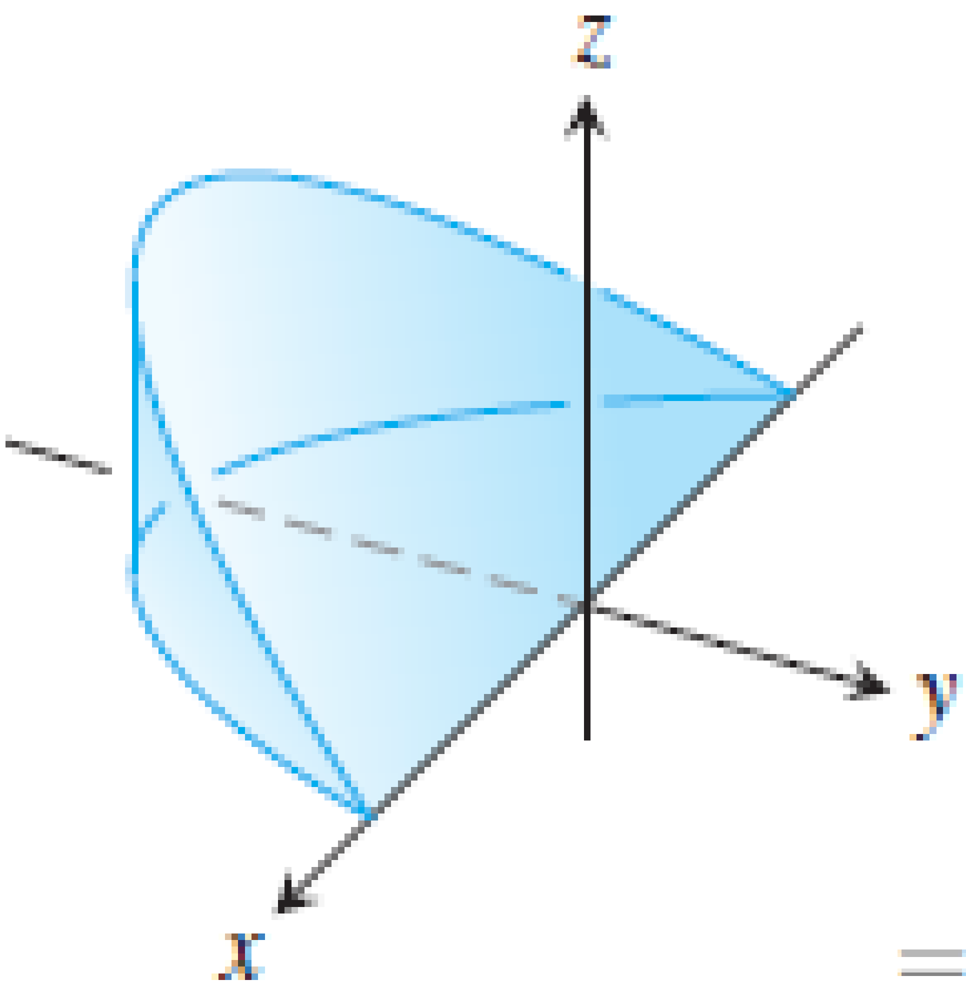
The wedge cut from the cylinder $x^2 + y^2 = 1$ by the planes $z = -y$ and $z = 0$

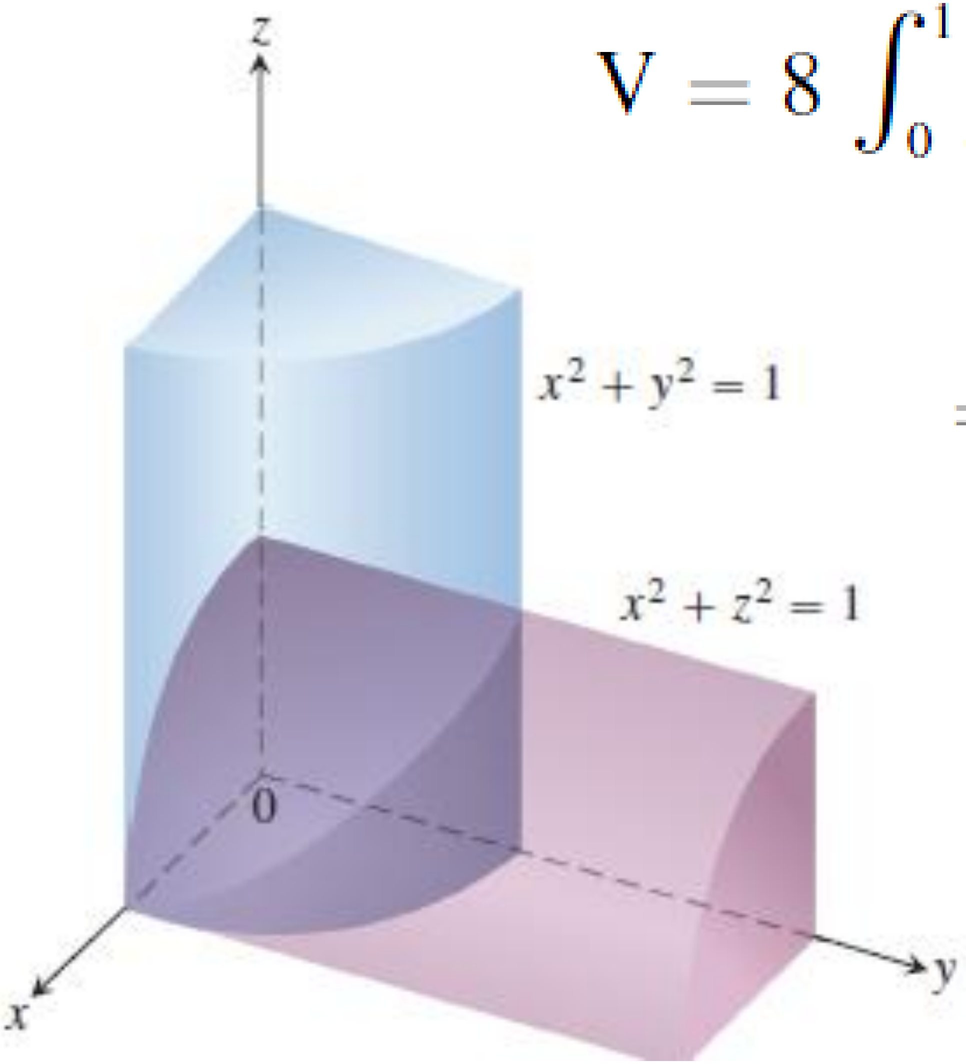


$$\int_0^1 \int_{-\sqrt{1-x^2}}^0 \int_0^{-y} dz\, dy\, dx$$

$$= -2 \int_0^1 \int_{-\sqrt{1-x^2}}^0 y\, dy\, dx =$$

$$= \int_0^1 (1 - x^2)\, dx = \frac{2}{3}$$

The region common to the interiors of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$, one-eighth of which is shown in the accompanying figure.



$$V = 8 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dz\, dy\, dx$$

$$= 8 \int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1 - x^2}\, dy\, dx$$

$$= 8 \int_0^1 (1 - x^2)\, dx = \frac{16}{3}$$

15.5 Masses and Moments in Three Dimensions

Mass and moment formulas for solid objects in space

Mass: $M = \iiint_D \delta \, dV \qquad (\delta = \delta(x, y, z) = \text{density})$

First moments about the coordinate planes:

$$M_{yz} = \iiint_D x\,\delta\, dV, \qquad M_{xz} = \iiint_D y\,\delta\, dV, \qquad M_{xy} = \iiint_D z\,\delta\, dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \qquad \bar{y} = \frac{M_{xz}}{M}, \qquad \bar{z} = \frac{M_{xy}}{M}$$

Moments of inertia (second moments) about the coordinate axes:

$$I_x = \iiint (y^2 + z^2)\,\delta\,dV$$

$$I_y = \iiint (x^2 + z^2)\,\delta\,dV$$

$$I_z = \iiint (x^2 + y^2)\,\delta\,dV$$

Moments of inertia about a line *L*:

$$I_L = \iiint r^2\,\delta\,dV \qquad (r(x, y, z) = \text{distance from the point } (x, y, z) \text{ to line } L)$$

Radius of gyration about a line *L*:

$$R_L = \sqrt{I_L/M}$$

a. **Centroid and moments of inertia** Find the centroid and the moments of inertia I_x, I_y, and I_z of the tetrahedron whose vertices are the points (0, 0, 0), (1, 0, 0), (0, 1, 0), and (0, 0, 1).

b. **Radius of gyration** Find the radius of gyration of the tetrahedron about the x-axis. Compare it with the distance from the centroid to the x-axis.

