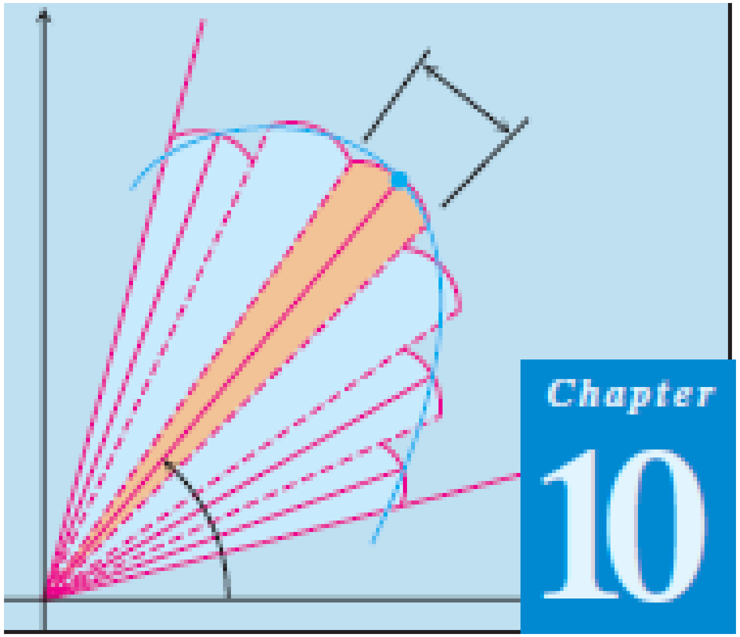
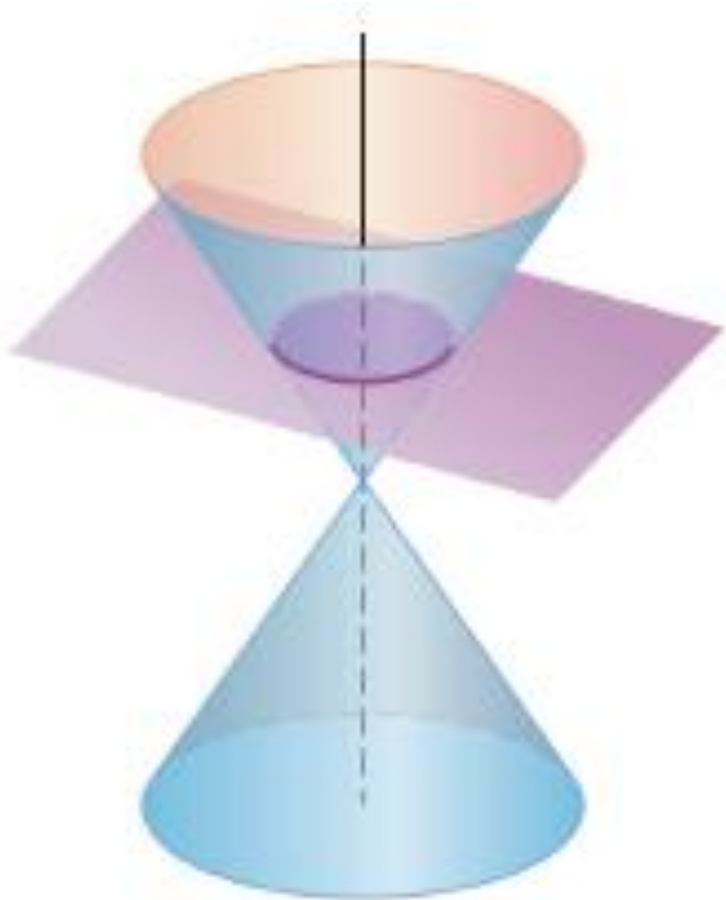


Chapter 10 Conic Sections and Polar Coordinates

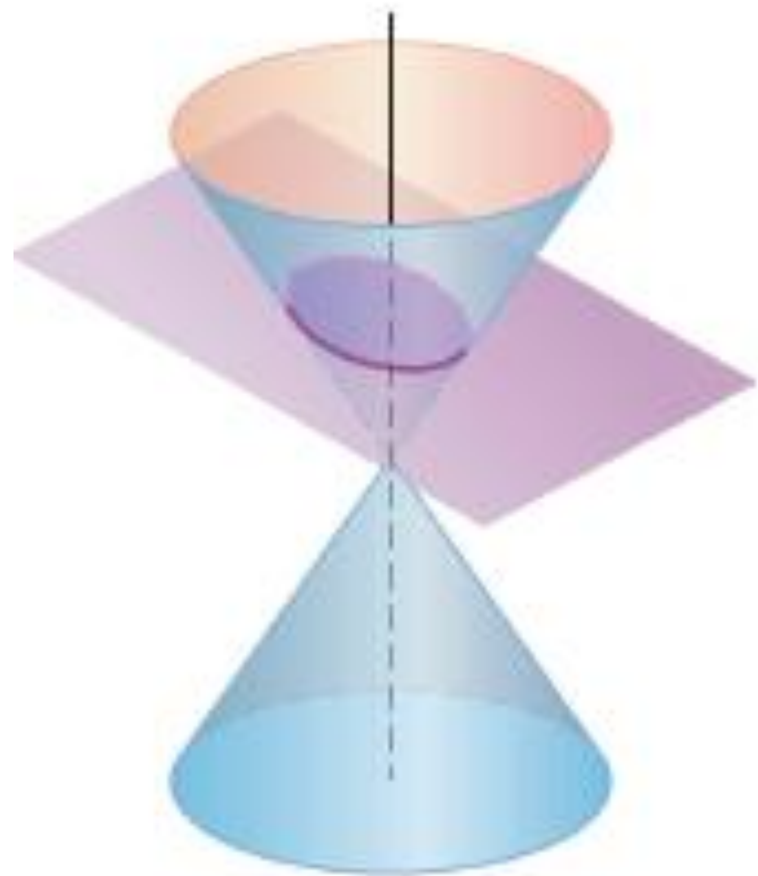
geometric definitions and standard equations of **parabolas**, **ellipses**, and **hyperbolas**.

