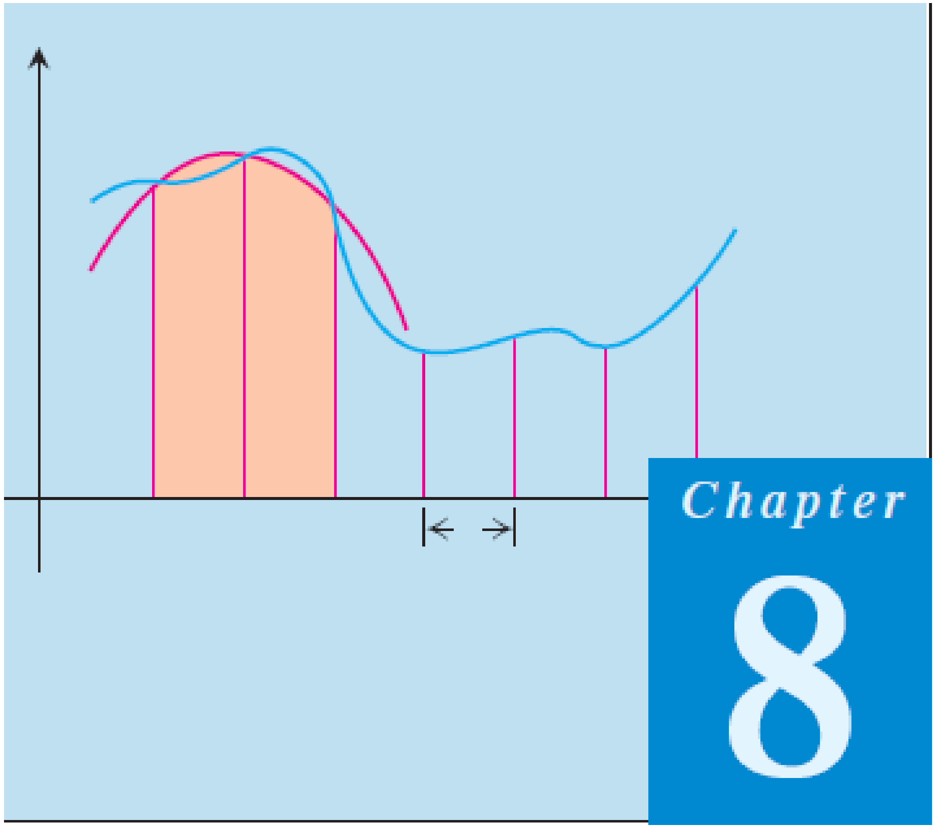


Chapter 8

TECHNIQUES OF INTEGRATION

a number of important techniques:
indefinite integrals of more complicated functions

- how to change unfamiliar integrals
- find in a table,
- evaluate with a computer.

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

Basic integration formulas

1. $\int du = u + C$

2. $\int k\,du = ku + C \qquad$ (any number k)

3. $\int (du + dv) = \int du + \int dv$

4. $\int u^n\,du = \frac{u^{n+1}}{n+1} + C \qquad (n \neq -1)$

5. $\int \frac{du}{u} = \ln|u| + C$

6. $\int \sin u\,du = -\cos u + C$

7. $\int \cos u\,du = \sin u + C$

8. $\int \sec^2 u\,du = \tan u + C$

9. $\int \csc^2 u\,du = -\cot u + C$

10. $\int \sec u \tan u\,du = \sec u + C$

11. $\int \csc u \cot u\,du = -\csc u + C$

12. $\int \tan u\,du = -\ln|\cos u| + C$
$= \ln|\sec u| + C$

13. $\int \cot u\,du = \ln|\sin u| + C$
$= -\ln|\csc u| + C$

14. $\int e^u\,du = e^u + C$

15. $\int a^u\,du = \frac{a^u}{\ln a} + C \qquad (a > 0, a \neq 1)$

16. $\int \sinh u\,du = \cosh u + C$

17. $\int \cosh u\,du = \sinh u + C$

18. $\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$

19. $\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$

20. $\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$

21. $\int \frac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C \qquad (a > 0)$

22. $\int \frac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C \qquad (u > a > 0)$

Evaluate $\int \sin(3x + 5)\, dx$

Let u = 3x + 5.