(a) $M = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\,dy\,dx = \int_0^1 \int_0^{1-x} (1-x-y)\,dy\,dx = \int_0^1 \left(\frac{x^2}{2} - x + \frac{1}{2}\right) dx = \frac{1}{6}$;

$M_{yz} = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x\,dz\,dy\,dx = \int_0^1 \int_0^{1-x} x(1-x-y)\,dy\,dx = \frac{1}{2} \int_0^1 (x^3 - 2x^2 + x)\,dx = \frac{1}{24}$

$\Rightarrow \bar{x} = \bar{y} = \bar{z} = \frac{1}{4}$, by symmetry; $I_x = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} (y^2 + z^2)\,dz\,dy\,dx$

$= \int_0^1 \int_0^{1-x} \left[y^2 - xy^2 - y^3 + \frac{(1-x-y)^3}{3}\right] dy\,dx = \frac{1}{6} \int_0^1 (1-x)^4\,dx = \frac{1}{30} \Rightarrow I_y = I_x = \frac{1}{30}$, by symmetry

(b) $R_x = \sqrt{\frac{I_x}{M}} = \sqrt{\frac{1}{5}} = \frac{\sqrt{5}}{5} \approx 0.4472$; the distance from the centroid to the x-axis is $\sqrt{0^2 + \frac{1}{16} + \frac{1}{16}} = \sqrt{\frac{1}{8}} = \frac{\sqrt{2}}{4}$ ≈ 0.3536

Center of mass and moments of inertia A solid "trough" of constant density is bounded below by the surface $z = 4y^2$, above by the plane $z = 4$, and on the ends by the planes $x = 1$ and $x = -1$. Find the center of mass and the moments of inertia with respect to the three axes.

$$M = 4\int_0^1\int_0^1\int_{4y^2}^4 dz\,dy\,dx = 4\int_0^1\int_0^1 (4-4y^2)\,dy\,dx = 16\int_0^1 \tfrac{2}{3}\,dx = \tfrac{32}{3}\,;\ M_{xy} = 4\int_0^1\int_0^1\int_{4y^2}^4 z\,dz\,dy\,dx$$

$$= 2\int_0^1\int_0^1 (16-16y^4)\,dy\,dx = \tfrac{128}{5}\int_0^1 dx = \tfrac{128}{5} \Rightarrow \bar{z} = \tfrac{12}{5}\text{, and } \bar{x} = \bar{y} = 0\text{, by symmetry;}$$

$$I_x = 4\int_0^1\int_0^1\int_{4y^2}^4 (y^2+z^2)\,dz\,dy\,dx = 4\int_0^1\int_0^1 \left[\left(4y^2+\tfrac{64}{3}\right)-\left(4y^4+\tfrac{64y^6}{3}\right)\right] dy\,dx = 4\int_0^1 \tfrac{1976}{105}\,dx = \tfrac{7904}{105}\,;$$

$$I_y = 4\int_0^1\int_0^1\int_{4y^2}^4 (x^2+z^2)\,dz\,dy\,dx = 4\int_0^1\int_0^1 \left[\left(4x^2+\tfrac{64}{3}\right)-\left(4x^2y^2+\tfrac{64y^6}{3}\right)\right] dy\,dx = 4\int_0^1 \left(\tfrac{8}{3}x^2+\tfrac{128}{7}\right) dx$$

$$= \tfrac{4832}{63}\,;\ I_z = 4\int_0^1\int_0^1\int_{4y^2}^4 (x^2+y^2)\,dz\,dy\,dx = 16\int_0^1\int_0^1 (x^2-x^2y^2+y^2-y^4)\,dy\,dx$$

$$= 16\int_0^1 \left(\tfrac{2x^2}{3}+\tfrac{2}{15}\right) dx = \tfrac{256}{45}$$

A solid cube in the first octant is bounded by the coordinate planes and by the planes $x = 1$, $y = 1$, and $z = 1$. The density of the cube is $\delta(x, y, z) = x + y + z + 1$.

(a) $M = \int_0^1 \int_0^1 \int_0^1 (x + y + z + 1)\, dz\, dy\, dx = \int_0^1 \int_0^1 \left(x + y + \frac{3}{2}\right) dy\, dx = \int_0^1 (x + 2)\, dx = \frac{5}{2}$

(b) $M_{xy} = \int_0^1 \int_0^1 \int_0^1 z(x + y + z + 1)\, dz\, dy\, dx = \frac{1}{2} \int_0^1 \int_0^1 \left(x + y + \frac{5}{3}\right) dy\, dx = \frac{1}{2} \int_0^1 \left(x + \frac{13}{6}\right) dx = \frac{4}{3}$

$\Rightarrow M_{xy} = M_{yz} = M_{xz} = \frac{4}{3}$, by symmetry $\Rightarrow \bar{x} = \bar{y} = \bar{z} = \frac{8}{15}$

(c) $I_z = \int_0^1 \int_0^1 \int_0^1 (x^2 + y^2)(x + y + z + 1)\, dz\, dy\, dx = \int_0^1 \int_0^1 (x^2 + y^2)\left(x + y + \frac{3}{2}\right) dy\, dx$

$= \int_0^1 \left(x^3 + 2x^2 + \frac{1}{3}x + \frac{3}{4}\right) dx = \frac{11}{6} \Rightarrow I_x = I_y = I_z = \frac{11}{6}$, by symmetry