curves are called ***conic sections*** or *conics*

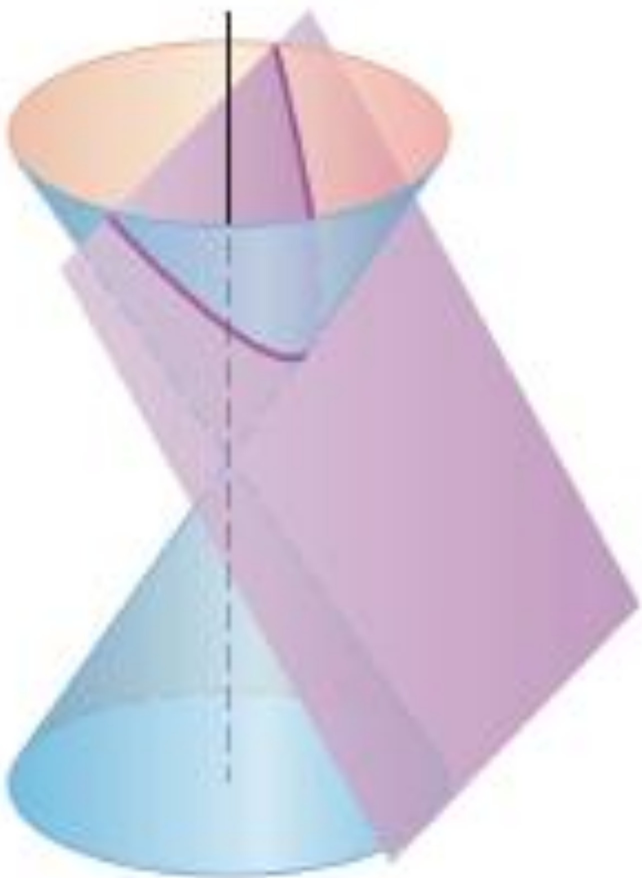
10.1 Conic Sections and Quadratic Equations



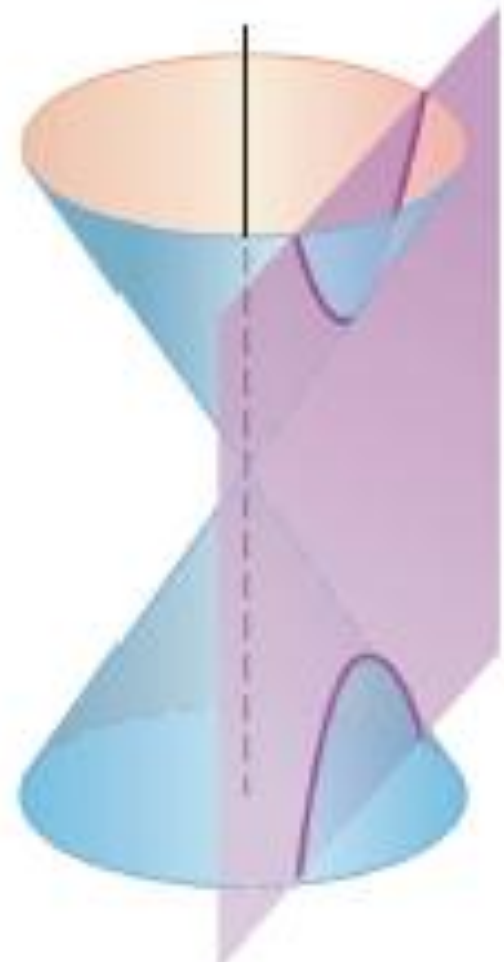
Circle: plane perpendicular to cone axis



