Chapter

2

LIMITS AND CONTINUITY

LIMIT IS A CENTRAL IDEA THAT DISTINGUISHES CALCULUS FROM ALGEBRA

FUNDAMENTAL TO FINDING THE TANGENT TO A CURVE

Rates of Change and Limits

average and instantaneous rates of change.

Example: A rock breaks loose from the top of a tall cliff. What is its average speed

(a) during the first 2 sec of fall?

(b) during the 1-sec interval between second 1 and second 2?

we use the fact, discovered by Galileo in the late 16th century, that a solid object dropped from rest to fall freely near the surface of the earth will fall a distance proportional to the square of the time it has been falling.

(This assumes negligible air resistance to slow the object down and that gravity is the only force acting on the falling body. We call this type of motion **free fall**.)

If y denotes the distance fallen in feet after t seconds, then Galileo's law is

$y = 16\,t^2$ (ft) $y = 4.9\,t^2$ (m)

where 16 is the constant of proportionality.

The average speed of the rock during a given time interval Δy, divided by the length of the time interval, Δt.

(a) For the first 2 sec: $$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(0)^2}{2 - 0} = 32\,\frac{\text{ft}}{\text{sec}}$$

(b) From sec 1 to sec 2: $$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(1)^2}{2 - 1} = 48\,\frac{\text{ft}}{\text{sec}}$$

EXAMPLE 2 Finding an Instantaneous Speed

Find the speed of the falling rock at $t = 1$ and $t = 2$ sec.

Solution We can calculate the average speed of the rock over a time interval $[t_0, t_0 + h]$, having length $\Delta t = h$, as

$$\frac{\Delta y}{\Delta t} = \frac{16(t_0 + h)^2 - 16t_0^2}{h}. \quad (1)$$

We cannot use this formula to calculate the "instantaneous" speed at t_0 by substituting $h = 0$, because we cannot divide by zero. But we *can* use it to calculate average speeds over increasingly short time intervals starting at $t_0 = 1$ and $t_0 = 2$. When we do so, we see a pattern (Table).

TABLE Average speeds over short time intervals

$$\text{Average speed: } \frac{\Delta y}{\Delta t} = \frac{16(t_0 + h)^2 - 16t_0^2}{h}$$

Length of time interval h	Average speed over interval of length h starting at $t_0 = 1$	Average speed over interval of length h starting at $t_0 = 2$
1	48	80
0.1	33.6	65.6
0.01	32.16	64.16
0.001	32.016	64.016
0.0001	32.0016	64.0016

Average Rates of Change and Secant Lines

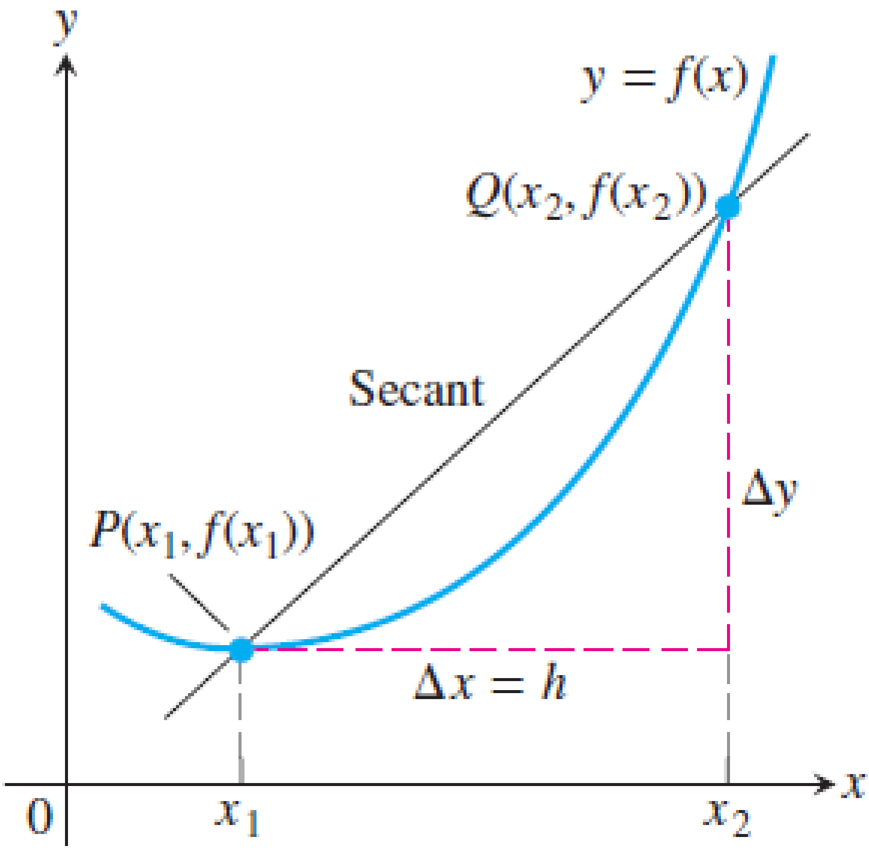
Given an arbitrary function $y = f(x)$, we calculate the average rate of change of y with respect to x over the interval $[x_1, x_2]$ by dividing the change in the value of y, $\Delta y = f(x_2) - f(x_1)$, by the length $\Delta x = x_2 - x_1 = h$ of the interval over which the change occurs.

DEFINITION Average Rate of Change over an Interval

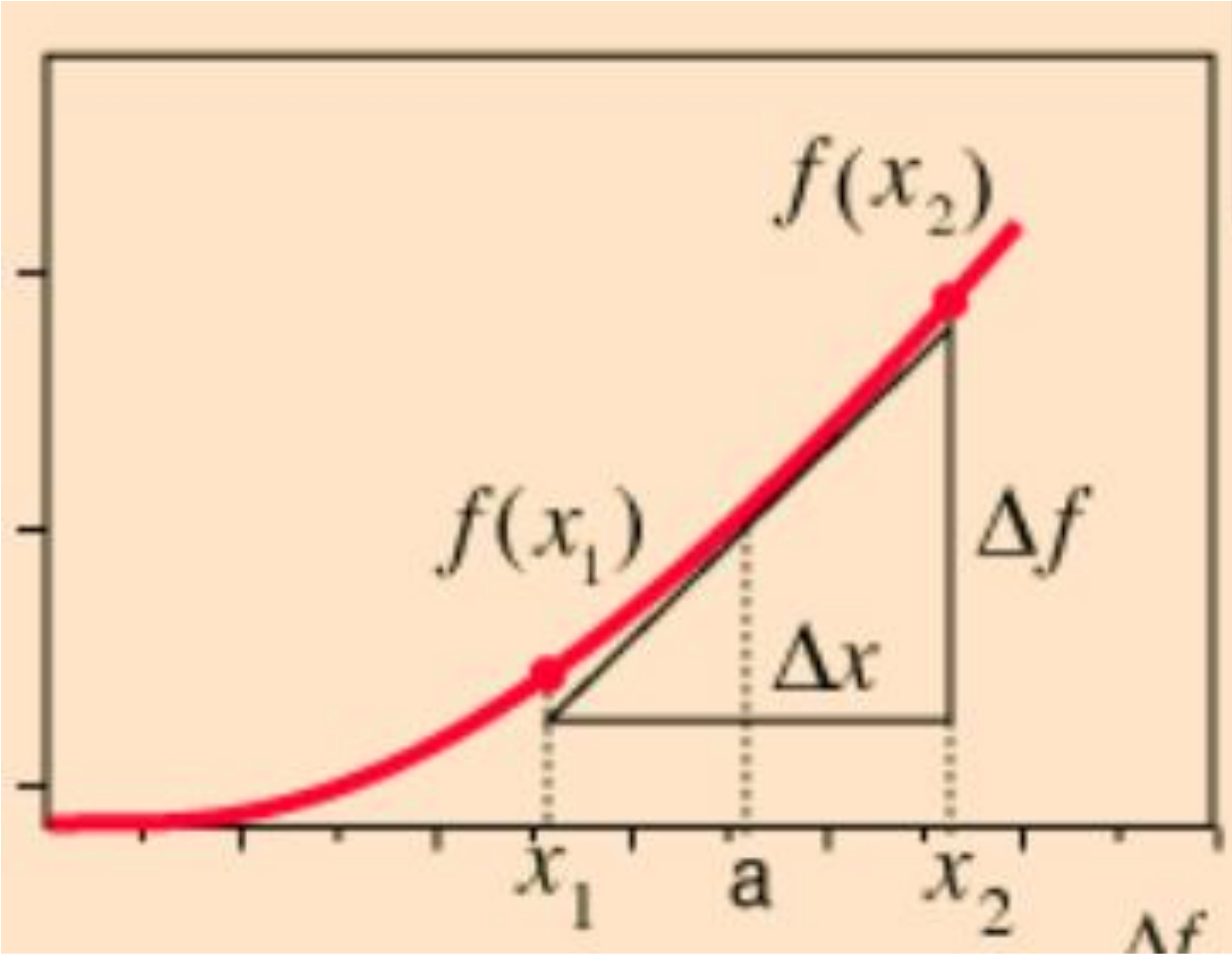
The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