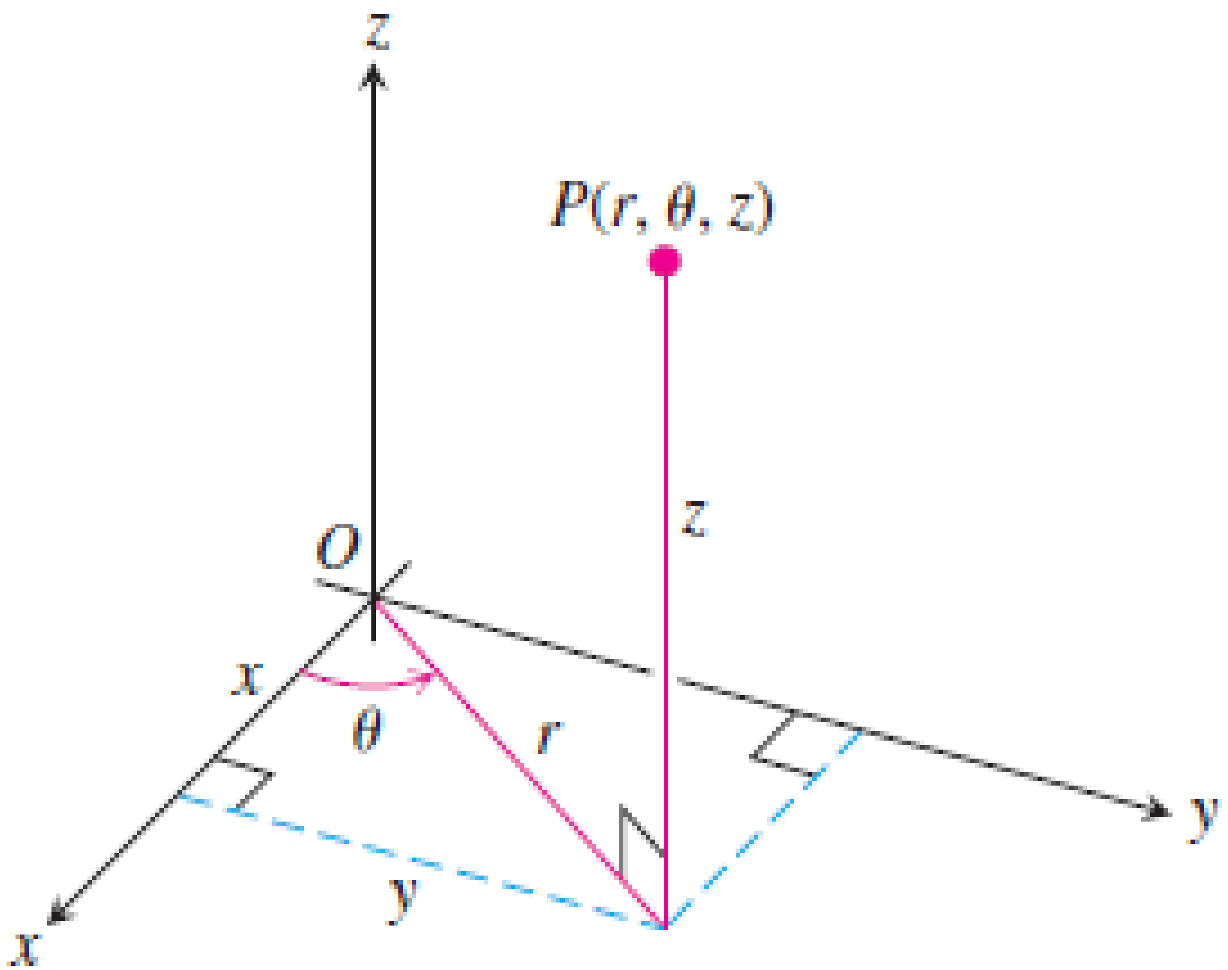
(d) $R_x = R_y = R_z = \sqrt{\frac{I_z}{M}} = \sqrt{\frac{11}{15}}$

15.6 Triple Integrals in Cylindrical and Spherical Coordinates

Equations Relating Rectangular (x, y, z) and Cylindrical (r, θ, z) Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z,$$

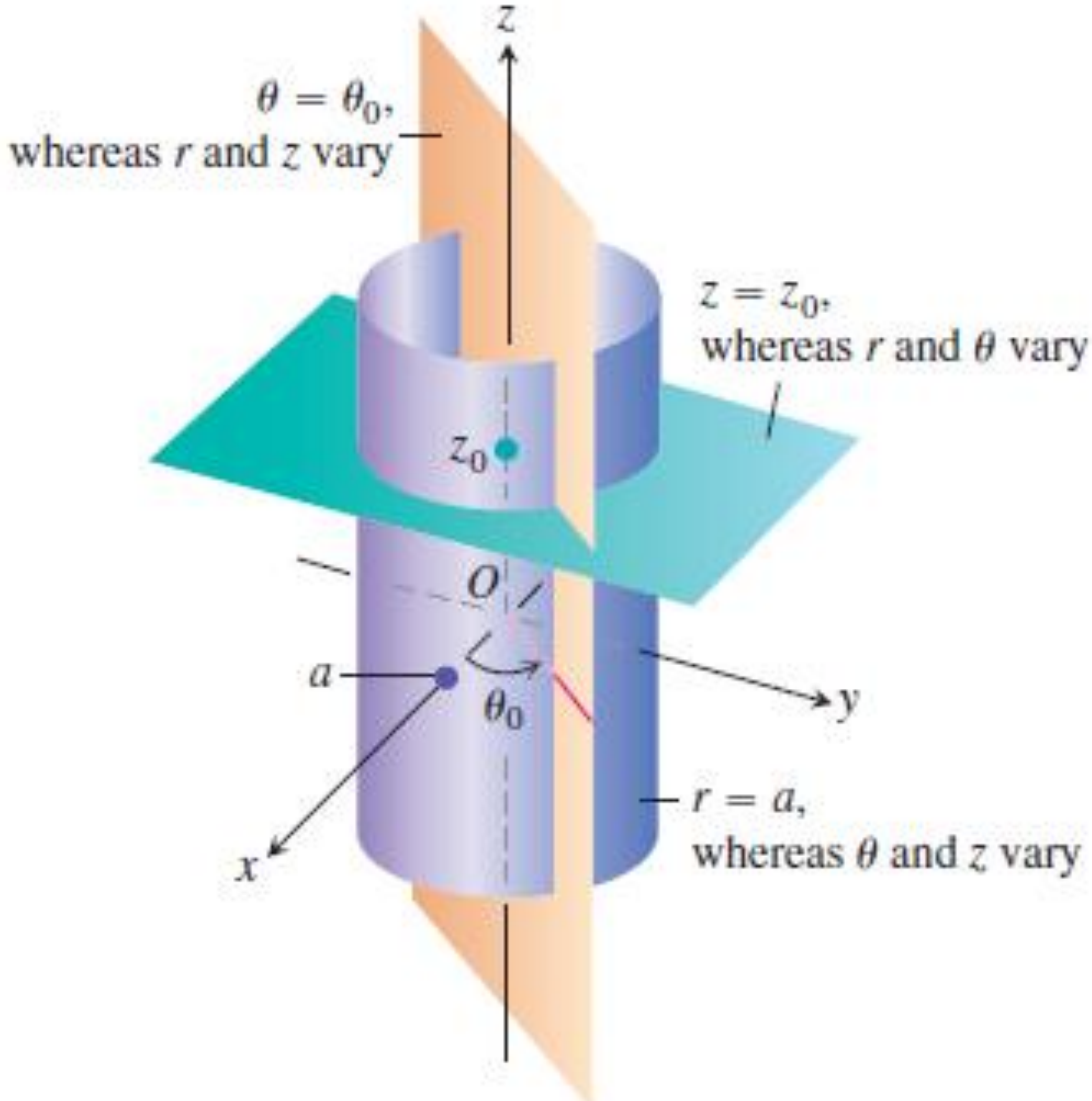
$$r^2 = x^2 + y^2, \qquad \tan\theta = y/x$$