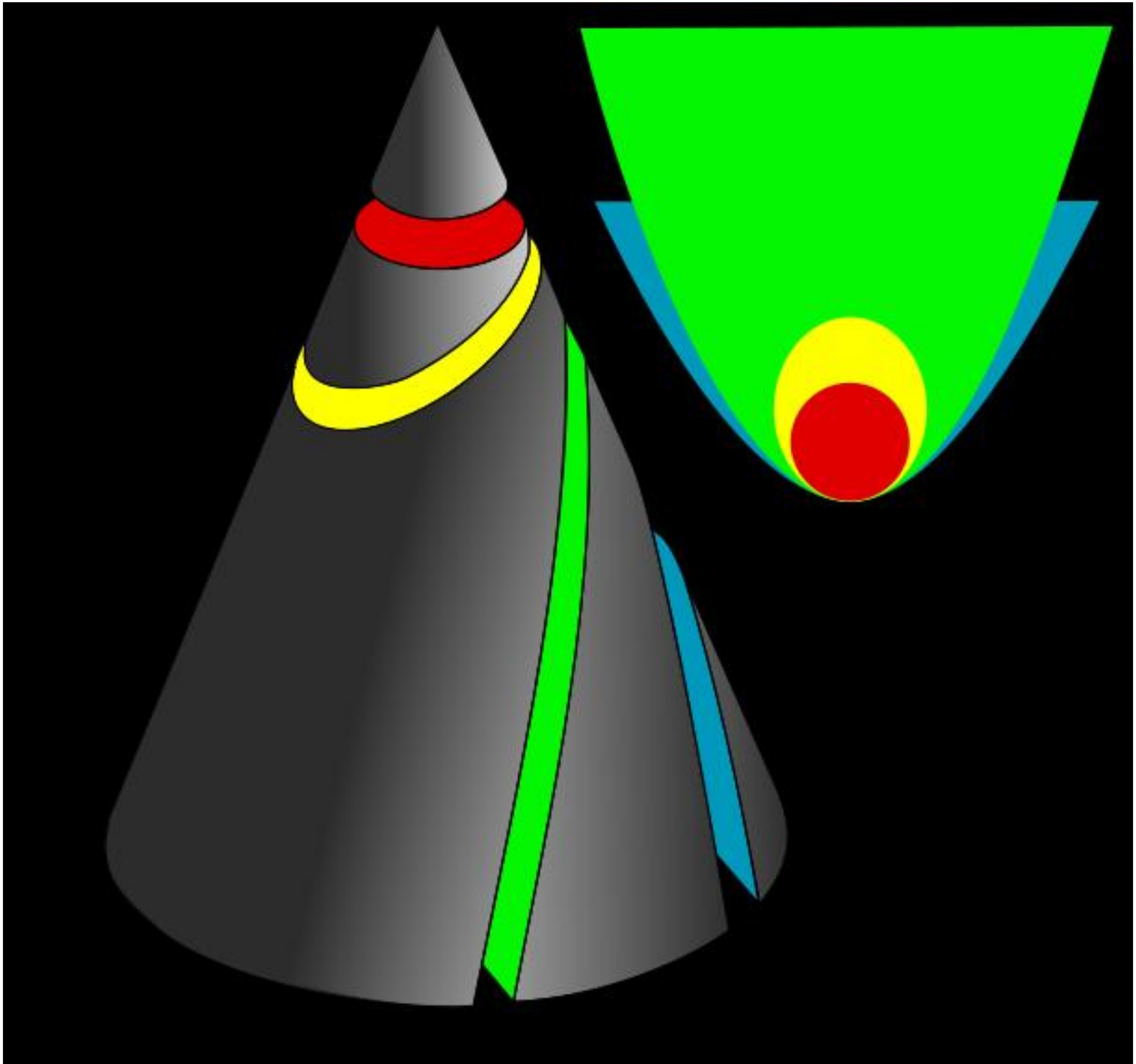
Ellipse: plane oblique to cone axis

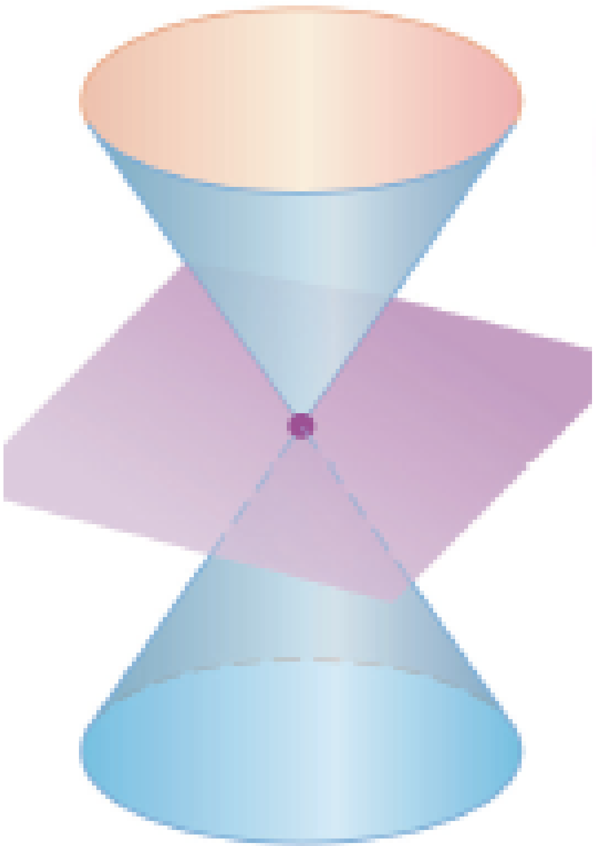


Parabola: plane parallel to side of cone



Hyperbola: plane cuts both halves of cone





Point: plane through cone vertex only

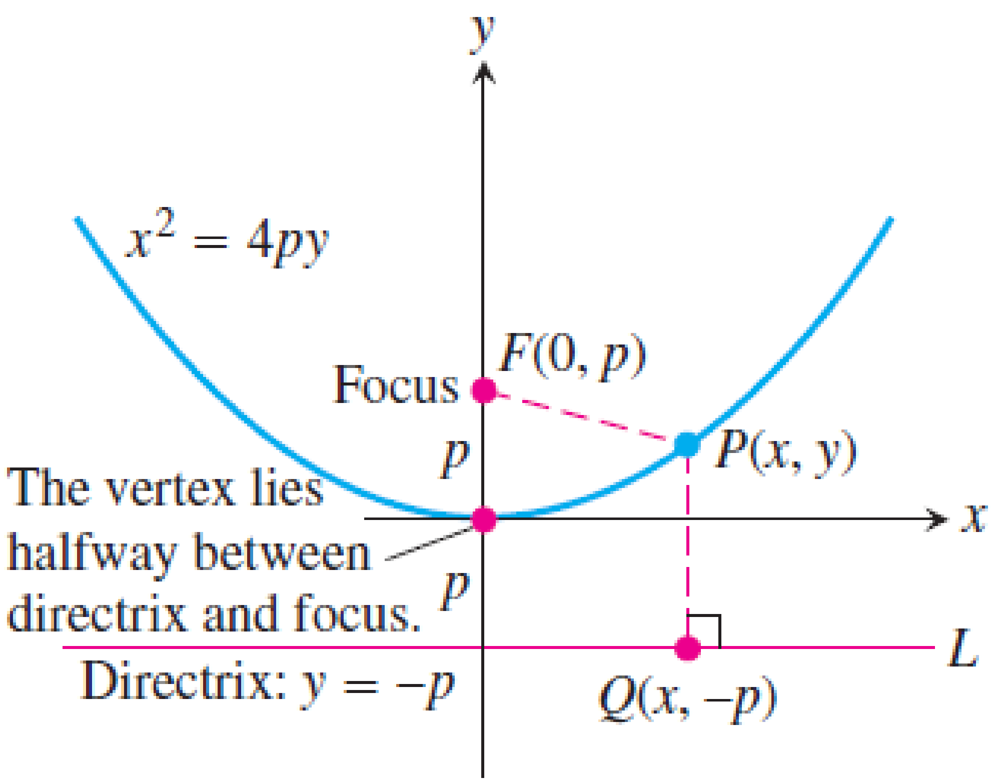
Single line: plane tangent to cone

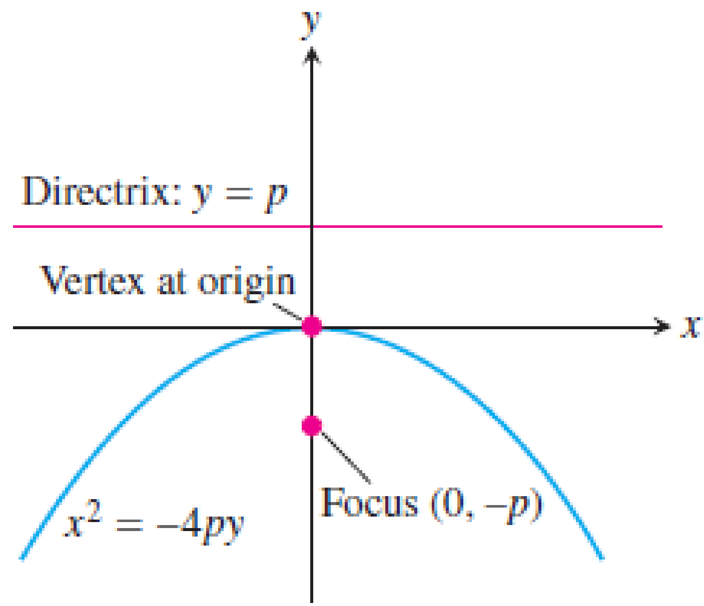
Pair of intersecting lines

Parabolas

DEFINITIONS Parabola, Focus, Directrix

A set that consists of all the points in a plane equidistant from a given fixed point