Average Rates of Change and Secant Lines

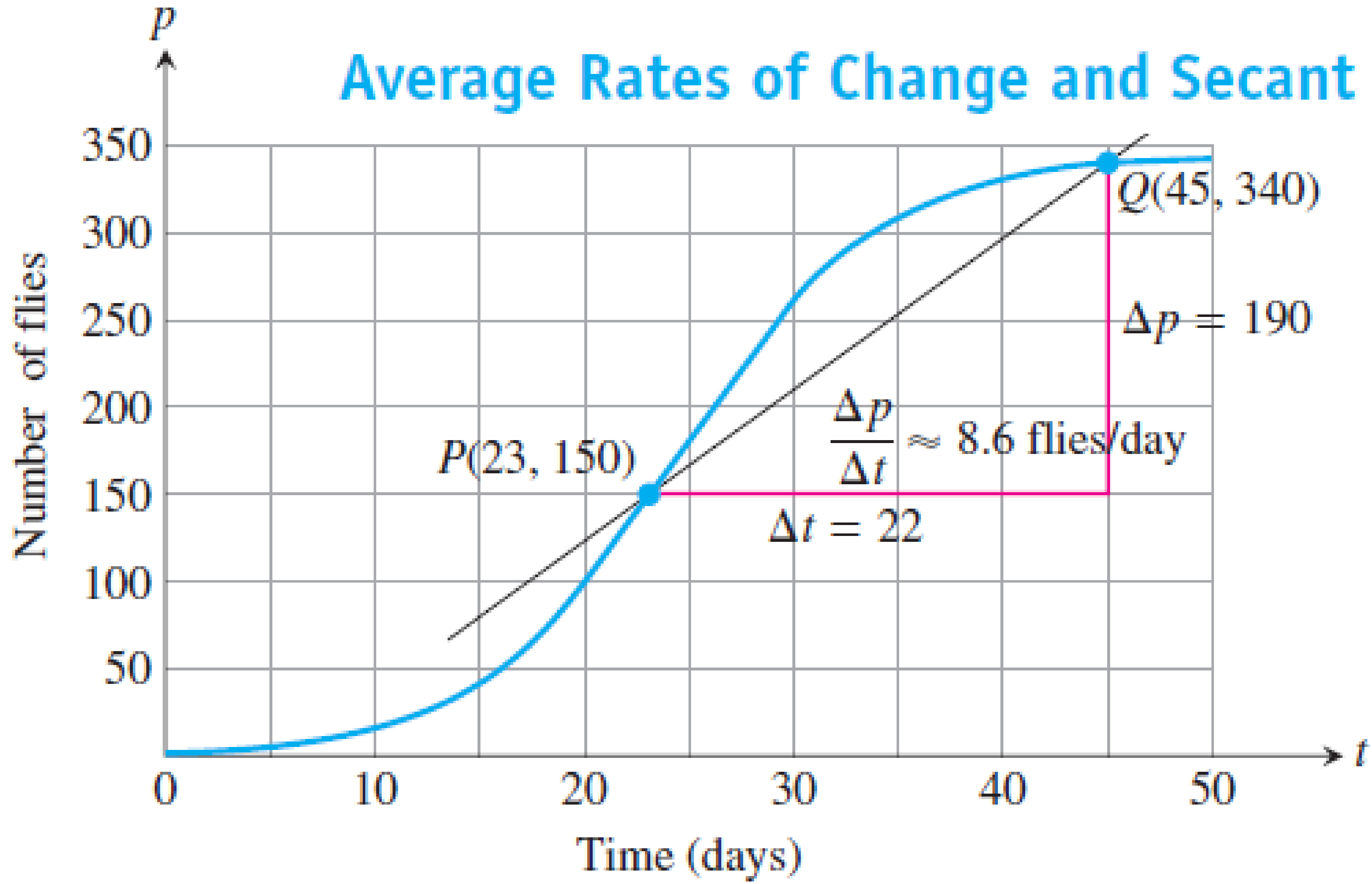


A secant to the graph $y = f(x)$. Its slope is $\Delta y / \Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.



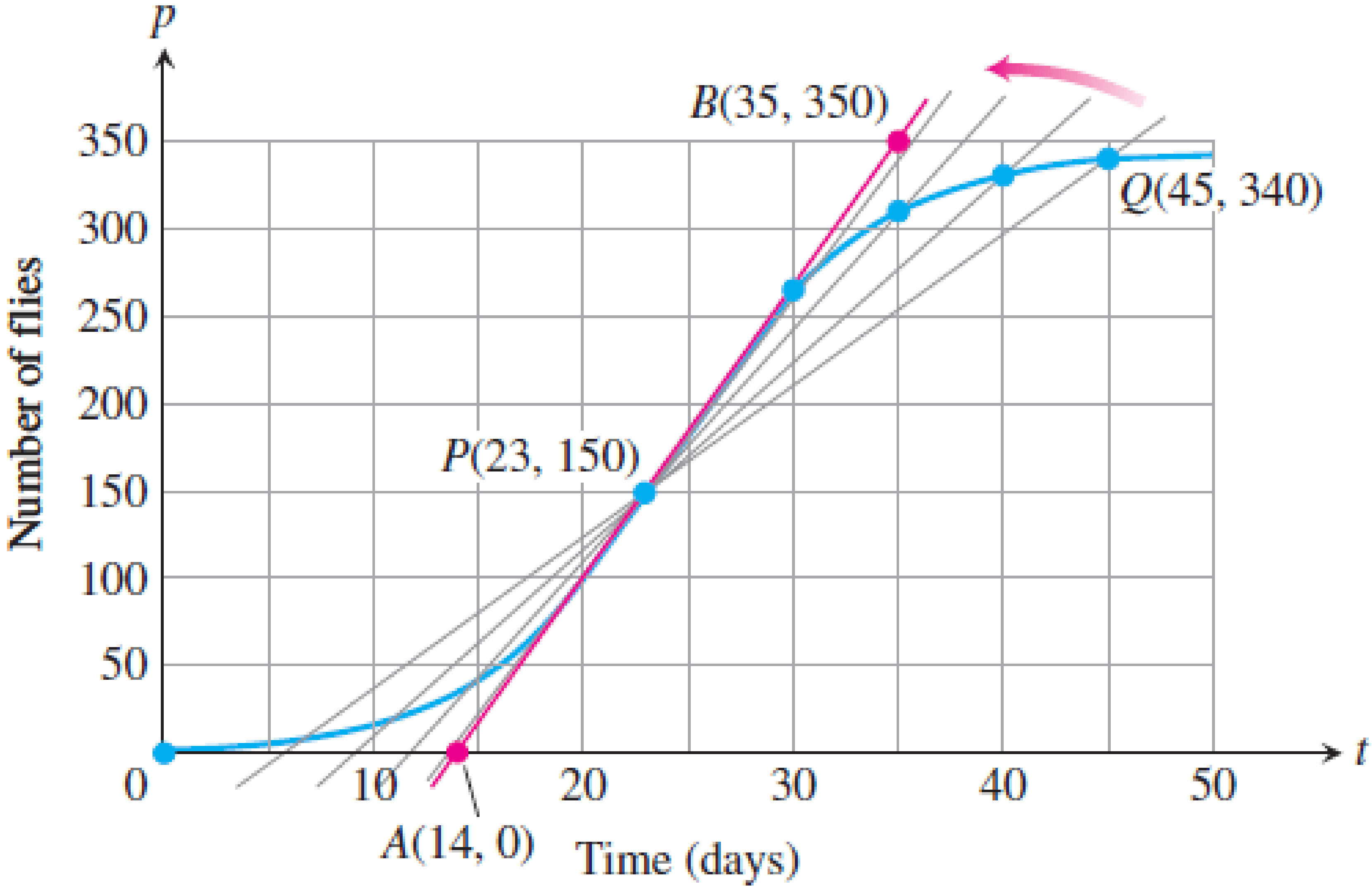
$f(x_2)$
$f(x_1)$
Δf
Δx
x_1
a
x_2

Average Rates of Change and Secant Lines



Growth of a fruit fly population in a controlled experiment. The average rate of change over 22 days is the slope $\Delta p / \Delta t$ of the secant line.

How fast was the number of flies in the population growing on day 23?



Limits of Function Values

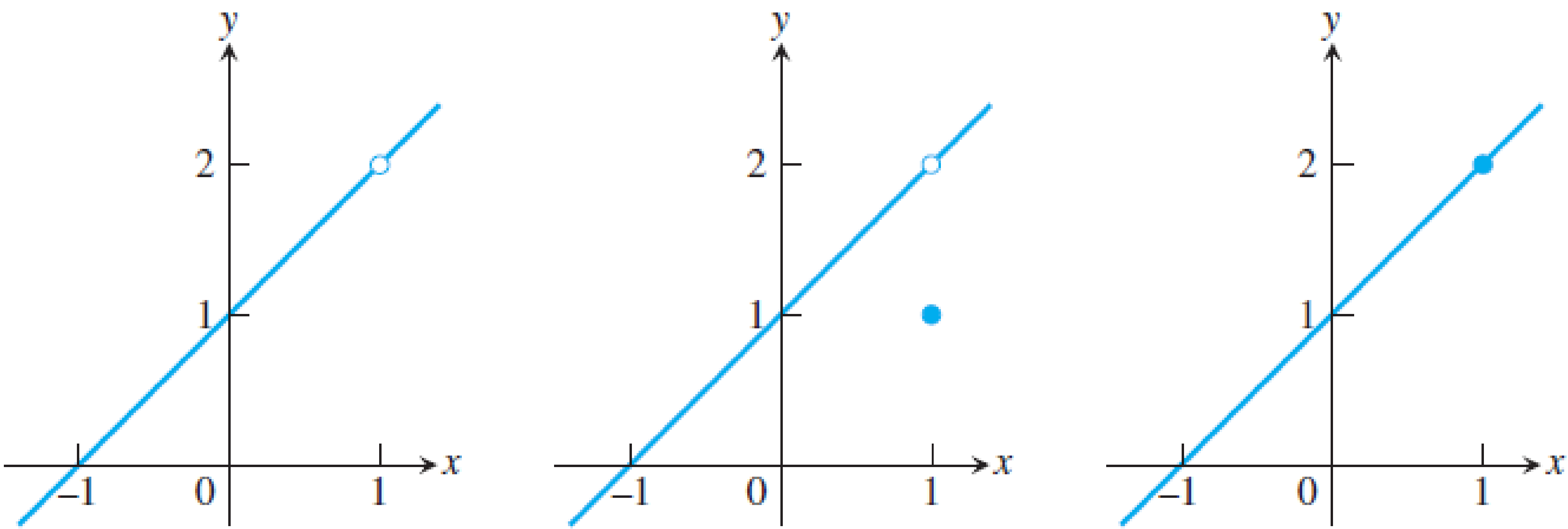
Let f(x) be defined on an open interval about $\boldsymbol{x_0}$ *except possibly at* $\boldsymbol{x_0}$ *itself*. If f(x) gets arbitrarily close to L (as close to L as we like) for all x sufficiently close to $\boldsymbol{x_0}$ we say that f approaches the **limit** L as x approaches $\boldsymbol{x_0}$ and we write

$$\lim_{x \to x_0} f(x) = L$$

which is read "the limit of f(x) as x approaches $\boldsymbol{x_0}$ is L". Essentially, the definition says that the values of f(x) are close to the number L whenever x is close to (on either side of). This definition is "informal" because phrases like *arbitrarily close* and *sufficiently close* are imprecise;

The Limit Value Does Not Depend on How the Function Is Defined at x_0

The function f in Figure 2.5 has limit 2 as $x \to 1$ even though f is not defined at $x = 1$. The function g has limit 2 as $x \to 1$ even though $2 \neq g(1)$. The function h is the only one



(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$

(b) $g(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$

(c) $h(x) = x + 1$

The limits of $f(x)$, $g(x)$, and $h(x)$ all equal 2 as x approaches 1. However, only $h(x)$ has the same function value as its limit at $x = 1$ (Example 6).