r

θ

z

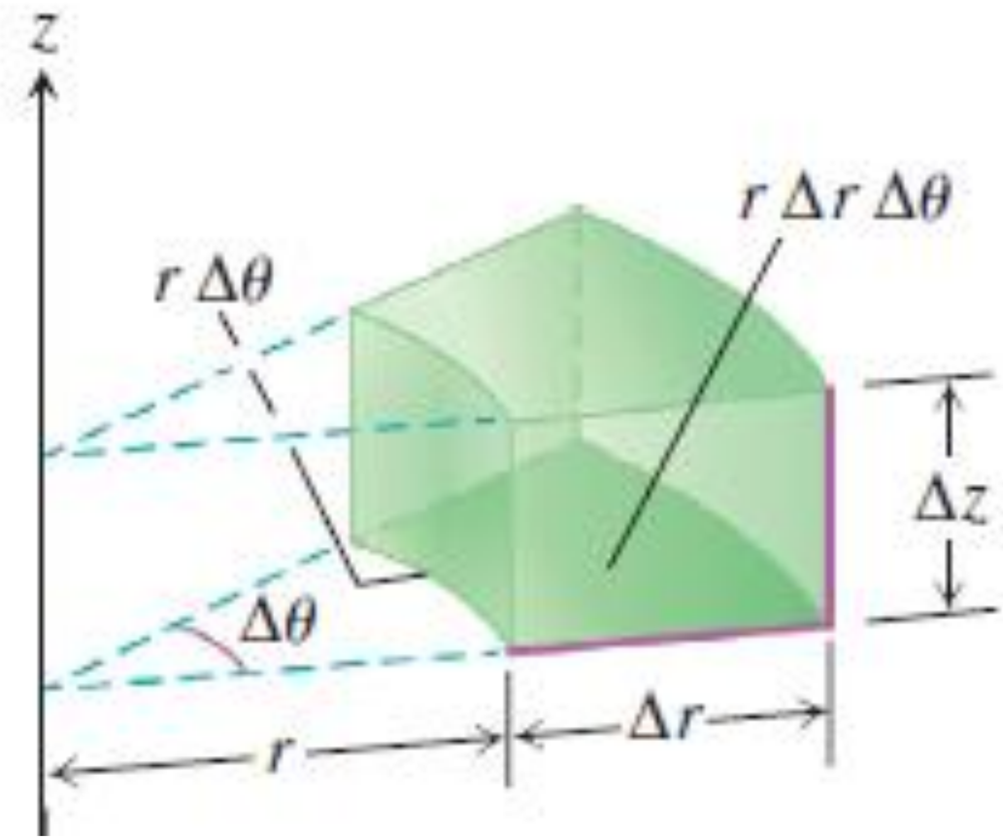


Constant-coordinate equations in cylindrical coordinates yield cylinders and planes.

r

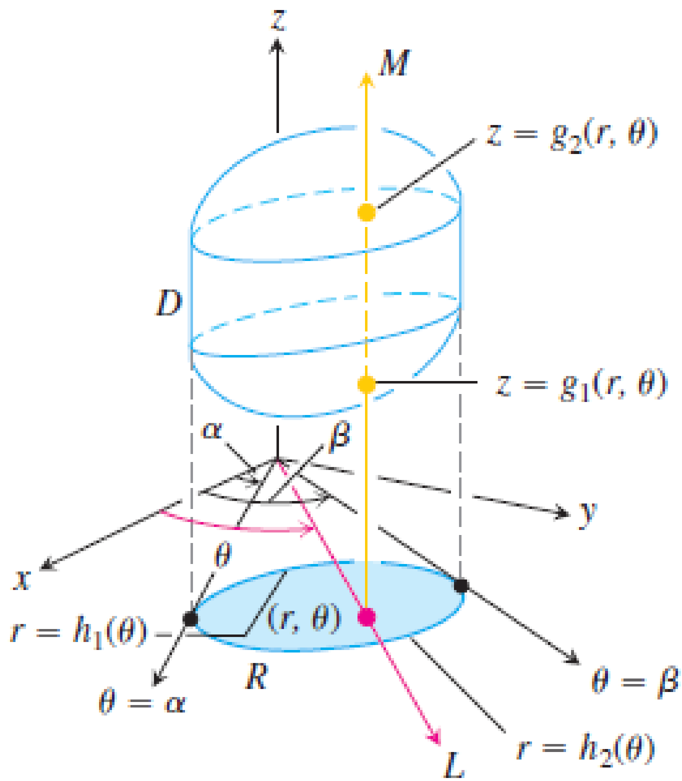
θ

z



In cylindrical coordinates the volume of the wedge is approximated by the product $\Delta V = \Delta z\, r\, \Delta r\, \Delta\theta$.

Find the r-limits of integration. Draw a ray L through (r, θ) from the origin. The ray enters R at $r = h_1(\theta)$ and leaves at $r = h_2(\theta)$. These are the r-limits of integration.



r

θ

z

Find the θ-limits of integration. As L sweeps across R, the angle θ it makes with the positive x-axis runs from $\theta = \alpha$ to $\theta = \beta$. These are the θ-limits of integration. The integral is

$$\iiint_D f(r, \theta, z)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r, \theta, z)\, dz\, r\, dr\, d\theta.$$

Cylinder and paraboloids Find the volume of the region bounded below by the paraboloid $z = x^2 + y^2$, laterally by the cylinder $x^2 + y^2 = 1$, and above by the paraboloid $z = x^2 + y^2 + 1$.

$$V = 4 \int_0^{\pi/2} \int_0^1 \int_{r^2}^{r^2+1} dz\, r\, dr\, d\theta$$

$$= 4 \int_0^{\pi/2} \int_0^1 r\, dr\, d\theta$$

$$= 2 \int_0^{\pi/2} d\theta \qquad = \pi$$

. **Water in a hemispherical bowl** A hemispherical bowl of radius 5 cm is filled with water to within 3 cm of the top. Find the volume of water in the bowl.

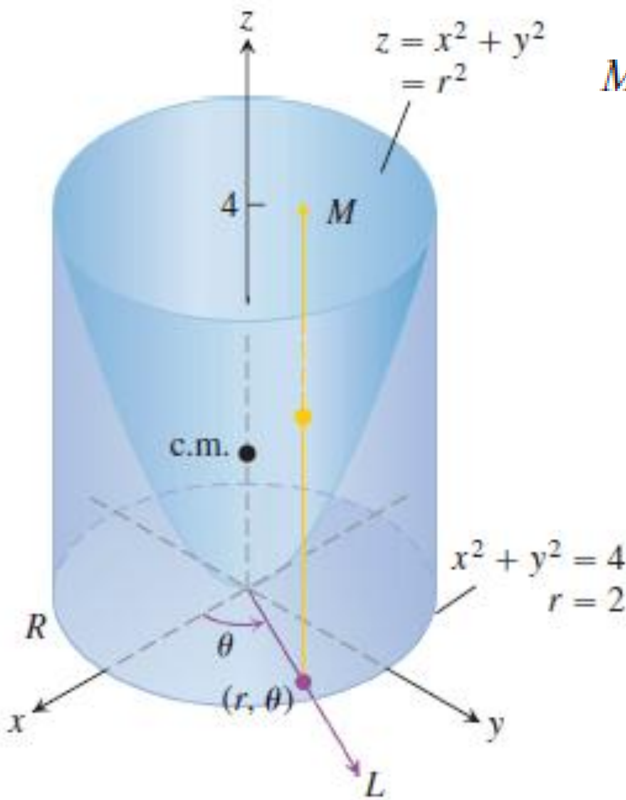
$$\Rightarrow V = \int_0^{2\pi}\int_0^4\int_{-\sqrt{25-r^2}}^{-3} dz\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^4 \left(r\sqrt{25-r^2} - 3r\right) dr\, d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{3}\left(25-r^2\right)^{3/2} - \frac{3}{2}r^2\right]_0^4 d\theta$$

$$= \int_0^{2\pi} \frac{26}{3}\, d\theta = \frac{52\pi}{3}$$

Find the centroid ($\delta = 1$) of the solid enclosed by the cylinder $x^2 + y^2 = 4$, bounded above by the paraboloid $z = x^2 + y^2$, and bounded below by the xy-plane.