and a given fixed line in the plane is a **parabola**. The fixed point is the **focus** of the parabola. The fixed line is the **directrix**.





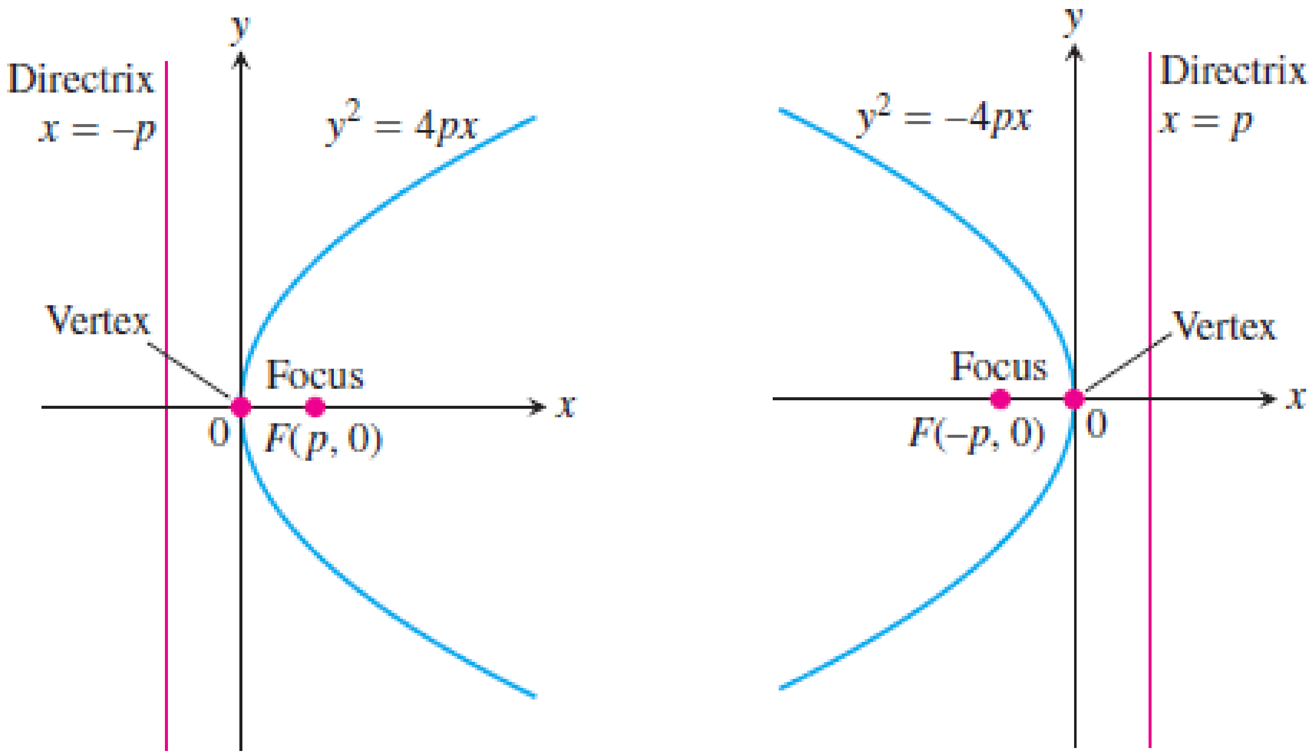
a point $P(x, y)$ lies on the parabola if and only if $PF = PQ$. From the distance

$$PF = \sqrt{(x-0)^2 + (y-p)^2} = \sqrt{x^2 + (y-p)^2}$$

$$PQ = \sqrt{(x-x)^2 + (y-(-p))^2} = \sqrt{(y+p)^2}.$$

When we equate these expressions, square, and simplify, we get

$$y = \frac{x^2}{4p} \quad \text{or} \quad x^2 = 4py. \qquad \text{Standard form}$$