Then du = 3 dx.

$$\int \sin(3x + 5)\, dx = \int \sin(u) \frac{du}{3}$$

$$= \frac{1}{3} \int \sin(u)\, du$$

$$= \frac{1}{3} [- \cos(u)] + C$$

$$= \frac{1}{3} [- \cos(3x + 5)] + C$$

Completing the Square

$$\int \frac{dx}{\sqrt{8x - x^2}}.$$

$$\int \frac{dx}{\sqrt{8x - x^2}} = \int \frac{dx}{\sqrt{16 - (x - 4)^2}}$$

$$= \int \frac{du}{\sqrt{a^2 - u^2}}$$

$a = 4, u = (x - 4),$
$du = dx$

$$= \sin^{-1}\left(\frac{u}{a}\right) + C \qquad = \sin^{-1}\left(\frac{x - 4}{4}\right) + C.$$

Reducing an Improper Fraction

$$\int \frac{3x^2 - 7x}{3x + 2}\,dx.$$

$$\frac{3x^2 - 7x}{3x + 2} = x - 3 + \frac{6}{3x + 2}.$$

$$= \frac{x^2}{2} - 3x + 2\ln|3x + 2| + C.$$

Separating a Fraction

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\,dx$$

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\,dx = 3\int \frac{x\,dx}{\sqrt{1 - x^2}} + 2\int \frac{dx}{\sqrt{1 - x^2}}.$$

$$\int \frac{3x + 2}{\sqrt{1 - x^2}}\,dx = -3\sqrt{1 - x^2} + 2\sin^{-1} x + C.$$

8.2 Integration by Parts

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int [f'(x)g(x) + f(x)g'(x)]\,dx$$

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

$$\int u\,dv = uv - \int v\,du$$

Using Integration by Parts

$$\int x \cos x \, dx.$$

$$\text{formula} \int u \, dv = uv - \int v \, du$$

$$u = x, \qquad dv = \cos x \, dx,$$

$$du = dx, \qquad v = \sin x.$$

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C.$$

Using Tabular Integration

$$\int x^2 e^x \, dx.$$

$$f(x) = x^2 \text{ and } g(x) = e^x$$

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

$$\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C$$

Evaluate the following integral.

$$\int_{-1}^{2} x\mathbf{e}^{6x}\,dx$$

$$\int_{-1}^{2} x\mathbf{e}^{6x}\,dx = \frac{x}{6}\mathbf{e}^{6x}\Big|_{-1}^{2} - \frac{1}{6}\int_{-1}^{2} \mathbf{e}^{6x}\,dx$$

$$= \frac{x}{6}\mathbf{e}^{6x}\Big|_{-1}^{2} - \frac{1}{36}\mathbf{e}^{6x}\Big|_{-1}^{2}$$

$$= \frac{11}{36}\mathbf{e}^{12} + \frac{7}{36}\mathbf{e}^{-6}$$

how to express a rational function (a quotient of polynomials) as a sum of simpler fractions, called *partial fractions*, which are easily integrated

$$\int \frac{3x+11}{x^2-x-6}\,dx \qquad \frac{3x+11}{x^2-x-6} = \frac{4}{x-3} - \frac{1}{x+2}$$

$$\int \frac{3x+11}{x^2-x-6}\,dx = \int \frac{4}{x-3} - \frac{1}{x+2}\,dx$$

$$= 4\ln|x-3| - 1\ln|x+2| + c$$

Method of Partial Fractions ($f(x)/g(x)$ Proper)

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$

   Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be a quadratic factor of $g(x)$. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

$$\frac{B_1x + C_1}{x^2 + px + q} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$