Finding Limits by Calculating $f(x_0)$

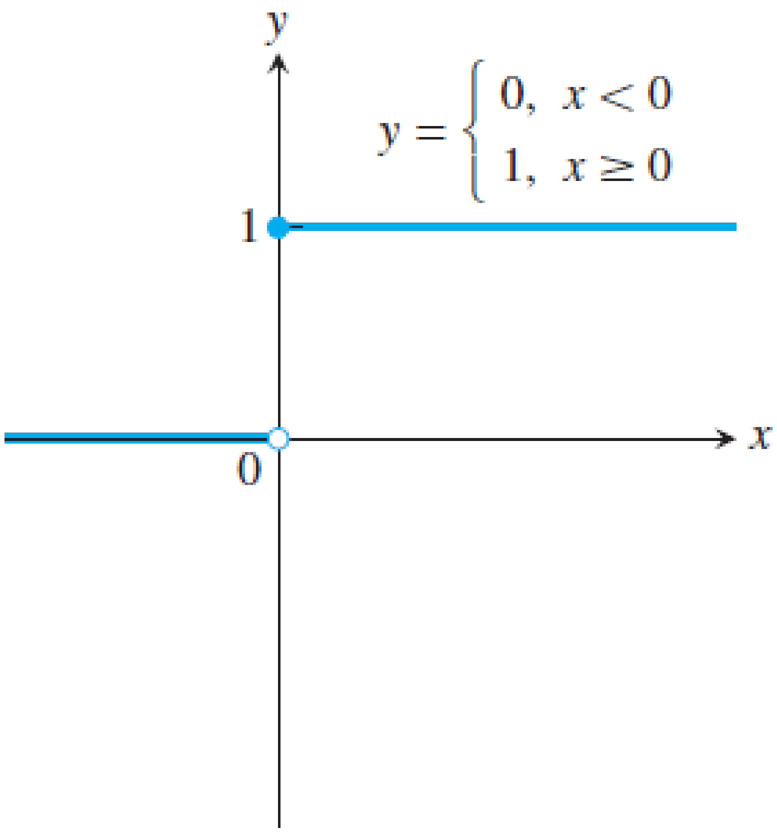
(a) $\lim_{x \to 2} (4) = 4$

(b) $\lim_{x \to -13} (4) = 4$

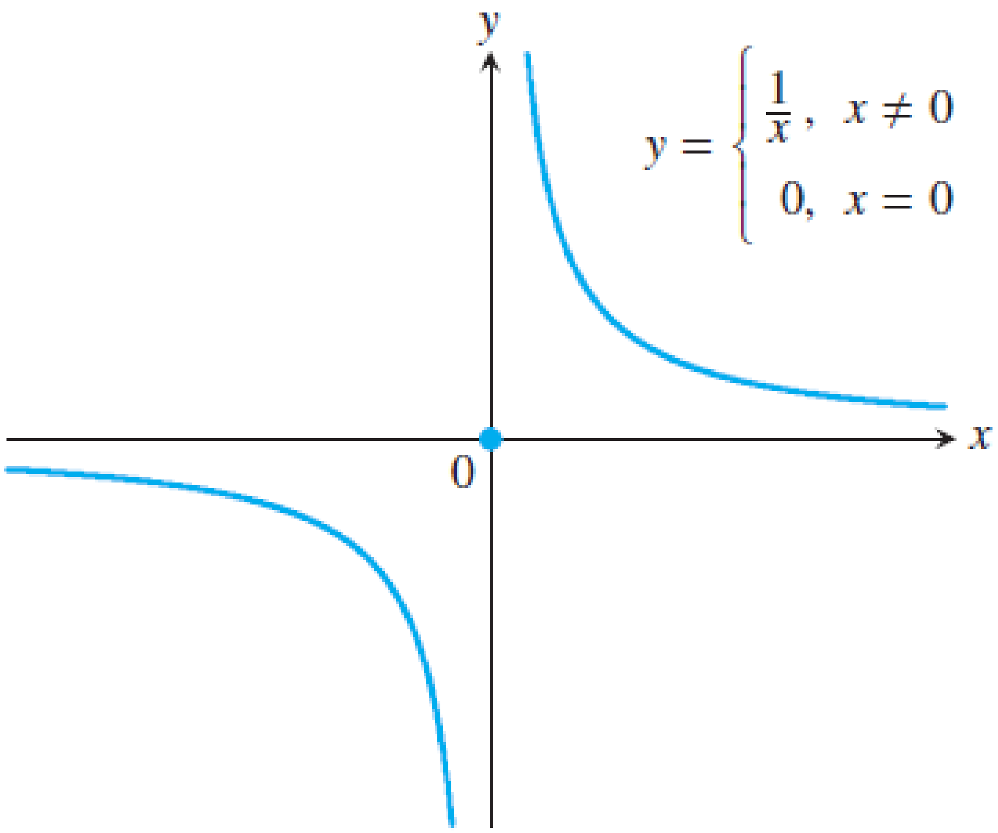
(c) $\lim_{x \to 3} x = 3$

(d) $\lim_{x \to 2} (5x - 3) = 10 - 3 = 7$

(e) $\lim_{x \to -2} \frac{3x + 4}{x + 5} = \frac{-6 + 4}{-2 + 5} = -\frac{2}{3}$



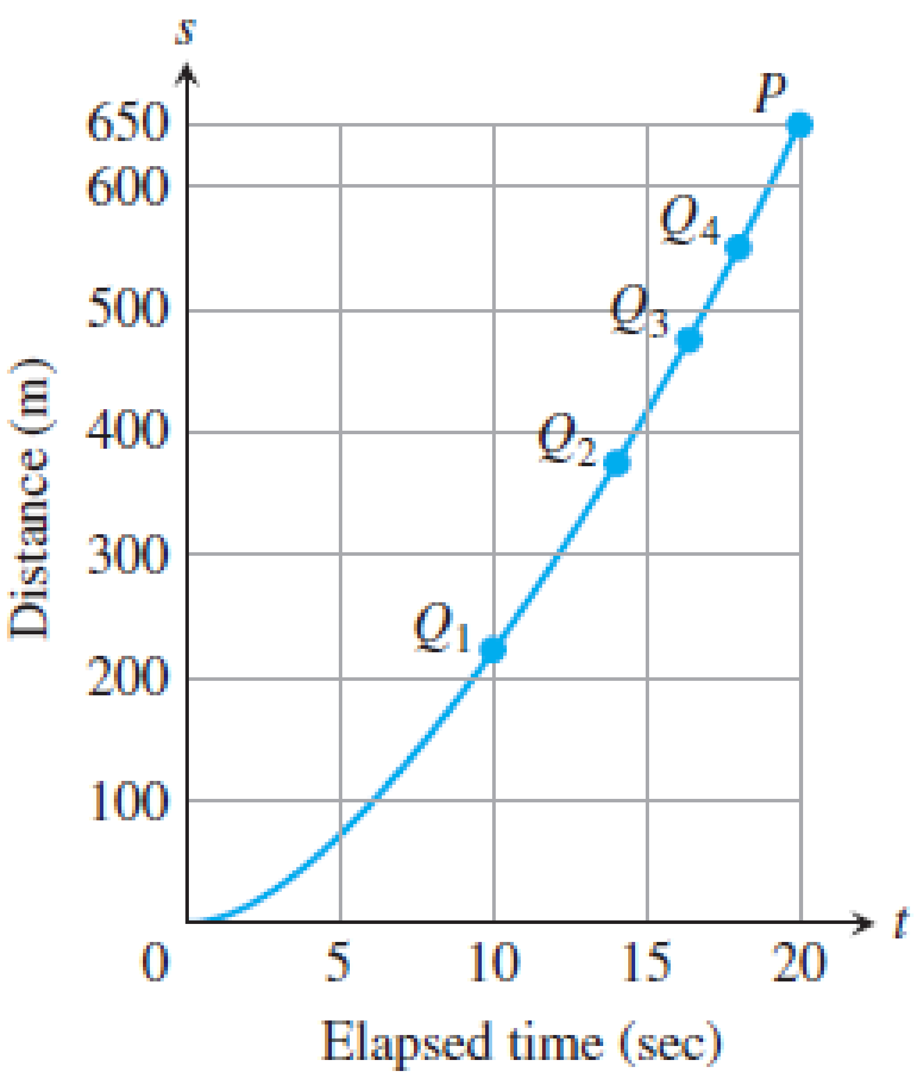
(a) Unit step function $U(x)$



(b) $g(x)$

None of these functions has a limit as x approaches 0 (Example 9).

A Ford Mustang Cobra's speed The accompanying figure shows the time-to-distance graph for a 1994 Ford Mustang Cobra accelerating from a standstill.