Directrix
$x = -p$
$y^2 = 4px$
Vertex
Focus
0
$F(p, 0)$
x
y
$y^2 = -4px$
Directrix
$x = p$
Vertex
Focus
$F(-p, 0)$
0
x
y

Standard-form equations for parabolas with vertices at the origin ($p > 0$)

Equation	Focus	Directrix	Axis	Opens
$x^2 = 4py$	$(0, p)$	$y = -p$	y-axis	Up
$x^2 = -4py$	$(0, -p)$	$y = p$	y-axis	Down
$y^2 = 4px$	$(p, 0)$	$x = -p$	x-axis	To the right
$y^2 = -4px$	$(-p, 0)$	$x = p$	x-axis	To the left

Find the focus and directrix of the parabola $y^2 = 10x$.

We find the value of p in the standard equation $y^2 = 4px$

$$4p = 10, \qquad \text{so} \qquad p = \frac{10}{4} = \frac{5}{2}$$

Focus: $(p, 0) = \left(\frac{5}{2}, 0\right)$

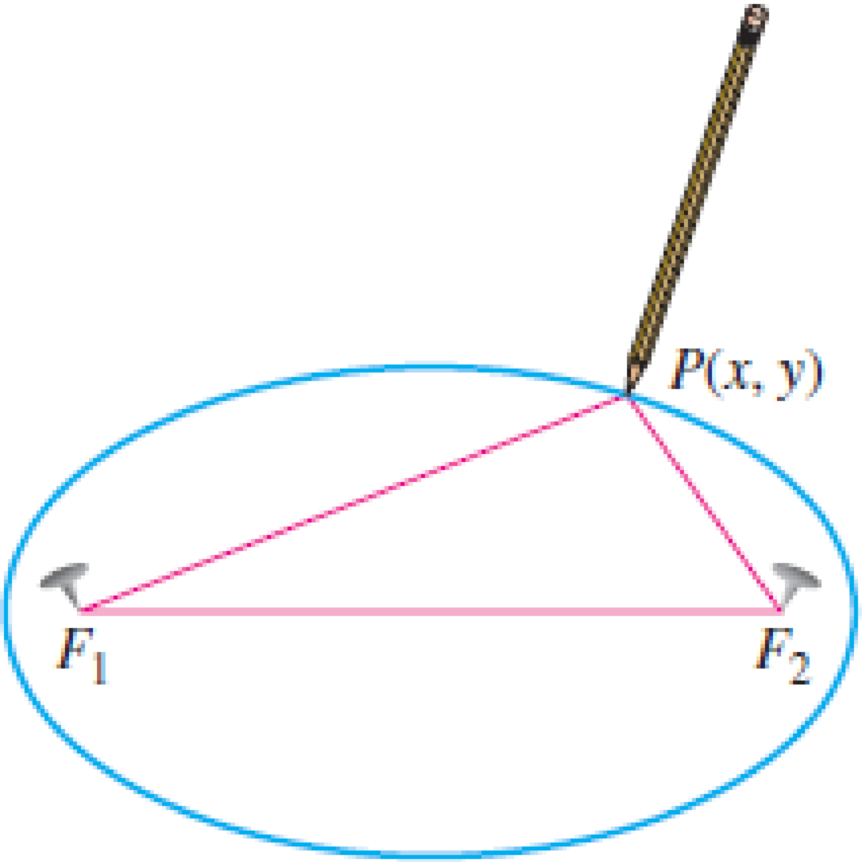
Directrix: $x = -p$

$$x = -\frac{5}{2}$$

Ellipses

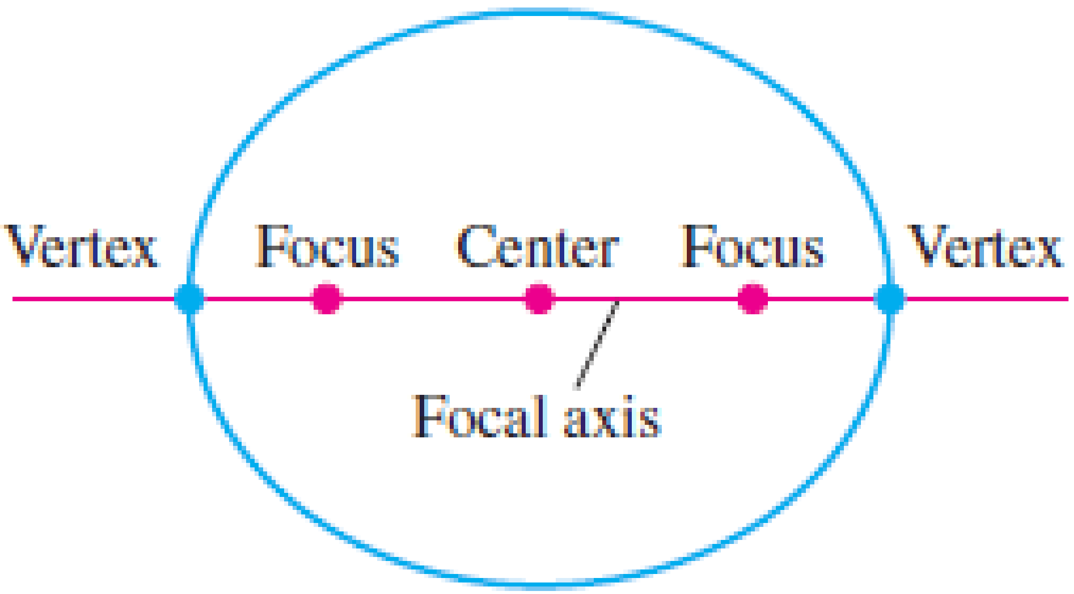
DEFINITIONS Ellipse, Foci

An **ellipse** is the set of points in a plane whose distances from two fixed points in the plane have a constant sum. The two fixed points are the **foci** of the ellipse.