   Do this for each distinct quadratic factor of $g(x)$ that cannot be factored into linear factors with real coefficients.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\cdots+\dfrac{A_k}{(ax+b)^k},\ k=1,2,3,\ldots$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\dfrac{A_kx+B_k}{(ax^2+bx+c)^k},\ k=1,2,3,\ldots$

$$\int \frac{x^2-29x+5}{(x-4)^2(x^2+3)}dx$$

$$\frac{x^2-29x+5}{(x-4)^2(x^2+3)}=\frac{A}{x-4}+\frac{B}{(x-4)^2}+\frac{Cx+D}{x^2+3}$$

$$\int \frac{x^2-29x+5}{(x-4)^2(x^2+3)}dx=\int \frac{1}{x-4}-\frac{5}{(x-4)^2}+\frac{-x+2}{x^2+3}dx$$

$$=\ln|x-4|+\frac{5}{x-4}-\frac{1}{2}\ln|x^2+3|+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+c$$

$$\int \frac{dx}{x(x^2+1)^2}.$$

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$= \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} - \int \frac{x\,dx}{(x^2+1)^2}$$

$$= \int \frac{dx}{x} - \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u^2}$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + K$$

8.4 Trigonometric Integrals

Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions. In principle, we can always express such integrals in terms of sines and cosines, but it is often simpler to work with other functions, as in the integral

Products of Powers of Sines and Cosines

We begin with integrals of the form:

$$\int \sin^m x \cos^n x \, dx,$$

where m and n are nonnegative integers (positive or zero). We can divide the work into three cases.

Case 1 If m is odd, we write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \tag{1}$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x \, dx$ equal to $-d(\cos x)$.

Case 2 If m is even and n is odd in $\int \sin^m x \cos^n x \, dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x \, dx$ equal to $d(\sin x)$.

Case 3 If both m and n are even in $\int \sin^m x \cos^n x \, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \tag{2}$$

to reduce the integrand to one in lower powers of $\cos 2x$.

Here are some examples illustrating each case.

m is Odd

$$\begin{aligned}
\int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx \\
&= \int (1 - \cos^2 x) \cos^2 x \, (-d(\cos x)) \\
&= \int (1 - u^2)(u^2)(-du) \qquad u = \cos x \\
&= \int (u^4 - u^2) \, du \\
&= \frac{u^5}{5} - \frac{u^3}{3} + C \\
&= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C.
\end{aligned}$$

m is Even and n is Odd

$$\int \cos^5 x \, dx.$$

$$= \int \cos^4 x \cos x \, dx = \int (1 - \sin^2 x)^2 \, d(\sin x)$$

$$= \int (1 - u^2)^2 \, du$$

$$= \int (1 - 2u^2 + u^4) \, du$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.$$

m and n are Both Even

$$\int \sin^2 x \cos^4 x \, dx.$$

$$= \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{8}\int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x)\, dx$$

$$= \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\, dx$$

$$= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \int (\cos^2 2x + \cos^3 2x)\, dx\right].$$

For the term involving $\cos^2 2x$ we use

$$\int \cos^2 2x\,dx = \frac{1}{2}\int (1 + \cos 4x)\,dx$$

$$= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right).$$

Omitting the constant of integration until the final result

For the $\cos^3 2x$ term we have

$$\int \cos^3 2x\,dx = \int (1 - \sin^2 2x)\cos 2x\,dx$$

$u = \sin 2x$,
$du = 2\cos 2x\,dx$

$$= \frac{1}{2}\int (1 - u^2)\,du = \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right).$$

Again omitting C

Combining everything and simplifying we get

$$\int \sin^2 x\cos^4 x\,dx = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right) + C.$$

■

8.5 Trigonometric Substitutions

Trigonometric substitutions can be effective in transforming integrals involving $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$ into integrals we can evaluate directly.

Three Basic Substitutions

The most common substitutions are $x = a\tan\theta$, $x = a\sin\theta$, and $x = a\sec\theta$. from the reference right triangles in Figure 8.2.

With $x = a\tan\theta$,

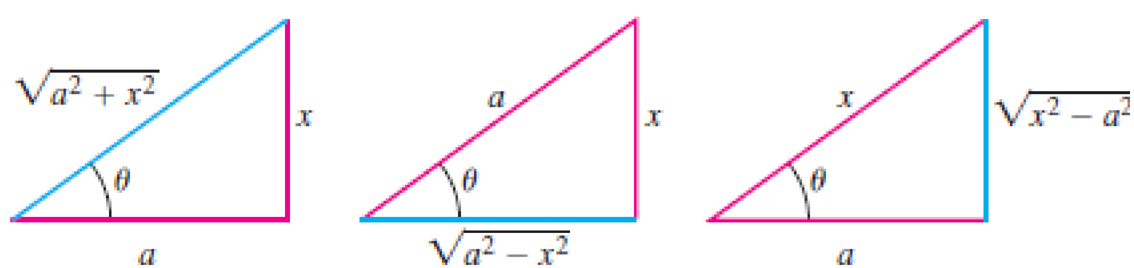
$$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1 + \tan^2\theta) = a^2\sec^2\theta.$$