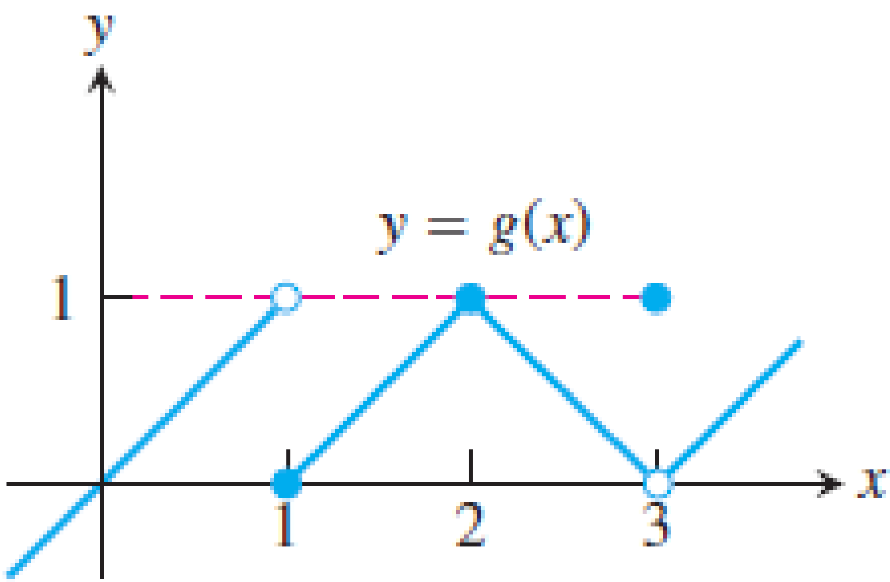
(a)

Q	Slope of PQ $= \frac{\Delta p}{\Delta t}$
$Q_1(10, 225)$	$\frac{650-225}{20-10} = 42.5$ m/sec
$Q_2(14, 375)$	$\frac{650-375}{20-14} = 45.83$ m/sec
$Q_3(16.5, 475)$	$\frac{650-475}{20-16.5} = 50.00$ m/sec
$Q_4(18, 550)$	$\frac{650-550}{20-18} = 50.00$ m/sec

(b) At $t = 20$, the Cobra was traveling approximately 50 m/sec or 180 km/h.

For the function $g(x)$ graphed here, find the following limits or explain why they do not exist.

a. $\lim_{x \to 1} g(x)$ b. $\lim_{x \to 2} g(x)$ c. $\lim_{x \to 3} g(x)$



a) At x = 1 limit does not exist
b) At x = 2 limit equals to 1
c) At x = 3 limit equals to 0

Calculating Limits Using the Limit Laws

how to calculate limits of functions that are arithmetic combinations of functions whose limits we already know.

THEOREM 1 Limit Laws

If L, M, c and k are real numbers and

$$\lim_{x \to c} f(x) = L \quad \text{and} \quad \lim_{x \to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule*: $$\lim_{x \to c}(f(x) + g(x)) = L + M$$

The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule*: $$\lim_{x \to c}(f(x) - g(x)) = L - M$$

The limit of the difference of two functions is the difference of their limits.

3. *Product Rule*: $$\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$$

The limit of a product of two functions is the product of their limits.

4. *Constant Multiple Rule:* $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$

The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule*: $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$

The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. *Power Rule*: If r and s are integers with no common factor and $s \neq 0$, then

$$\lim_{x \to c}(f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

The limit of a rational power of a function is that power of the limit of the function, provided the latter is a real number.

THEOREM 2 Limits of Polynomials Can Be Found by Substitution

If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x \to c} P(x) = P(c) = a_n c^n + a_{n-1} c^{n-1} + \cdots + a_0.$$

THEOREM 3 Limits of Rational Functions Can Be Found by Substitution If the Limit of the Denominator Is Not Zero

If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Limit of a Rational Function

$$\lim_{x \to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0$$

Eliminating Zero Denominators Algebraically

Canceling a Common Factor

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}.$$

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x - 1)(x + 2)}{x(x - 1)} = \frac{x + 2}{x}, \qquad \text{if } x \neq 1.$$

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{x + 2}{x} = \frac{1 + 2}{1} = 3.$$

Creating and Canceling a Common Factor

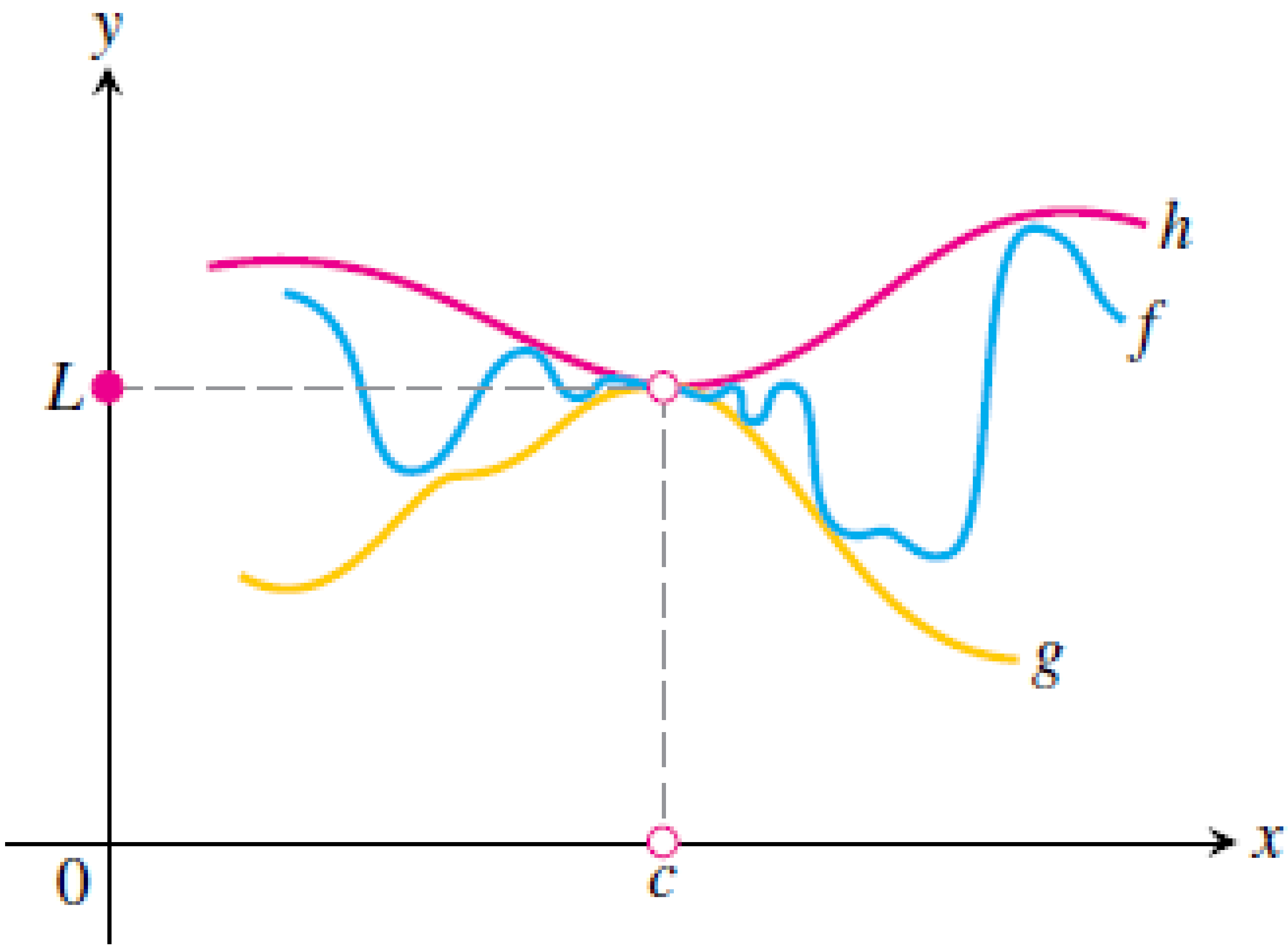
$$\lim_{x \to 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}.$$

$$\begin{aligned}
\lim_{x \to 0} \frac{\sqrt{x^2 + 100} - 10}{x^2} &= \lim_{x \to 0} \frac{1}{\sqrt{x^2 + 100} + 10} \\
&= \frac{1}{\sqrt{0^2 + 100} + 10} \\
&= \frac{1}{20} = 0.05.
\end{aligned}$$

The Sandwich Theorem

the Sandwich Theorem refers to a function f whose values are sandwiched between the values of two other functions *g* and *h* that have the same limit *L* at a point *c*.

Being trapped between the values of two functions that approach *L*, the values of f must also approach *L*



The graph of f is
sandwiched between the graphs of g and h.

THEOREM 4 The Sandwich Theorem

Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

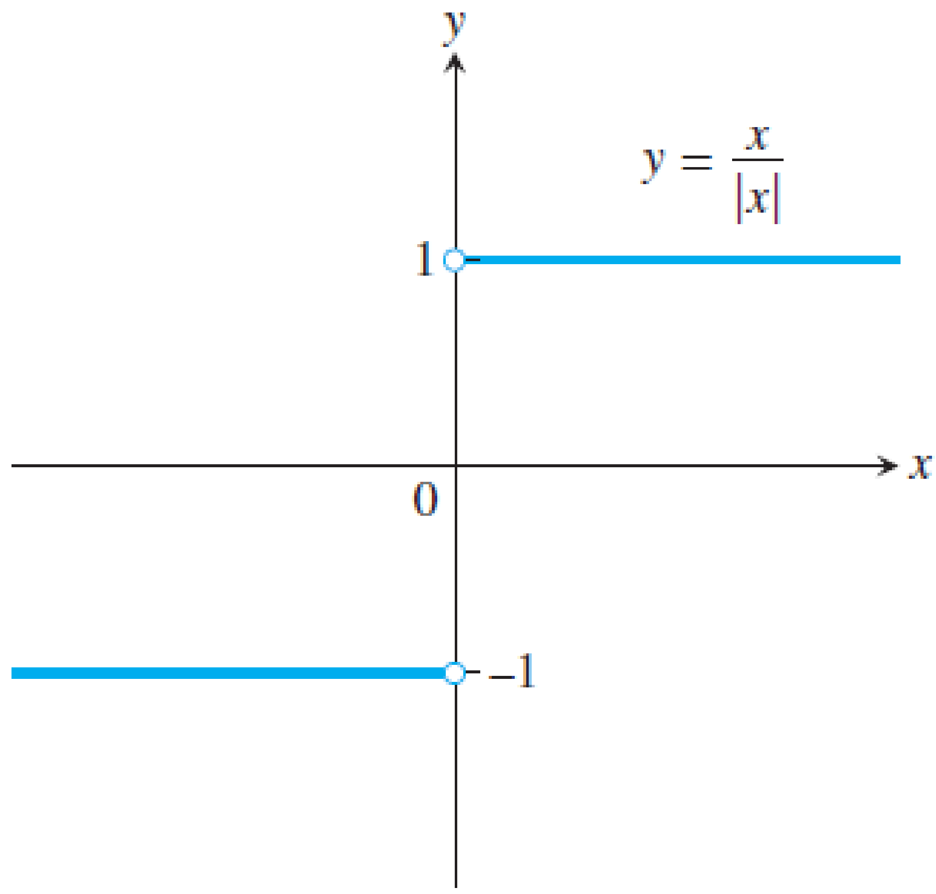
$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

THEOREM 5 If $f(x) \leq g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x \to c} f(x) \leq \lim_{x \to c} g(x).$$

One-Sided Limits and Limits at Infinity



Different right-hand and left-hand limits at the origin.