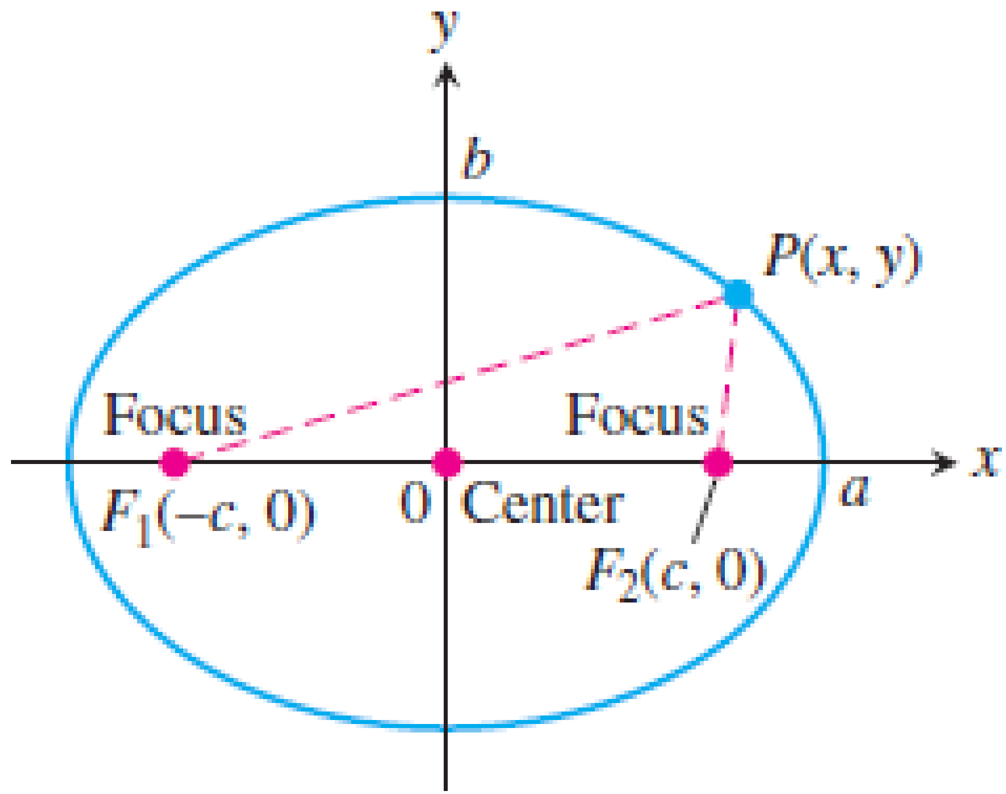


DEFINITIONS **Focal Axis, Center, Vertices**

The line through the foci of an ellipse is the ellipse's **focal axis**. The point on the axis halfway between the foci is the **center**. The points where the focal axis and ellipse cross are the ellipse's **vertices**



Points on the focal axis of an ellipse.



The ellipse defined by the equation $PF_1 + PF_2 = 2a$ is the graph of the equation $(x^2/a^2) + (y^2/b^2) = 1$, where $b^2 = a^2 - c^2$.

$$\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$$

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1$$

$$b=\sqrt{a^2-c^2}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Major Axis Horizontal

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

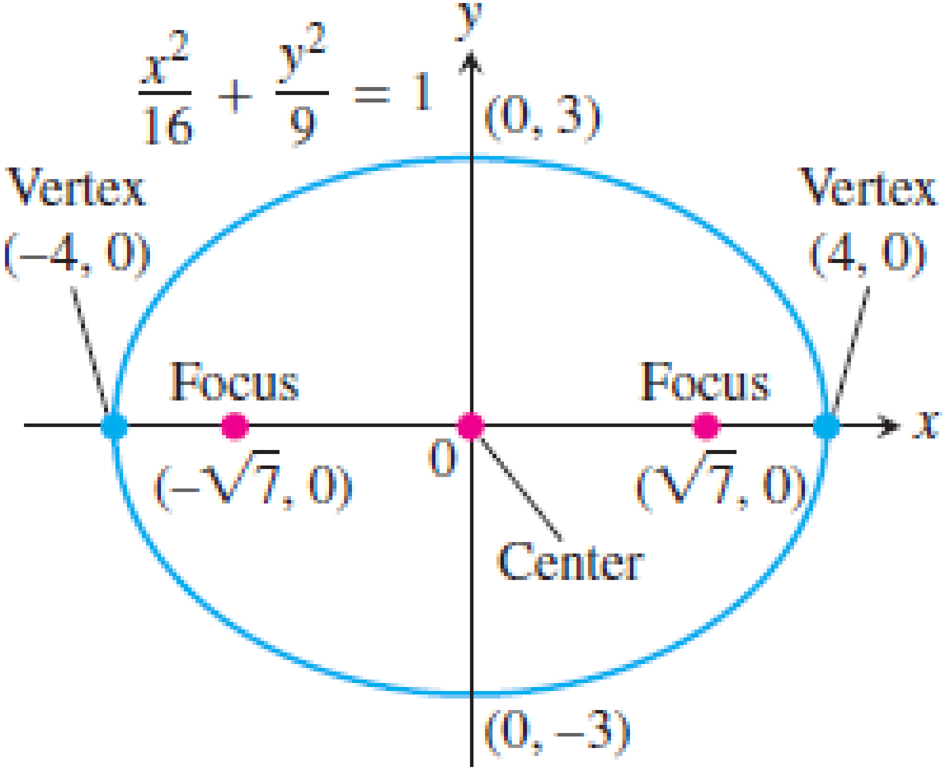
Semimajor axis: $a = \sqrt{16} = 4$, Semiminor axis: $b = \sqrt{9} = 3$

Center-to-focus distance: $c = \sqrt{16 - 9} = \sqrt{7}$

Foci: $(\pm c, 0) = \left(\pm\sqrt{7}, 0\right)$

Vertices: $(\pm a, 0) = (\pm 4, 0)$

Center: $(0, 0)$.