With $x = a\sin\theta$,

$$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1 - \sin^2\theta) = a^2\cos^2\theta.$$

With $x = a\sec\theta$,

$$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta.$$



$x = a\tan\theta$ $\quad$ $x = a\sin\theta$ $\quad$ $x = a\sec\theta$

$\sqrt{a^2 + x^2} = a|\sec\theta|$ $\quad$ $\sqrt{a^2 - x^2} = a|\cos\theta|$ $\quad$ $\sqrt{x^2 - a^2} = a|\tan\theta|$

Using the Substitution $x = a \tan\theta$

$$\int \frac{dx}{\sqrt{4+x^2}}.$$

$$x = 2\tan\theta, \qquad dx = 2\sec^2\theta\, d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$4 + x^2 = 4 + 4\tan^2\theta = 4(1+\tan^2\theta) = 4\sec^2\theta.$$

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{\sec^2\theta\, d\theta}{|\sec\theta|} \qquad \sqrt{\sec^2\theta} = |\sec\theta|$$

$$= \int \sec\theta\, d\theta \qquad \sec\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C \qquad \text{From Fig. 8.4}$$

$$= \ln\left|\sqrt{4+x^2} + x\right| + C'. \qquad \text{Taking } C' = C - \ln 2$$

Using the Substitution $x = a \sin\theta$

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}}.$$

$$x = 3\sin\theta, \qquad dx = 3\cos\theta\,d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$9 - x^2 = 9 - 9\sin^2\theta = 9(1-\sin^2\theta) = 9\cos^2\theta.$$

$$\int \frac{x^2\,dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta \cdot 3\cos\theta\,d\theta}{|3\cos\theta|}$$

$$= 9\int \sin^2\theta\,d\theta \qquad \cos\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

$$= 9\int \frac{1-\cos 2\theta}{2}\,d\theta$$

$$= \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C$$

$$= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C \qquad \text{Fig. 8.5}$$

$$= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C.$$

■

Using the Substitution $x = a\sec\theta$

$$\int \frac{dx}{\sqrt{25x^2-4}}, \qquad x > \frac{2}{5}.$$

$$\sqrt{25x^2-4} = \sqrt{25\left(x^2 - \frac{4}{25}\right)}$$
$$= 5\sqrt{x^2 - \left(\frac{2}{5}\right)^2}$$

put the radicand in the form $x^2 - a^2$. We then substitute

$$x = \frac{2}{5}\sec\theta, \qquad dx = \frac{2}{5}\sec\theta\tan\theta\, d\theta, \qquad 0 < \theta < \frac{\pi}{2}$$

$$x^2 - \left(\frac{2}{5}\right)^2 = \frac{4}{25}\sec^2\theta - \frac{4}{25}$$
$$= \frac{4}{25}(\sec^2\theta - 1) = \frac{4}{25}\tan^2\theta$$

$$\sqrt{x^2 - \left(\frac{2}{5}\right)^2} = \frac{2}{5}|\tan\theta| = \frac{2}{5}\tan\theta.$$

$\tan\theta > 0$ for
$0 < \theta < \pi/2$

With these substitutions, we have

$$\int \frac{dx}{\sqrt{25x^2 - 4}} = \int \frac{dx}{5\sqrt{x^2 - (4/25)}} = \int \frac{(2/5)\sec\theta\tan\theta\, d\theta}{5 \cdot (2/5)\tan\theta}$$

$$= \frac{1}{5}\int \sec\theta\, d\theta = \frac{1}{5}\ln\left|\sec\theta + \tan\theta\right| + C$$

$$= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2 - 4}}{2}\right| + C.$$

8.8 Improper Integrals

DEFINITION **Type I Improper Integrals**

Integrals with infinite limits of integration are **improper integrals of Type I.**

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx,$$

where c is any real number.