$$M_{xy} = \int_0^{2\pi}\int_0^2\int_0^{r^2} z\,dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \left[\frac{z^2}{2}\right]_0^{r^2} r\,dr\,d\theta$$

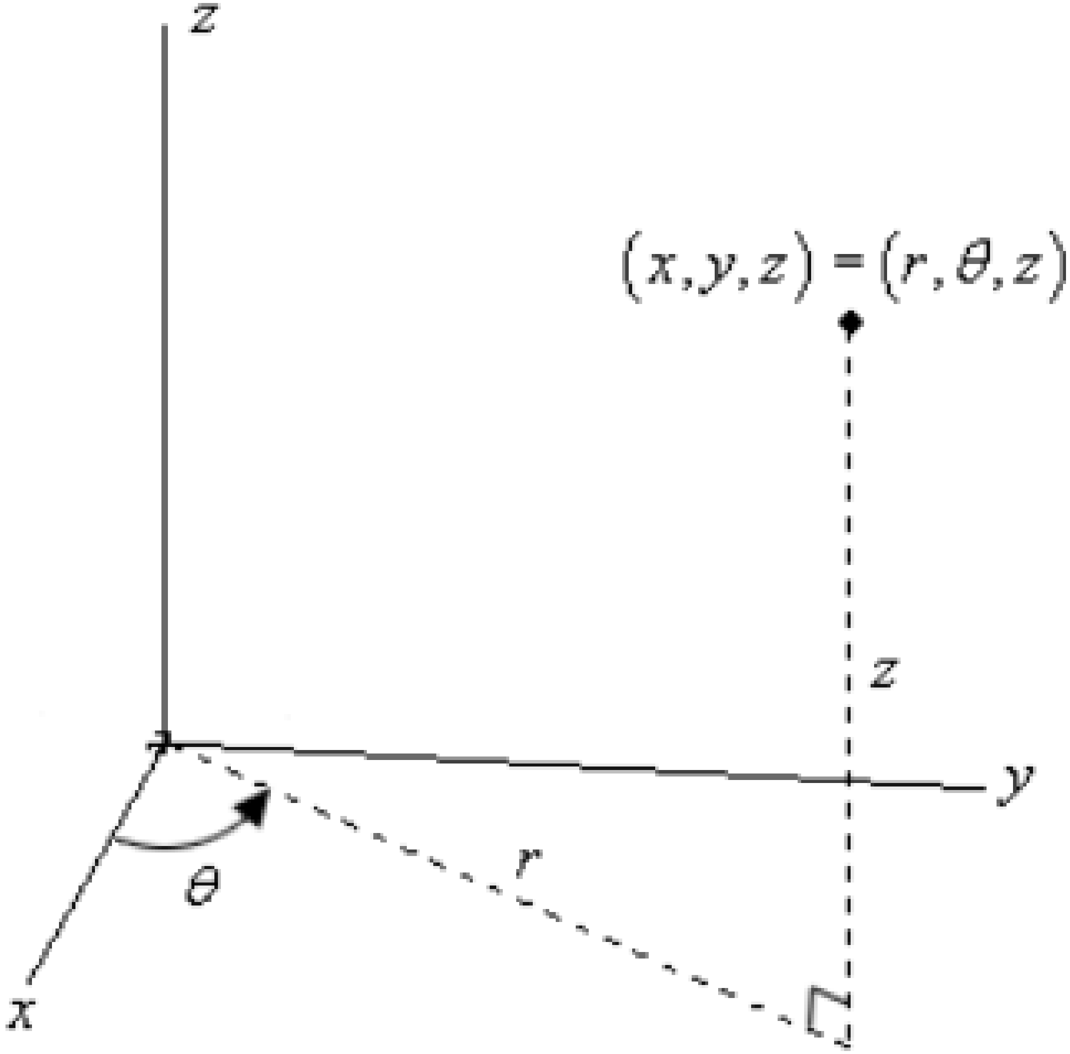
$$= \int_0^{2\pi}\int_0^2 \frac{r^5}{2}\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^6}{12}\right]_0^2 d\theta = \int_0^{2\pi}\frac{16}{3}\,d\theta = \frac{32\pi}{3}.$$

$$M = \int_0^{2\pi}\int_0^2\int_0^{r^2} dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 \Big[z\Big]_0^{r^2} r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r^3\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^2 d\theta = \int_0^{2\pi} 4\,d\theta = 8\pi.$$

$$\bar{z} = \frac{M_{xy}}{M} = \frac{32\pi}{3}\frac{1}{8\pi} = \frac{4}{3},$$

and the centroid is $(0, 0, 4/3)$. Notice that the centroid lies outside the solid.