Standard-Form Equations for Ellipses Centered at the Origin

Foci on the x-axis: $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \quad (a > b)$

Center-to-focus distance: $c = \sqrt{a^2 - b^2}$

Foci: $(\pm c, 0)$

Vertices: $(\pm a, 0)$

Foci on the y-axis: $\dfrac{x^2}{b^2} + \dfrac{y^2}{a^2} = 1 \quad (a > b)$

Center-to-focus distance: $c = \sqrt{a^2 - b^2}$

Foci: $(0, \pm c)$

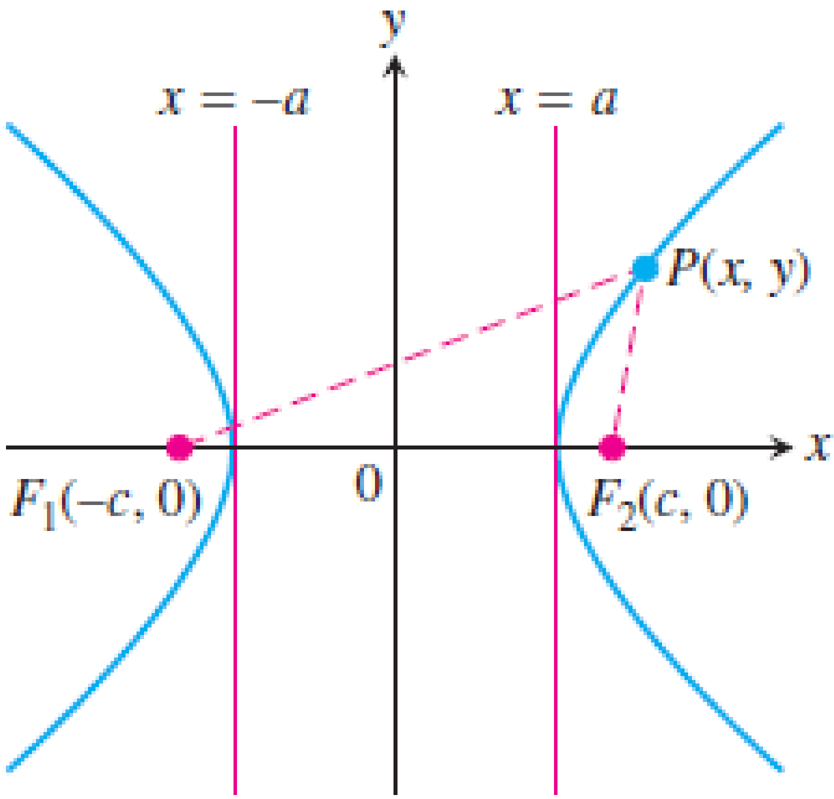
Vertices: $(0, \pm a)$

In each case, a is the semimajor axis and b is the semiminor axis.

Hyperbolas

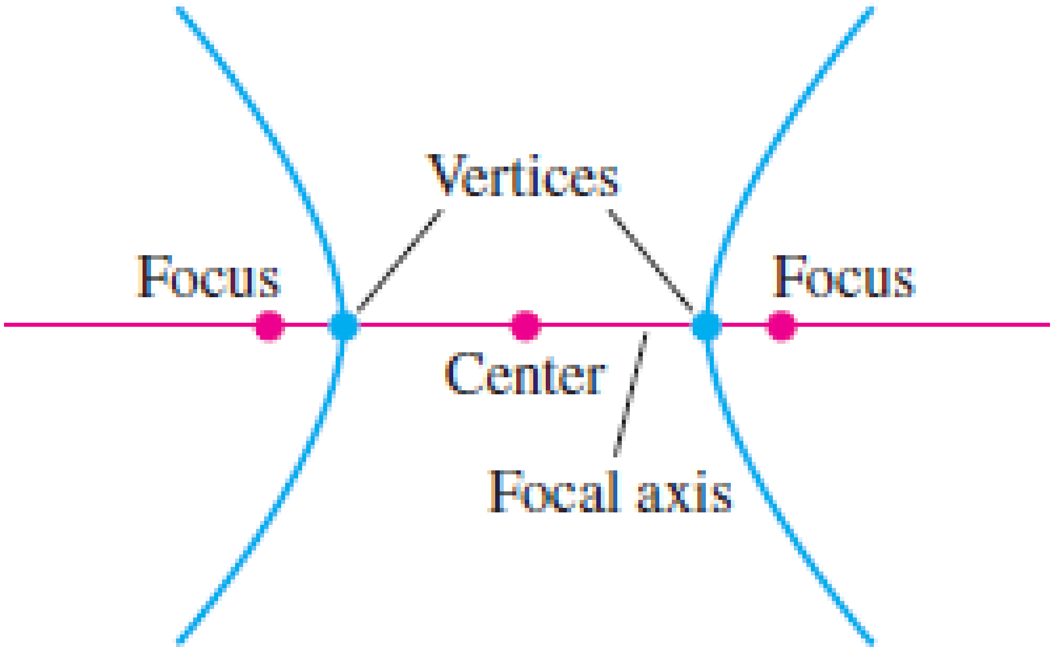
DEFINITIONS Hyperbola, Foci

A **hyperbola** is the set of points in a plane whose distances from two fixed points in the plane have a constant difference. The two fixed points are the **foci** of the hyperbola.