One-Sided Limits

To have a limit *L* as *x* approaches *c*, a function ƒ must be defined on *both sides* of *c* and its values ƒ(*x*) must approach *L* as *x* approaches *c* from either side.

ordinary limits are called **two-sided**

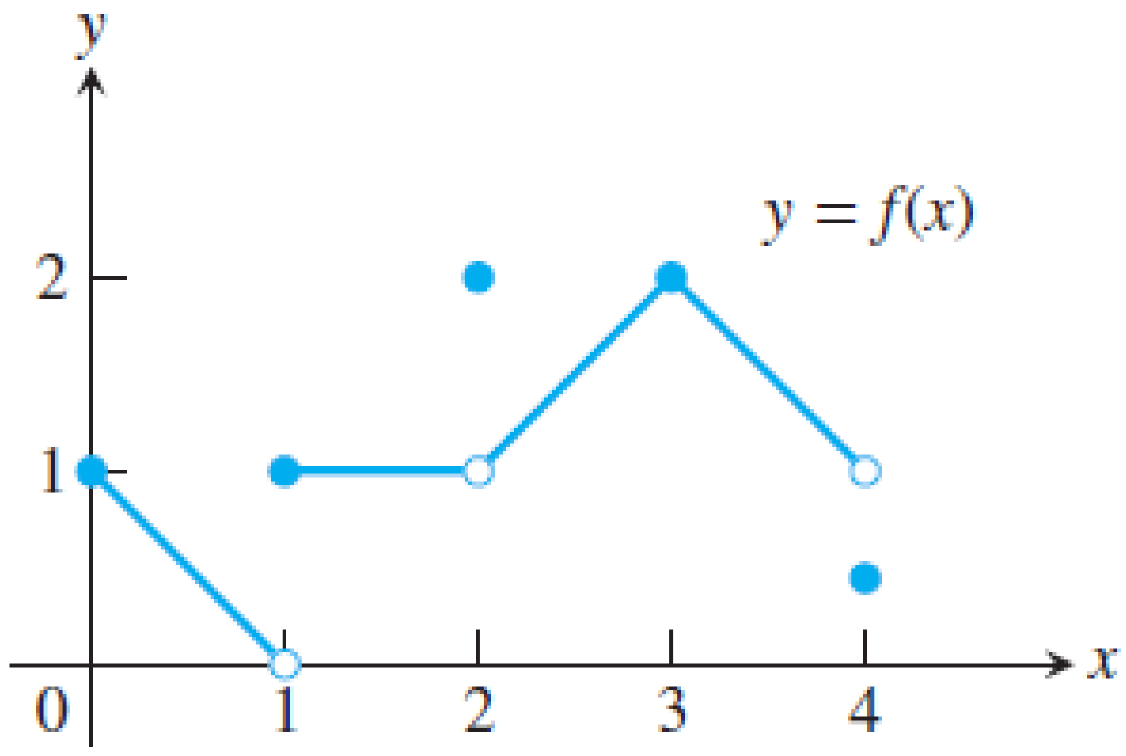
If ƒ fails to have a two-sided limit at *c*, it may still have a one-sided limit,

If the approach is from the right, the limit is a **right-hand limit**. From the left, it is a **left-hand limit**.

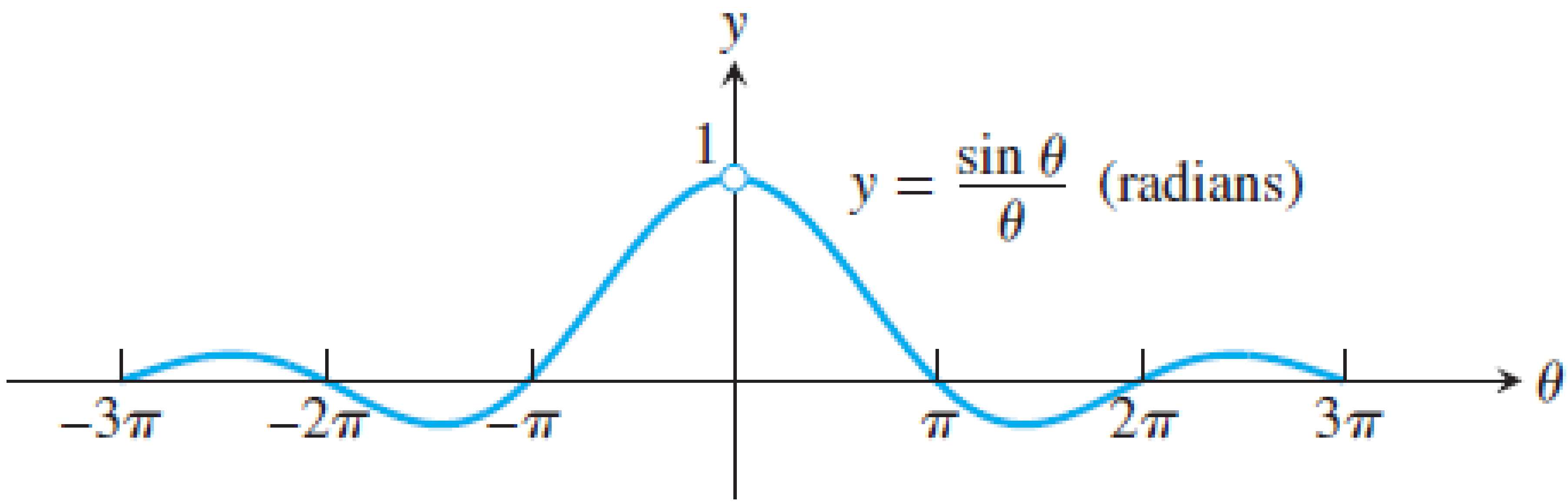
THEOREM 6

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$



Limits Involving $(\sin \theta)/\theta$

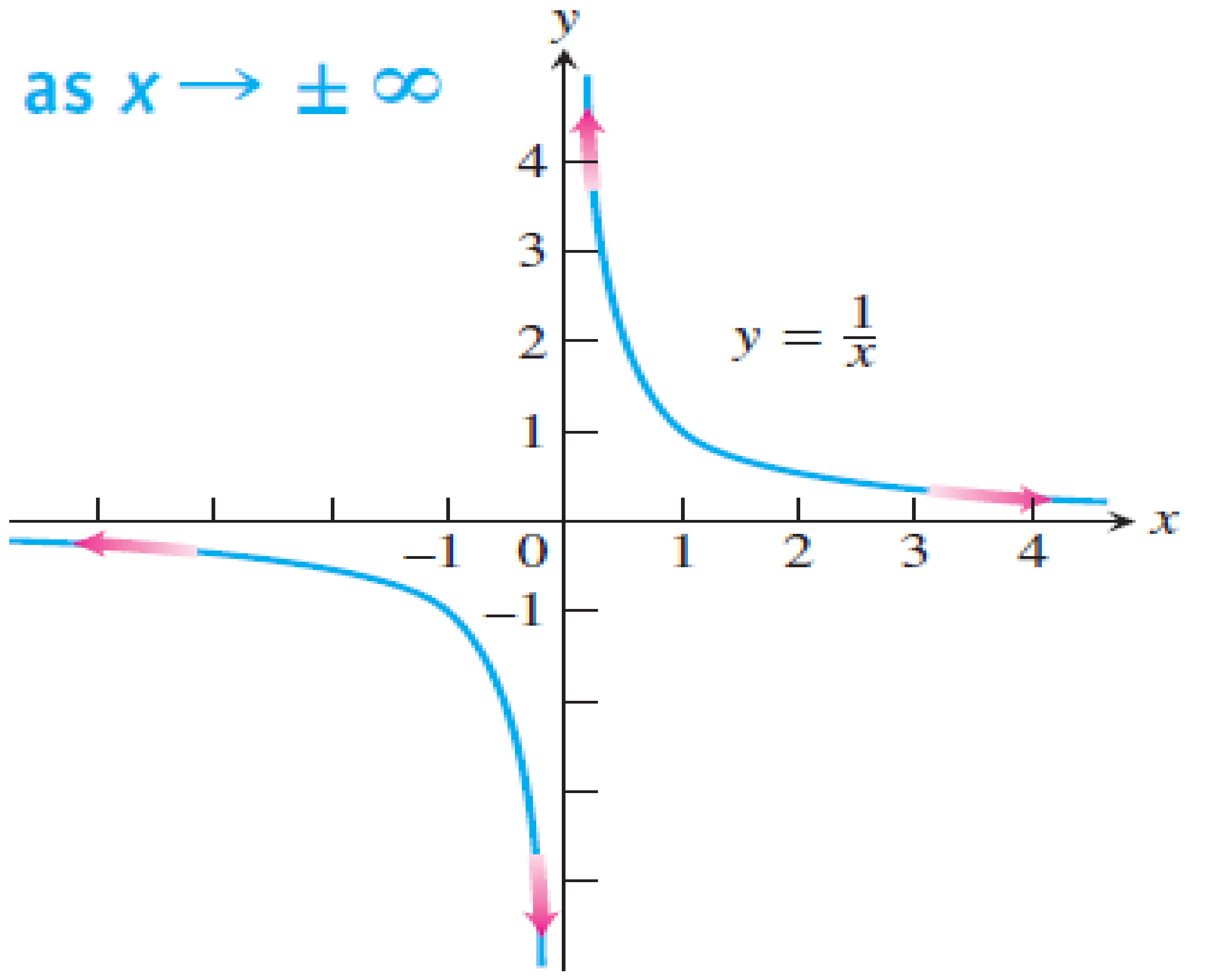


NOT TO SCALE

THEOREM 7

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \qquad (\theta \text{ in radians})$$

Finite Limits as $x \to \pm\infty$



$f(x) = 1/x$ has limit 0 as $x \to \pm\infty$ or that 0 is a *limit of $f(x) = 1/x$ at infinity and negative infinity*. Here is a precise definition.