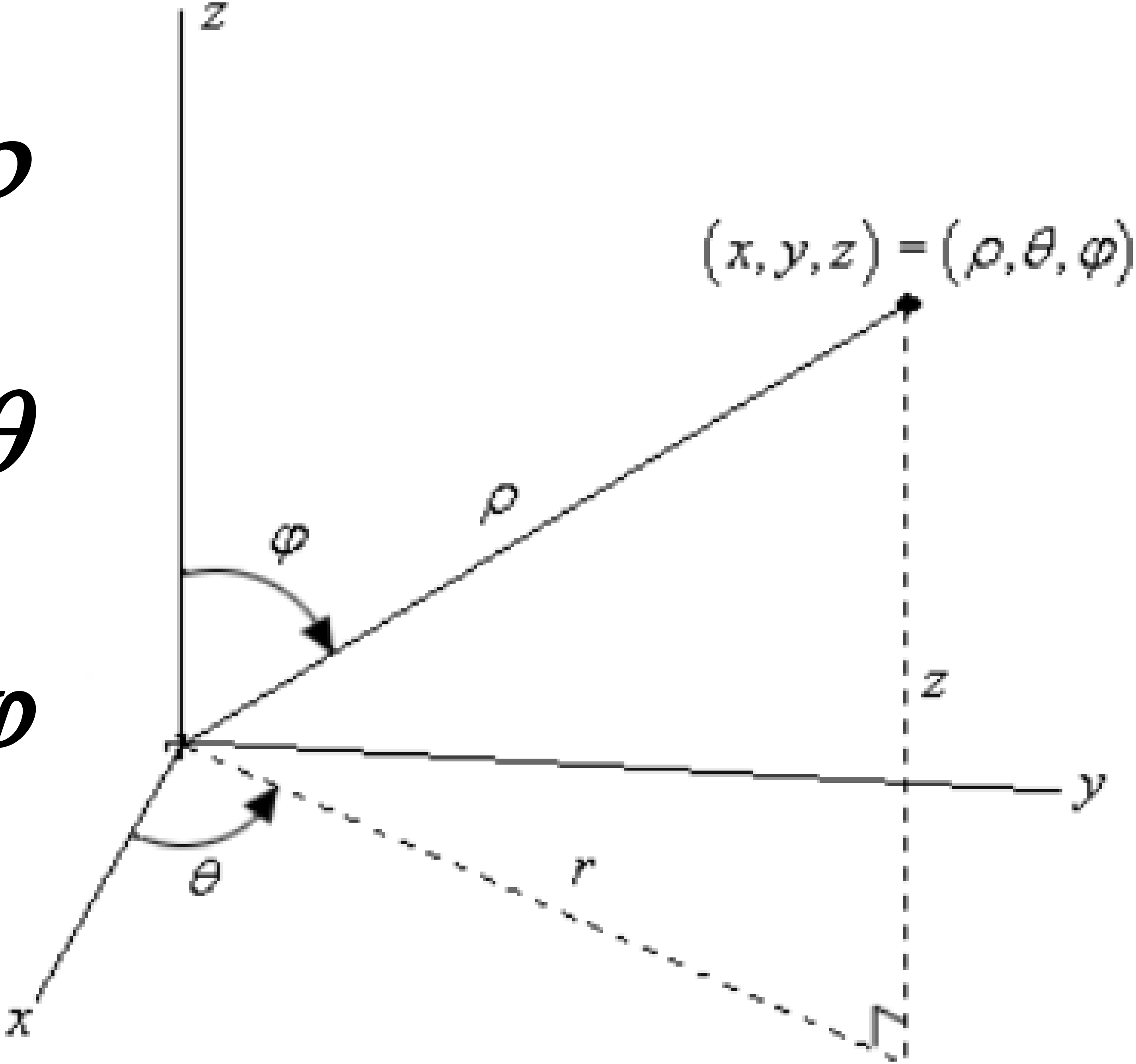
$$x = r\cos\theta$$
$$y = r\sin\theta$$

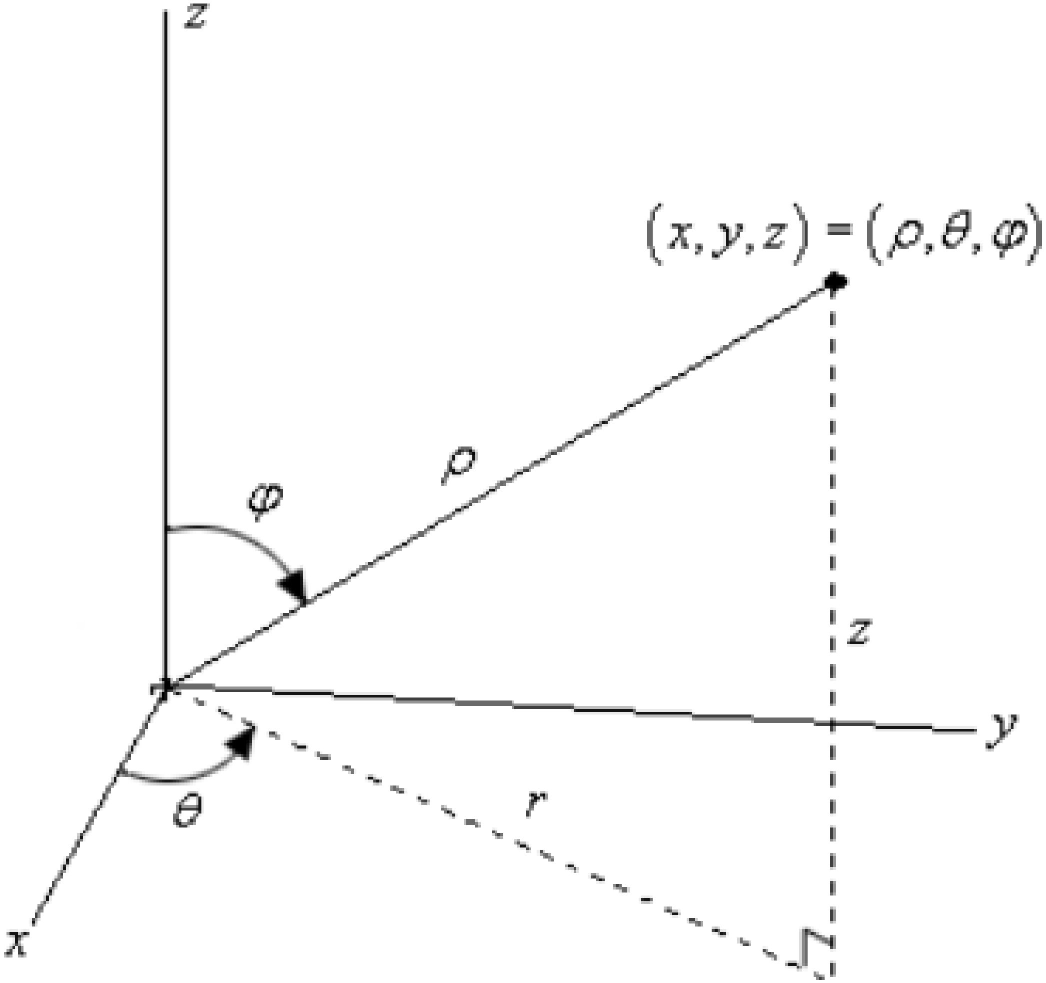
$$x^2 + y^2 = r^2$$
$$\tan\theta = \frac{y}{x}$$

$\boldsymbol{\rho}$

$\boldsymbol{\theta}$

$\boldsymbol{\varphi}$





$$
\begin{aligned}
x &= \rho \sin\varphi \cos\theta \\
y &= \rho \sin\varphi \sin\theta \\
z &= \rho \cos\varphi \\
\rho^2 &= x^2 + y^2 + z^2
\end{aligned}
$$

$\boldsymbol{\rho}$ $\boldsymbol{\theta}$ $\boldsymbol{\varphi}$

Identify the surfaces

$$\rho = 5$$

$$\rho^2 = 25$$

$$x^2 + y^2 + z^2 = 25$$

$$\rho = 5$$

a sphere of radius 5 centered at the origin.

$$\varphi = \frac{\pi}{3}$$

This is exactly what happens in a cone. All of the points on a cone are a fixed angle from the z-axis.
whose points are all at an angle of $\frac{\pi}{3}$ from the z-axis.

$$\theta = \frac{2\pi}{3}$$ a vertical plane that forms an angle of $\frac{2\pi}{3}$ with the positive x-axis.

$$\rho \sin \varphi = 2$$

$$\rho^2 \sin^2 \varphi = 4$$

add $\rho^2 \cos^2 \varphi$ to both sides.

$$\rho^2 \sin^2 \varphi + \rho^2 \cos^2 \varphi = 4 + \rho^2 \cos^2 \varphi$$

$$\rho^2 \left(\sin^2 \varphi + \cos^2 \varphi\right) = 4 + \rho^2 \cos^2 \varphi$$

$$\rho^2 = 4 + (\rho \cos \varphi)^2$$

$$x^2 + y^2 + z^2 = 4 + z^2$$

$$x^2 + y^2 = 4$$

Coordinate Conversion Formulas

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\cos\theta$	$x = \rho\sin\phi\cos\theta$	$r = \rho\sin\phi$
$y = r\sin\theta$	$y = \rho\sin\phi\sin\theta$	$z = \rho\cos\phi$
$z = z$	$z = \rho\cos\phi$	$\theta = \theta$

Corresponding formulas for dV in triple integrals:

$$\begin{aligned} dV &= dx\,dy\,dz \\ &= dz\,r\,dr\,d\theta \\ &= \rho^2\sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}$$