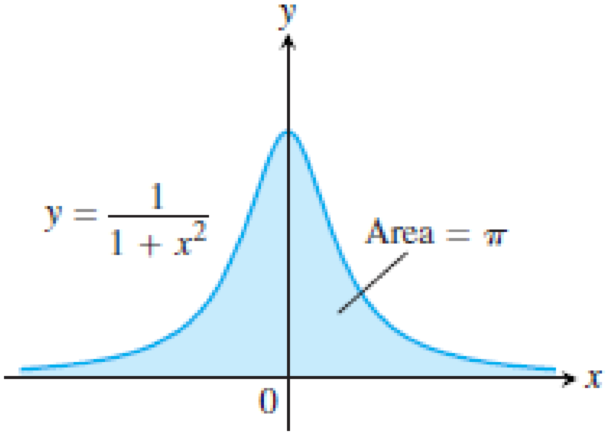
In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

Evaluating an Integral on $(-\infty, \infty)$

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2}.$$

According to the definition (Part 3), we can write

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \int_{-\infty}^{0} \frac{dx}{1 + x^2} + \int_{0}^{\infty} \frac{dx}{1 + x^2}.$$



$$\int_{-\infty}^{0} \frac{dx}{1 + x^2} = \lim_{a \to -\infty} \int_{a}^{0} \frac{dx}{1 + x^2}$$

$$= \lim_{a \to -\infty} \tan^{-1} x \Big]_a^0$$

$$= \lim_{a \to -\infty} (\tan^{-1} 0 - \tan^{-1} a) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

$$\int_{0}^{\infty} \frac{dx}{1 + x^2} = \lim_{b \to \infty} \int_{0}^{b} \frac{dx}{1 + x^2}$$

$$= \lim_{b \to \infty} \tan^{-1} x \Big]_0^b$$

$$= \lim_{b \to \infty} (\tan^{-1} b - \tan^{-1} 0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

The Integral $\int_1^\infty \frac{dx}{x^p}$

The function $y = 1/x$ is the boundary between the convergent and divergent improper integrals with integrands of the form $y = 1/x^p$. As the next example shows, the improper integral converges if $p > 1$ and diverges if $p \leq 1$.

$$\int_1^b \frac{dx}{x^p} = \frac{x^{-p+1}}{-p+1}\Big]_1^b = \frac{1}{1-p}(b^{-p+1} - 1) = \frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right).$$

Thus,

$$\int_1^\infty \frac{dx}{x^p} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^p}$$

$$= \lim_{b\to\infty} \left[\frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right)\right] = \begin{cases} \frac{1}{p-1}, & p > 1 \\ \infty, & p < 1 \end{cases}$$

because

$$\lim_{b\to\infty} \frac{1}{b^{p-1}} = \begin{cases} 0, & p > 1 \\ \infty, & p < 1. \end{cases}$$

Therefore, the integral converges to the value $1/(p-1)$ if $p > 1$ and it diverges if $p < 1$.

Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-\infty}^{0} \frac{1}{\sqrt{3-x}}\,dx$$

$$\begin{aligned}
\int_{-\infty}^{0} \frac{1}{\sqrt{3-x}}\,dx &= \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{\sqrt{3-x}}\,dx \\
&= \lim_{t\to-\infty} -2\sqrt{3-x}\,\Big|_{t}^{0} \\
&= \lim_{t\to-\infty} \left(-2\sqrt{3}+2\sqrt{3-t}\right) \\
&= -2\sqrt{3}+\infty \\
&= \infty
\end{aligned}$$

so this integral is divergent.

When does the integral of $1/x^p$ converge?