THEOREM 8 Limit Laws as $x \to \pm\infty$

If L, M, and k, are real numbers and

$$\lim_{x\to\pm\infty} f(x) = L \quad \text{and} \quad \lim_{x\to\pm\infty} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x\to\pm\infty} (f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to\pm\infty} (f(x) - g(x)) = L - M$
3. *Product Rule:* $\lim_{x\to\pm\infty} (f(x) \cdot g(x)) = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{x\to\pm\infty} (k \cdot f(x)) = k \cdot L$
5. *Quotient Rule:* $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* If r and s are integers with no common factors, $s \neq 0$, then

$$\lim_{x\to\pm\infty} (f(x))^{r/s} = L^{r/s}$$

provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

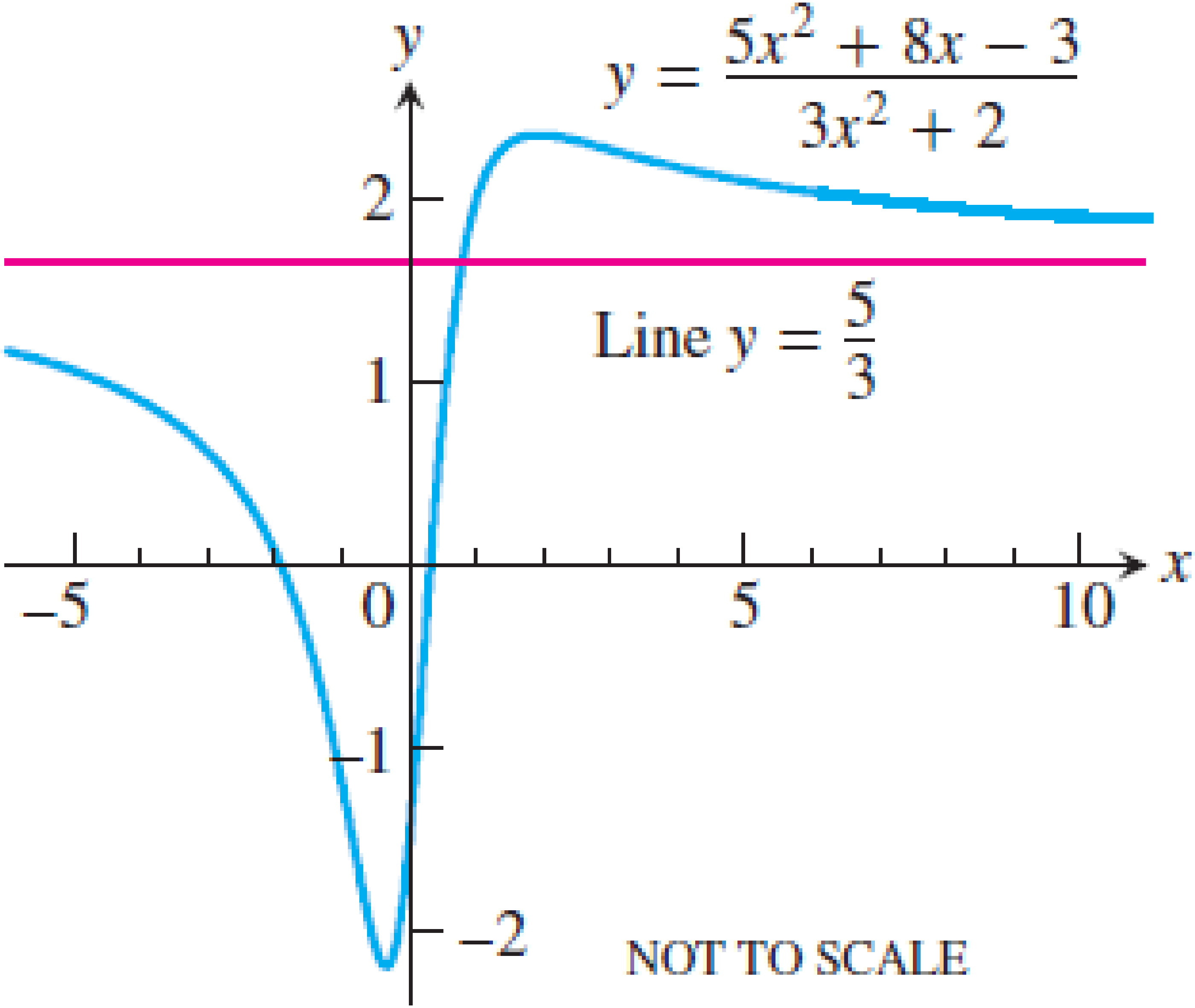
$$\lim_{x\to\infty}\left(5+\frac{1}{x}\right)=\lim_{x\to\infty}5+\lim_{x\to\infty}\frac{1}{x}$$

$$=5+0=5$$

Numerator and Denominator of Same Degree

$$\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2}=\lim_{x\to\infty}\frac{5+(8/x)-(3/x^2)}{3+(2/x^2)}$$

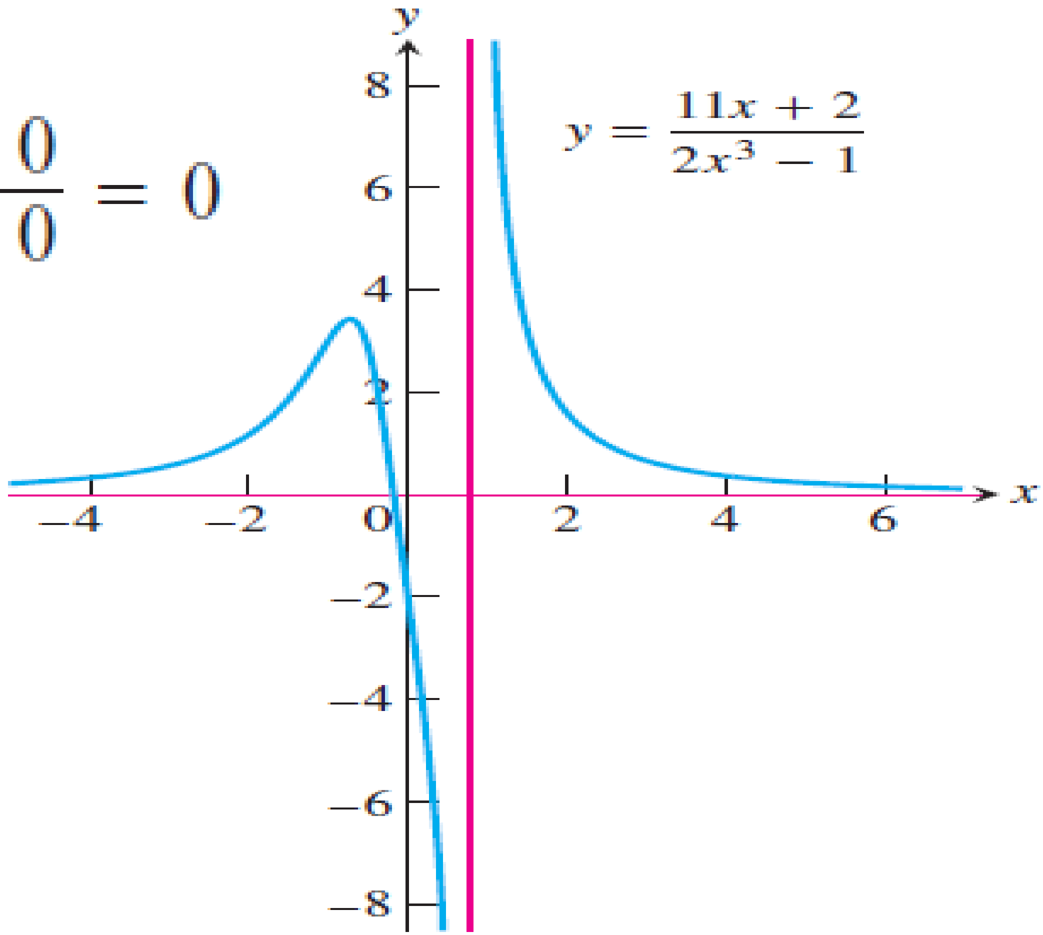
$$=\frac{5+0-0}{3+0}=\frac{5}{3}$$


y
$y = \frac{5x^2 + 8x - 3}{3x^2 + 2}$
2
1
Line $y = \frac{5}{3}$
−5
0
5
10
x
−1
−2
NOT TO SCALE

Degree of Numerator Less Than Degree of Denominator

$$\lim_{x \to -\infty} \frac{11x + 2}{2x^3 - 1} = \lim_{x \to -\infty} \frac{(11/x^2) + (2/x^3)}{2 - (1/x^3)}$$

$$= \frac{0 + 0}{2 - 0} = 0$$



DEFINITION **Horizontal Asymptote**

A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = b.$$

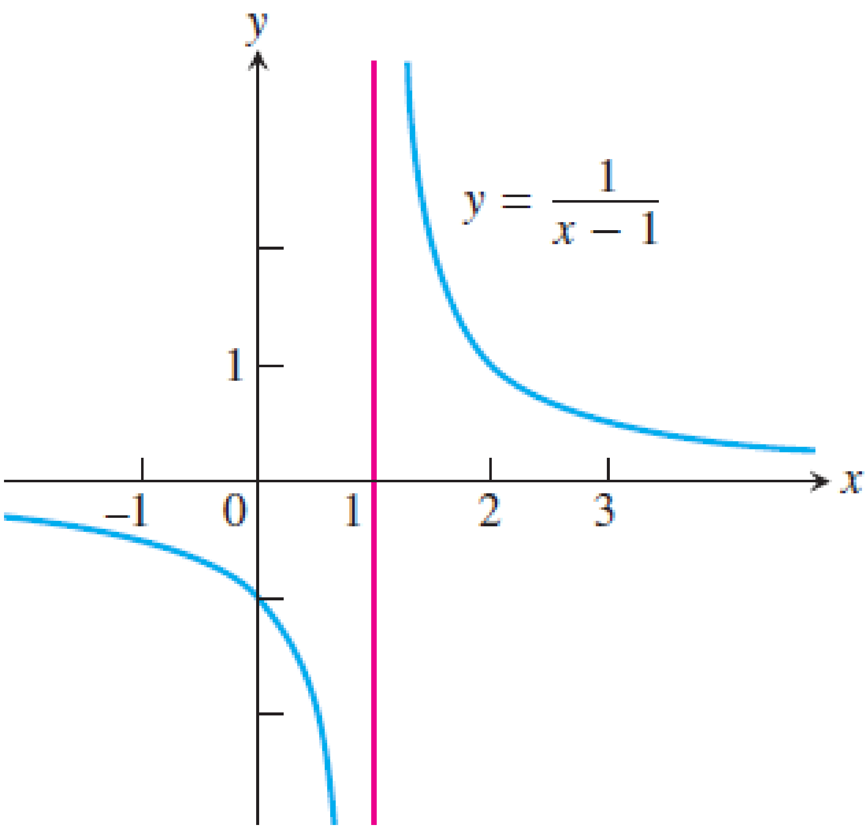
Find the oblique asymptote for the graph of

$$f(x) = \frac{2x^2 - 3}{7x + 4}$$

By long division, we find

$$\begin{aligned} f(x) &= \frac{2x^2 - 3}{7x + 4} \\ &= \underbrace{\left(\frac{2}{7}x - \frac{8}{49}\right)}_{\text{linear function } g(x)} + \underbrace{\frac{-115}{49(7x + 4)}}_{\text{remainder}} \end{aligned}$$