Convert

$$\left(\sqrt{2},\ \pi,\ \pi\right)$$

Rectangular coordinates

$$z = \sqrt{2}\cos\pi = -\sqrt{2}$$

$$x = \sqrt{2}\sin\pi\cos\pi = 0$$

$$y = \sqrt{2}\sin\pi\sin\pi = 0$$

$$\left(0,\ 0,\ -\sqrt{2}\ \right)$$

Cylindrical coordinates

$$r = \sqrt{2}\sin\pi = 0$$

$$\left(0,\ \pi,\ -\sqrt{2}\ \right)$$

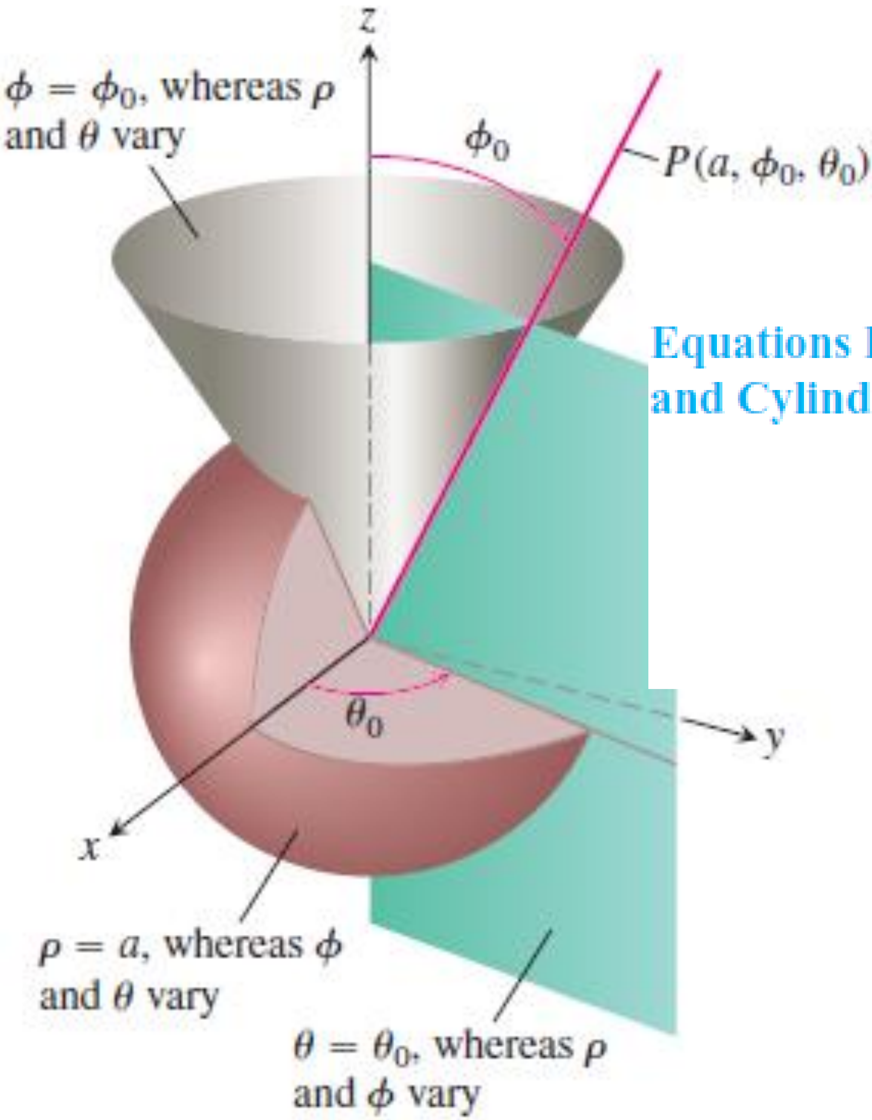
Convert

$$\rho = \sqrt{2} \sec \phi$$

into cylindrical and rectangular coordinates.

$$\rho = \sqrt{2} \sec \phi \rightarrow \rho \cos \phi = \sqrt{2} \rightarrow z = \sqrt{2}$$

The above equation is both the cylindrical form of the given equation and the rectangular form of the given equation.



Constant-coordinate equations in spherical coordinates yield spheres, single cones, and half-planes.

Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates

$$r = \rho \sin \phi, \qquad x = r \cos \theta = \rho \sin \phi \cos \theta,$$

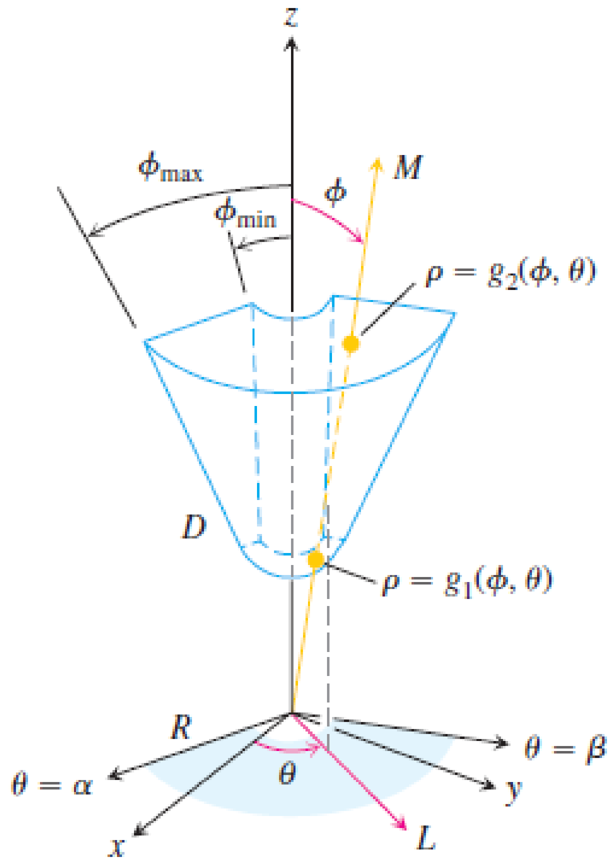
$$z = \rho \cos \phi, \qquad y = r \sin \theta = \rho \sin \phi \sin \theta,$$

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}.$$

$\boldsymbol{\rho}$

$\boldsymbol{\theta}$

$\boldsymbol{\varphi}$



ρ

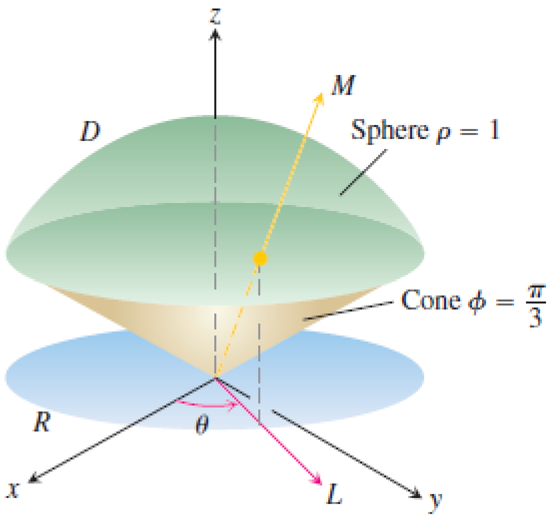
θ

φ

Find the ϕ-limits of integration. For any given θ, the angle ϕ that M makes with the z-axis runs from $\phi = \phi_{min}$ to $\phi = \phi_{max}$. These are the ϕ -limits of integration.

Find the θ-limits of integration. The ray L sweeps over R as θ runs from α to β. These are the θ-limits of integration. The integral is

$$\iiint_D f(\rho, \phi, \theta)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{\phi=\phi_{min}}^{\phi=\phi_{max}} \int_{\rho=g_1(\phi, \theta)}^{\rho=g_2(\phi, \theta)} f(\rho, \phi, \theta)\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta.$$