Here we consider an arbitrary power, p, that can be any real number. We ask when the corresponding improper integral converges or diverges. Let

$$I = \int_1^\infty \frac{1}{x^p}\, dx.$$

For $p = 1$ we have already established that this integral diverges and for $p = 2$ we have seen that it is convergent . By a similar calculation, we find that

$$I = \lim_{b\to\infty} \left.\frac{x^{1-p}}{(1-p)}\right|_1^b = \lim_{b\to\infty} \left(\frac{1}{1-p}\right)\left(b^{1-p} - 1\right).$$

Thus this integral converges provided that the term b^{1-p} does not "blow up" as b increases. For this to be true, we require that the exponent $(1 - p)$ should be negative, i.e. $1 - p < 0$ or $p > 1$. In this case, we have

$$I = \frac{1}{p-1}.$$

To summarize our result,

$$\int_1^\infty \frac{1}{x^p}\, dx \qquad \text{converges if } p > 1, \quad \text{diverges if } p \le 1.$$

The main points of this chapter can be summarized as follows:

1. We reviewed the definition of an improper integral

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx.$$

2. We computed some examples of improper integrals and discussed their convergence or divergence. We recalled (from earlier chapters) that

$$I = \int_0^\infty e^{-rt}\,dt \quad \text{converges,}$$

whereas

$$I = \int_1^\infty \frac{1}{x}\,dx \quad \text{diverges.}$$

3. More generally, we showed that

$$\int_1^\infty \frac{1}{x^p}\,dx \qquad \text{converges if } p > 1, \quad \text{diverges if } p \le 1.$$

4. Using a comparison between integrals and series we showed that the harmonic series,

$$\sum_{k=1}^\infty \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{k} + \ldots \qquad \text{diverges.}$$

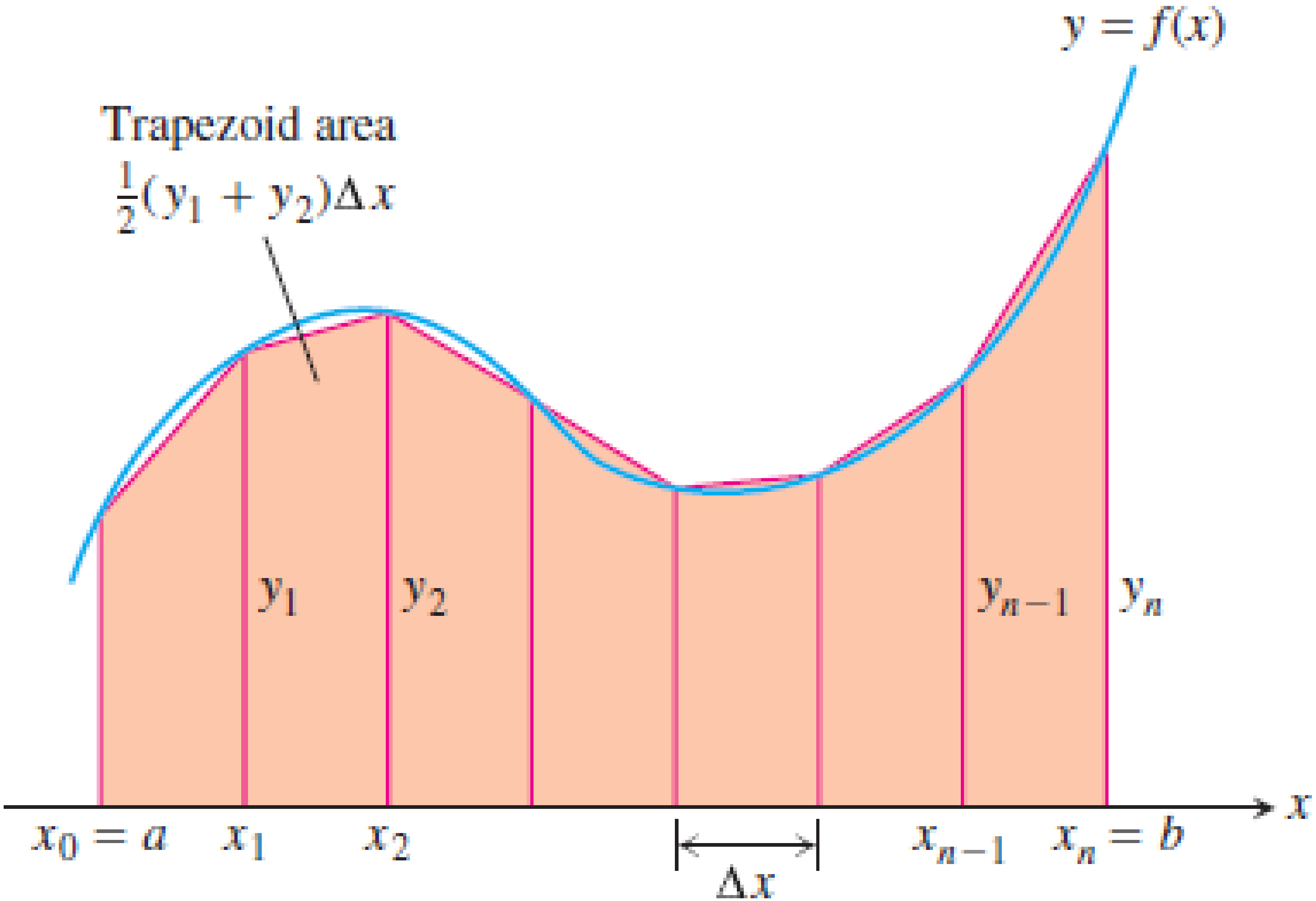
5. More generally, our results led to the conclusion that the "p" series,