Infinite Limits and Vertical Asymptotes

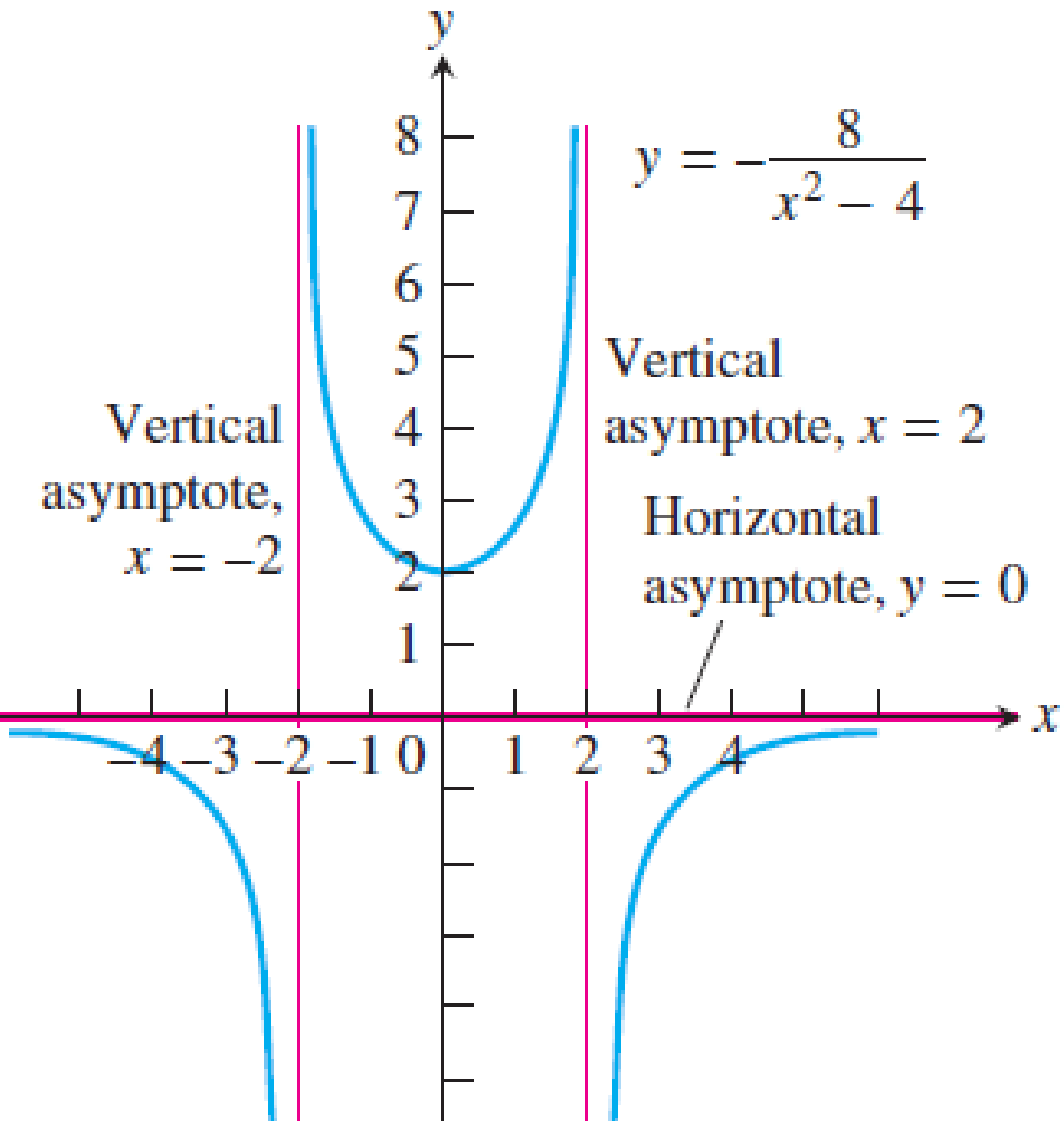


extend the concept of limit to *infinite limits*

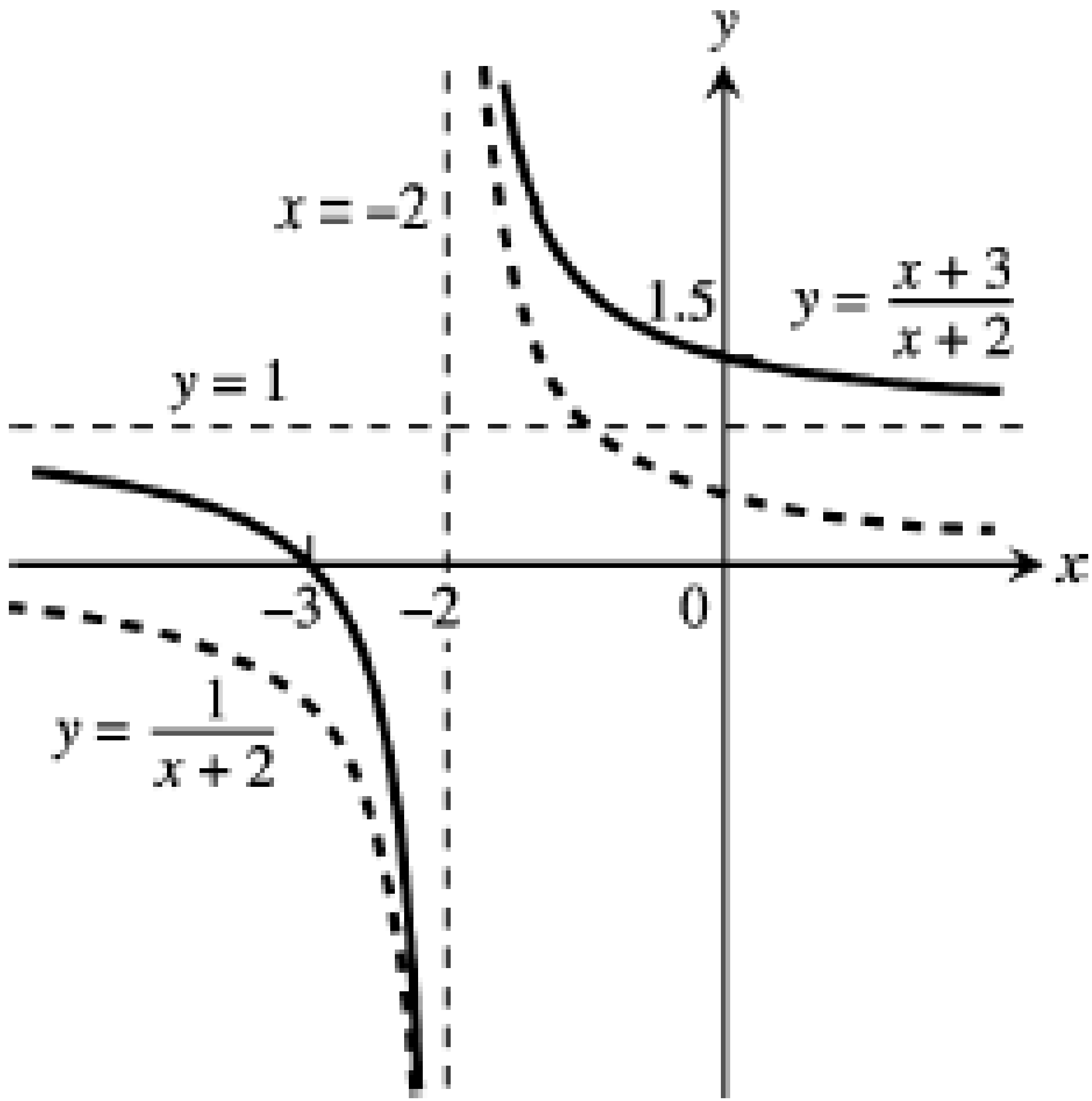
using vertical asymptotes and dominant terms for numerically large values of *x*.