Finding a Volume in Spherical Coordinates

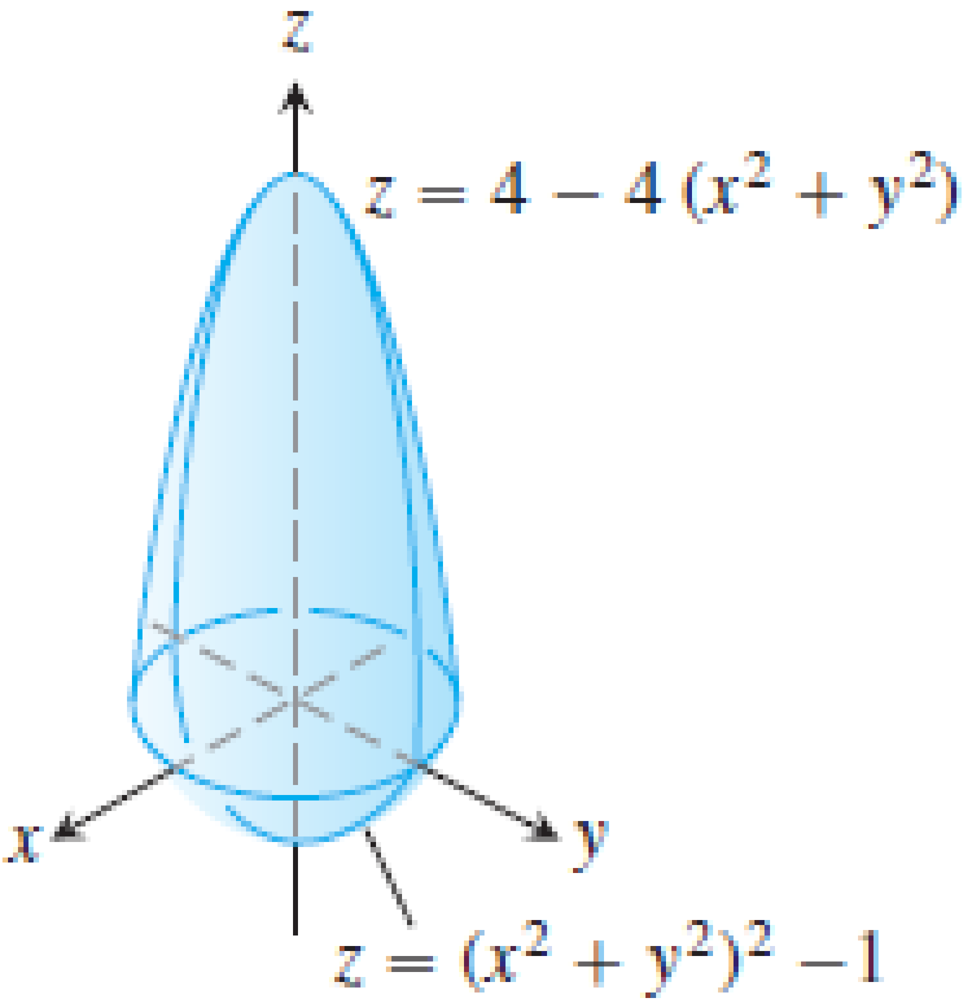
Find the volume of the "ice cream cone" D cut from the solid sphere $\rho \le 1$ by the cone $\phi = \pi/3$.

$$\rho \qquad \theta \qquad \varphi$$

$$V = \iiint_D \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

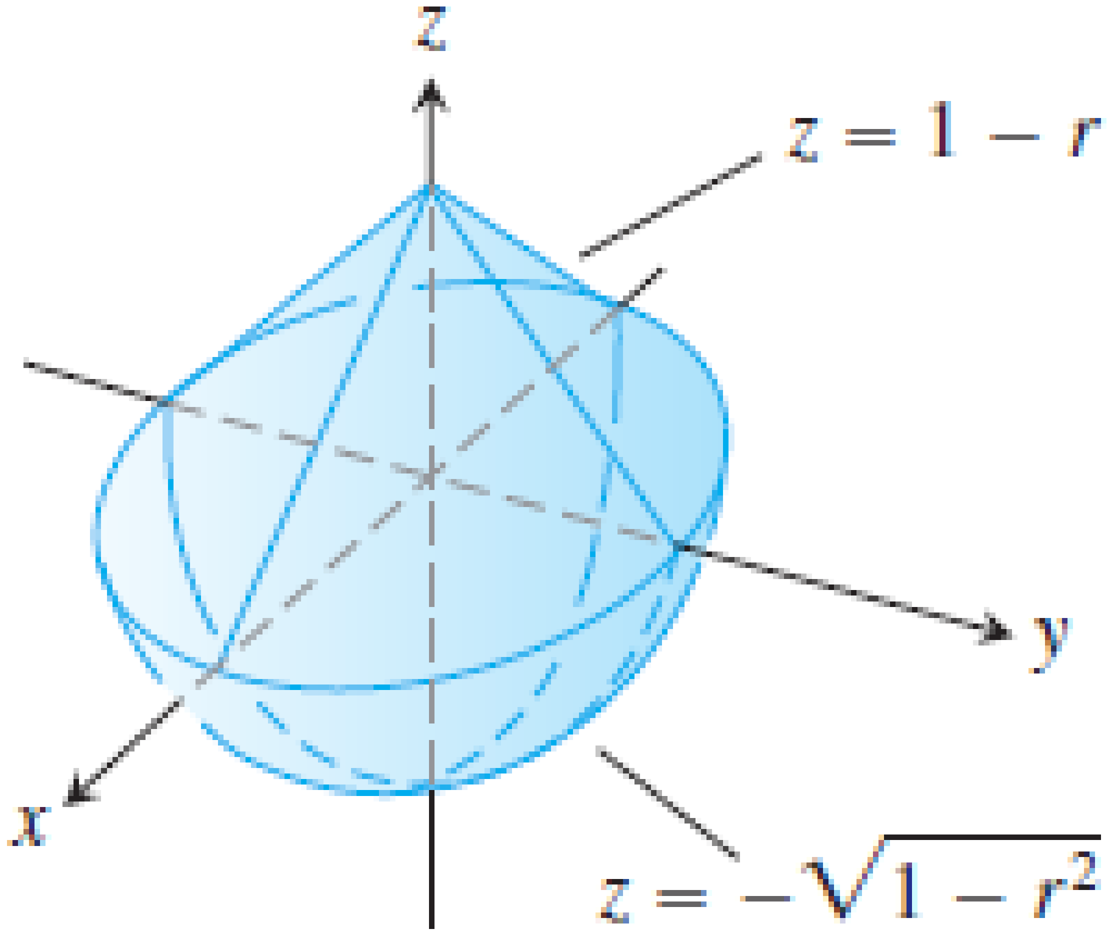
$$= \int_0^{2\pi} \int_0^{\pi/3} \left[\frac{\rho^3}{3}\right]_0^1 \sin\phi \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/3} \frac{1}{3} \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \left[-\frac{1}{3}\cos\phi\right]_0^{\pi/3} d\theta = \int_0^{2\pi} \left(-\frac{1}{6} + \frac{1}{3}\right) d\theta = \frac{1}{6}(2\pi) = \frac{\pi}{3}.$$



$$V = 4\int_0^{\pi/2}\int_0^1\int_{r^4-1}^{4-4r^2} dz\, r\, dr\, d\theta = 4\int_0^{\pi/2}\int_0^1 (5r - 4r^3 - r^5)\, dr\, d\theta = 4\int_0^{\pi/2}\left(\tfrac{5}{2} - 1 - \tfrac{1}{6}\right) d\theta$$

$$= 4\int_0^{\pi/2} d\theta = \tfrac{8\pi}{3}$$



$$V = 4\int_0^{\pi/2}\int_0^1\int_{-\sqrt{1-r^2}}^{1-r} dz\, r\, dr\, d\theta = 4\int_0^{\pi/2}\int_0^1\left(r - r^2 + r\sqrt{1-r^2}\right) dr\, d\theta = 4\int_0^{\pi/2}\left[\frac{r^2}{2} - \frac{r^3}{3} - \frac{1}{3}\left(1-r^2\right)^{3/2}\right]_0^1 d\theta$$

$$= 4\int_0^{\pi/2}\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{3}\right) d\theta = 2\int_0^{\pi/2} d\theta = 2\left(\frac{\pi}{2}\right) = \pi$$