$$\sum_{k=1}^\infty \frac{1}{k^p} \qquad \text{converges if } p > 1,\ \text{diverges if } p \le 1.$$

Numerical Integration

$$\Delta x = \frac{b - a}{n}.$$

the **step size** or **mesh size**.



$$
\begin{aligned}
T &= \frac{1}{2}(y_0 + y_1)\Delta x + \frac{1}{2}(y_1 + y_2)\Delta x + \cdots \\
&\qquad + \frac{1}{2}(y_{n-2} + y_{n-1})\Delta x + \frac{1}{2}(y_{n-1} + y_n)\Delta x \\
&= \Delta x \left(\frac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \frac{1}{2}y_n\right) \\
&= \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n),
\end{aligned}
$$

$$
y_0 = f(a), \qquad y_1 = f(x_1), \qquad \ldots, \qquad y_{n-1} = f(x_{n-1}), \qquad y_n = f(b)
$$

The Trapezoidal Rule

To approximate $\int_a^b f(x)\,dx$, use

$$T = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right).$$

The y's are the values of f at the partition points

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots, x_{n-1} = a + (n-1)\Delta x, x_n = b,$$

where $\Delta x = (b - a)/n$.

Using the trapezoidal rule, estimate the integral

$$\int_0^2 (t^3 + t)\,dt$$

with n = 4 steps.

Let $f(x) = x^3 + x$, $a = 0$, and $b = 2$. Now to find the step size.

$$h = \frac{2-0}{4} = \frac{2}{4} = \frac{1}{2}$$

	x_n	y_n
x_0	0	f (0) = 0
x_1	0.5	f (0.5) = 0.625
x_2	1	f (1) = 2
x_3	1.5	f (1.5) = 4.875
x_4	2	f (2) = 10

$$T_4 = \frac{\frac{1}{2}}{2}(0 + 2(0.625) + 2(2) + 2(4.875) + 10)$$

$$= \frac{1}{4}(0 + 1.25 + 4 + 9.75 + 10) = 6.26$$

Use the trapezoidal rule with $n = 8$ to estimate

$$\int_1^5 \sqrt{1+x^2}\,dx.$$

. For $n = 8$, we have $\Delta x = \frac{5-1}{8} = 0.5$. We compute the values of $y_0,\ y_1,\ y_2, \ldots,\ y_8$.

x	1	1.5	2	2.5	3	3.5	4	4.5	5
$y = \sqrt{1+x^2}$	$\sqrt{2}$	$\sqrt{3.25}$	$\sqrt{5}$	$\sqrt{7.25}$	$\sqrt{10}$	$\sqrt{13.25}$	$\sqrt{17}$	$\sqrt{21.25}$	$\sqrt{26}$

$$\approx \frac{0.5}{2}\left(\sqrt{2} + 2\sqrt{3.25} + 2\sqrt{5} + 2\sqrt{7.25} + 2\sqrt{10} + 2\sqrt{13.25} + 2\sqrt{17} + 2\sqrt{21.25} + \sqrt{26}\right)$$

$$\int_1^5 \sqrt{1+x^2}\,dx \approx$$

$$\approx \boxed{12.76}$$