Find the horizontal and vertical asymptotes of the graph of

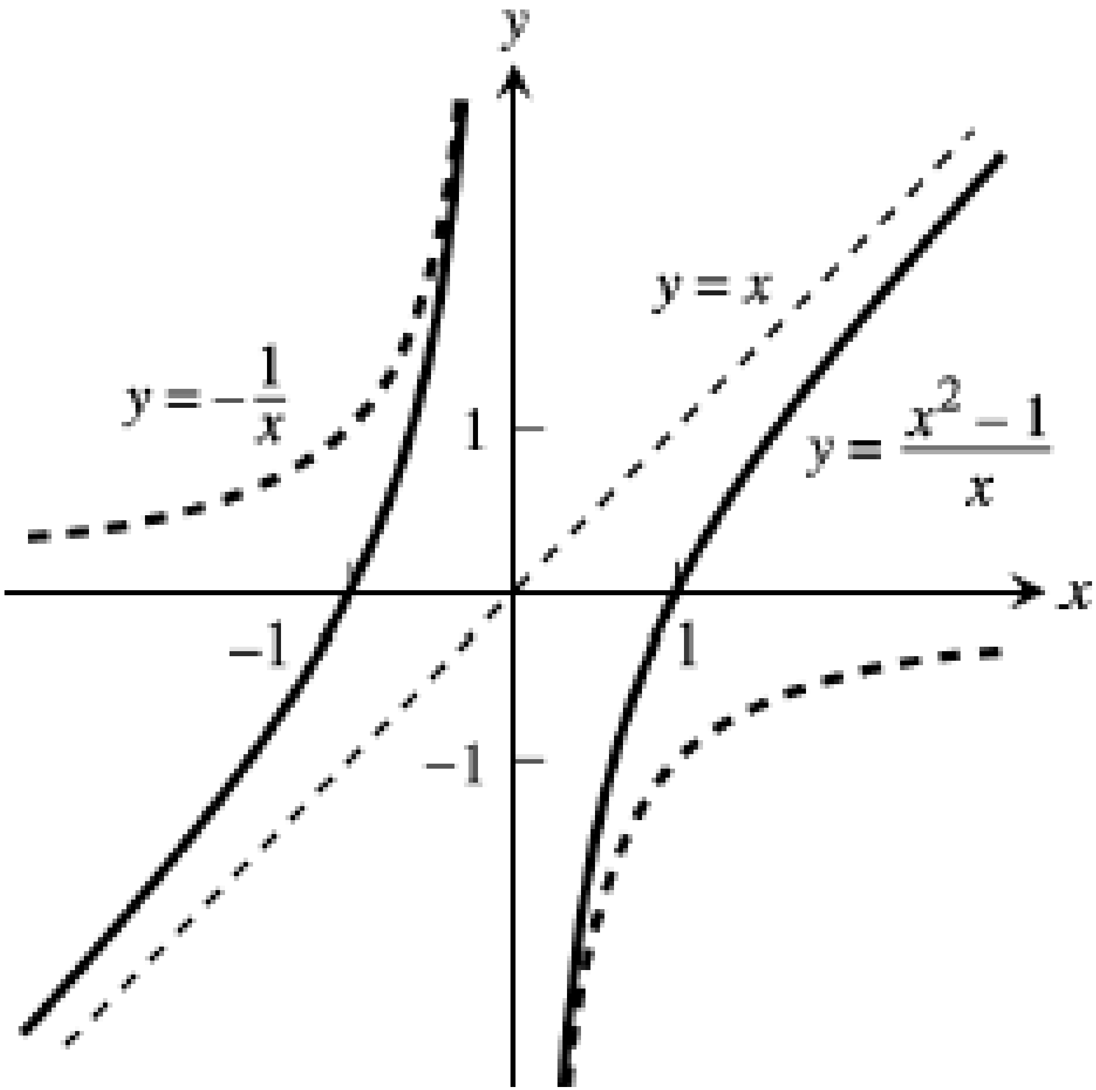
$$f(x) = -\frac{8}{x^2 - 4}.$$



$$y = \frac{x+3}{x+2} = 1 + \frac{1}{x+2}$$

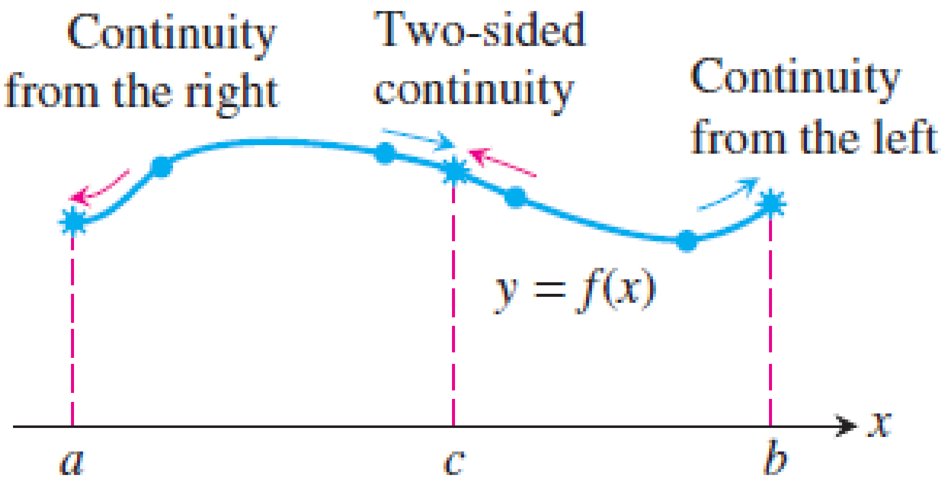


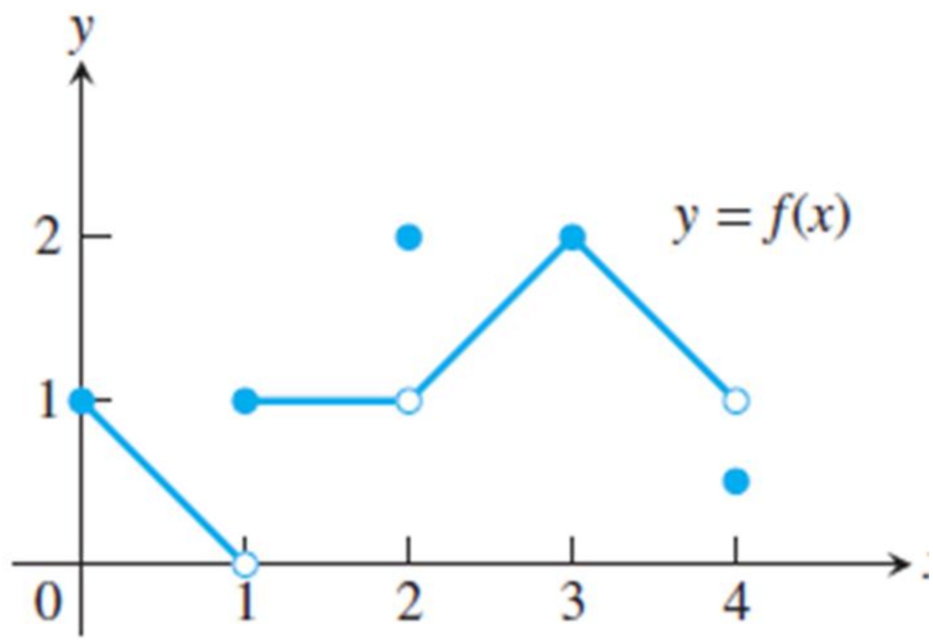
$$y = \frac{x^2 - 1}{x} = x - \frac{1}{x}$$



Continuity

Continuity at a Point





Points at which f is continuous:

At $x = 0$, $\lim_{x \to 0^+} f(x) = f(0)$.

At $x = 3$, $\lim_{x \to 3} f(x) = f(3)$.

At $0 < c < 4,\ c \neq 1, 2$, $\lim_{x \to c} f(x) = f(c)$.

Points at which f is discontinuous:

At $x = 1$, $\lim_{x \to 1} f(x)$ does not exist.

At $x = 2$, $\lim_{x \to 2} f(x) = 1$, but $1 \neq f(2)$.

At $x = 4$, $\lim_{x \to 4^-} f(x) = 1$, but $1 \neq f(4)$.

DEFINITION Continuous at a Point

Interior point: A function $y = f(x)$ is **continuous at an interior point** c of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint** a or is **continuous at a right endpoint** b of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \qquad \text{or} \qquad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively.}$$

Continuity Test

A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f)
2. $\lim_{x \to c} f(x)$ exists (f has a limit as $x \to c$)
3. $\lim_{x \to c} f(x) = f(c)$ (the limit equals the function value)

Continuous Functions

A function is **continuous on an interval** if and only if it is continuous at every point of the interval.

THEOREM 9 Properties of Continuous Functions

If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums:* $f + g$
2. *Differences:* $f - g$
3. *Products:* $f \cdot g$
4. *Constant multiples:* $k \cdot f$, for any number k
5. *Quotients:* f/g provided $g(c) \neq 0$
6. *Powers:* $f^{r/s}$, provided it is defined on an open interval containing c, where r and s are integers

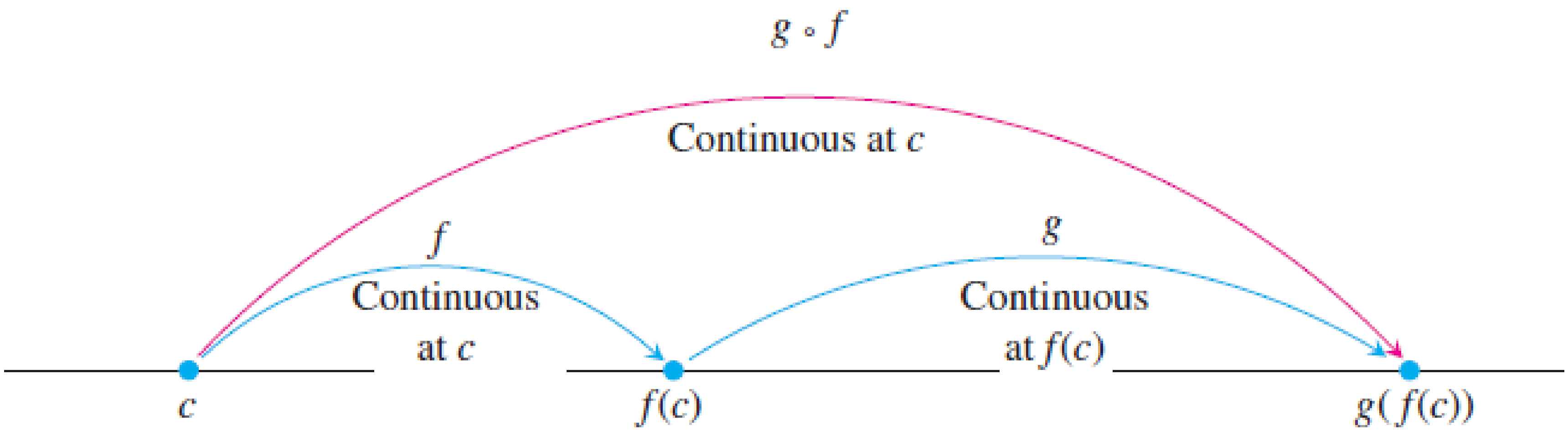


FIGURE 2.57 Composites of continuous functions are continuous.

THEOREM 10 Composite of Continuous Functions

If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.