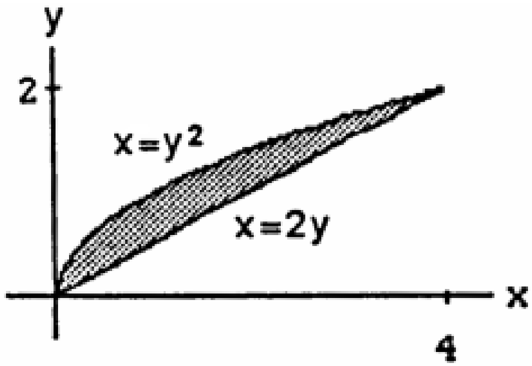
(c) *shell method*:

$$V = \int_a^b 2\pi \binom{\text{shell}}{\text{radius}} \binom{\text{shell}}{\text{height}} dx = 2\pi \int_{-1}^{1} (1-x)\,(3x^4)\,dx = 2\pi \left[\tfrac{3x^5}{5} - \tfrac{x^6}{2}\right]_{-1}^{1} = 2\pi \left[\left(\tfrac{3}{5} - \tfrac{1}{2}\right) - \left(-\tfrac{3}{5} - \tfrac{1}{2}\right)\right] = \tfrac{12\pi}{5}$$

(d) *washer method*:

$$R(x) = 3,\ r(x) = 3 - 3x^4 = 3\,(1 - x^4) \Rightarrow V = \int_a^b \pi\,[R^2(x) - r^2(x)]\,dx = \int_{-1}^{1} \pi \left[9 - 9\,(1 - x^4)^2\right] dx$$
$$= 9\pi \int_{-1}^{1} [1 - (1 - 2x^4 + x^8)]\,dx = 9\pi \int_{-1}^{1} (2x^4 - x^8)\,dx = 9\pi \left[\tfrac{2x^5}{5} - \tfrac{x^9}{9}\right]_{-1}^{1} = 18\pi \left[\tfrac{2}{5} - \tfrac{1}{9}\right] = \tfrac{2\pi \cdot 13}{5} = \tfrac{26\pi}{5}$$

Find the center of mass of a thin, flat plate covering the region enclosed by the parabola $y^2 = x$ and the line $x = 2y$ if the density function is $\delta(y) = 1 + y$. (Use horizontal strips.)



A typical horizontal strip has: center of mass: $(\tilde{x}, \tilde{y}) = \left(\frac{y^2+2y}{2}, y\right)$, length: $2y - y^2$, width: dy, area: $dA = (2y - y^2)\,dy$, mass: $dm = \delta \cdot dA = (1+y)(2y - y^2)\,dy \Rightarrow$ the moment about the x-axis is $\tilde{y}\,dm = y(1+y)(2y-y^2)\,dy = (2y^2 + 2y^3 - y^3 - y^4)\,dy = (2y^2 + y^3 - y^4)\,dy$; the moment about the y-axis is

$\tilde{x}\,dm = \left(\frac{y^2+2y}{2}\right)(1+y)(2y-y^2)\,dy = \frac{1}{2}(4y^2 - y^4)(1+y)\,dy = \frac{1}{2}(4y^2 + 4y^3 - y^4 - y^5)\,dy$

$\Rightarrow M_x = \int \tilde{y}\,dm = \int_0^2 (2y^2 + y^3 - y^4)\,dy = \left[\frac{2}{3}y^3 + \frac{y^4}{4} - \frac{y^5}{5}\right]_0^2 = \left(\frac{16}{3} + \frac{16}{4} - \frac{32}{5}\right) = 16\left(\frac{1}{3} + \frac{1}{4} - \frac{2}{5}\right)$

$= \frac{16}{60}(20 + 15 - 24) = \frac{4}{15}(11) = \frac{44}{15}$; $M_y = \int \tilde{x}\,dm = \int_0^2 \frac{1}{2}(4y^2 + 4y^3 - y^4 - y^5)\,dy = \frac{1}{2}\left[\frac{4}{3}y^3 + y^4 - \frac{y^5}{5} - \frac{y^6}{6}\right]_0^2$

$= \frac{1}{2}\left(\frac{4 \cdot 2^3}{3} + 2^4 - \frac{2^5}{5} - \frac{2^6}{6}\right) = 4\left(\frac{4}{3} + 2 - \frac{4}{5} - \frac{8}{6}\right) = 4\left(2 - \frac{4}{5}\right) = \frac{24}{5}$; $M = \int dm = \int_0^2 (1+y)(2y - y^2)\,dy$

$= \int_0^2 (2y + y^2 - y^3)\,dy = \left[y^2 + \frac{y^3}{3} - \frac{y^4}{4}\right]_0^2 = \left(4 + \frac{8}{3} - \frac{16}{4}\right) = \frac{8}{3} \Rightarrow \bar{x} = \frac{M_y}{M} = \left(\frac{24}{5}\right)\left(\frac{3}{8}\right) = \frac{9}{5}$ and $\bar{y} = \frac{M_x}{M}$

$= \left(\frac{44}{15}\right)\left(\frac{3}{8}\right) = \frac{44}{40} = \frac{11}{10}$. Therefore, the center of mass is $(\bar{x}, \bar{y}) = \left(\frac{9}{5}, \frac{11}{10}\right)$.