DEFINITIONS **Focal Axis, Center, Vertices**

The line through the foci of a hyperbola is the **focal axis**. The point on the axis halfway between the foci is the hyperbola's **center**. The points where the focal axis and hyperbola cross are the **vertices**



$$\sqrt{(x+c)^2+y^2}-\sqrt{(x-c)^2+y^2}=\pm 2a.$$

$$\frac{x^2}{a^2}+\frac{y^2}{a^2-c^2}=1$$

$$b=\sqrt{c^2-a^2}$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$$\underbrace{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}_{\text{hyperbola}}$$

$$\underbrace{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0}_{\text{0 for 1}}$$

$$\underbrace{y = \pm\frac{b}{a}x}_{\text{asymptotes}}$$

Standard-Form Equations for Hyperbolas Centered at the Origin

Foci on the x-axis: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Center-to-focus distance: $c = \sqrt{a^2 + b^2}$

Foci: $(\pm c, 0)$

Vertices: $(\pm a, 0)$

Asymptotes: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ or $y = \pm \frac{b}{a} x$

Find the foci and asymptotes of the hyperbola $9x^2 - 16y^2 = 144$ and sketch its graph.

SOLUTION If we divide both sides of the equation by 144,

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

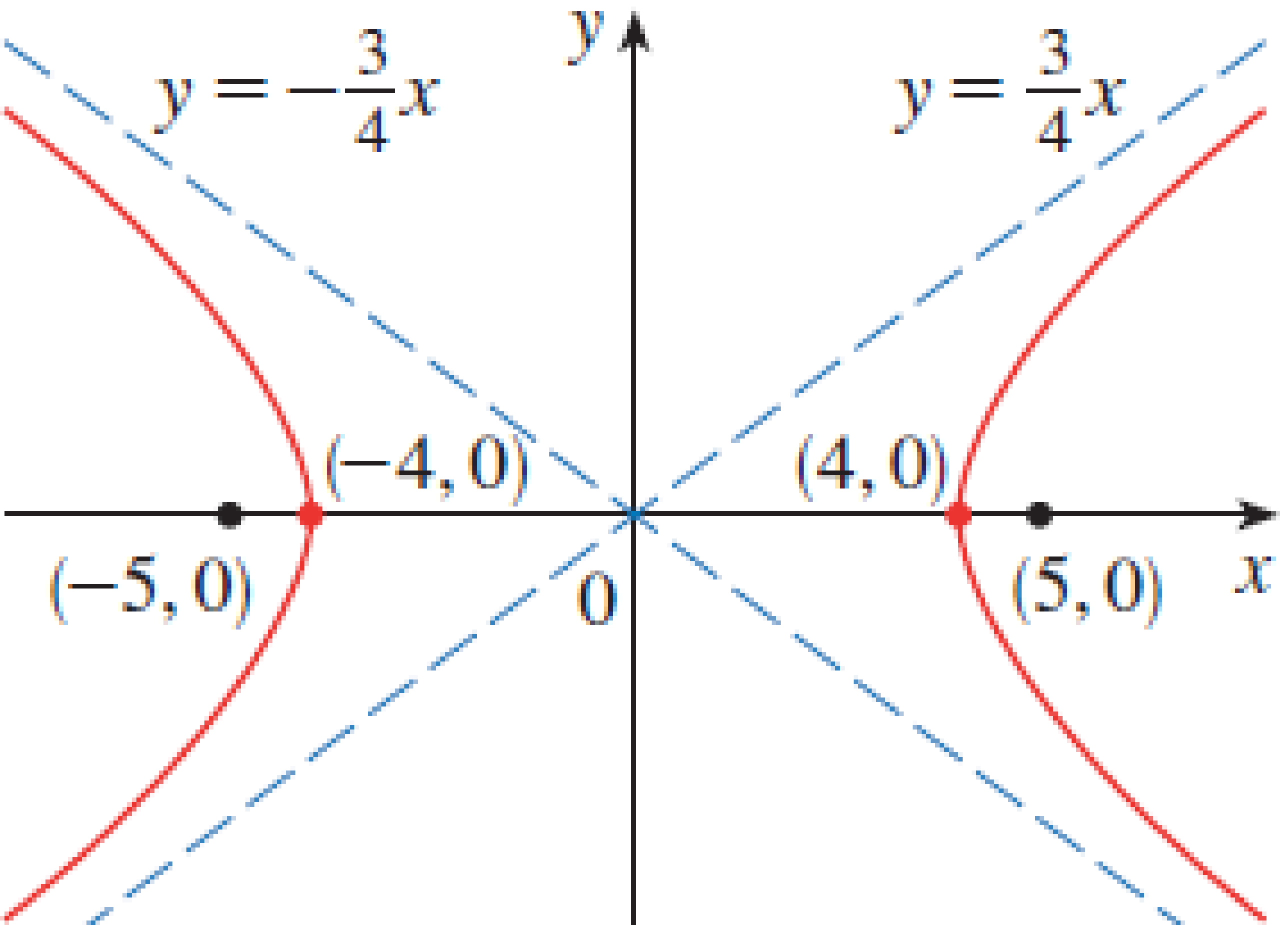
$a = 4$ and $b = 3$. Since $c^2 = 16 + 9 = 25$

foci are $(\pm 5, 0)$

$$y = \tfrac{3}{4}x \text{ and } y = -\tfrac{3}{4}x.$$

hyperbola $9x^2 - 16y^2 = 144$

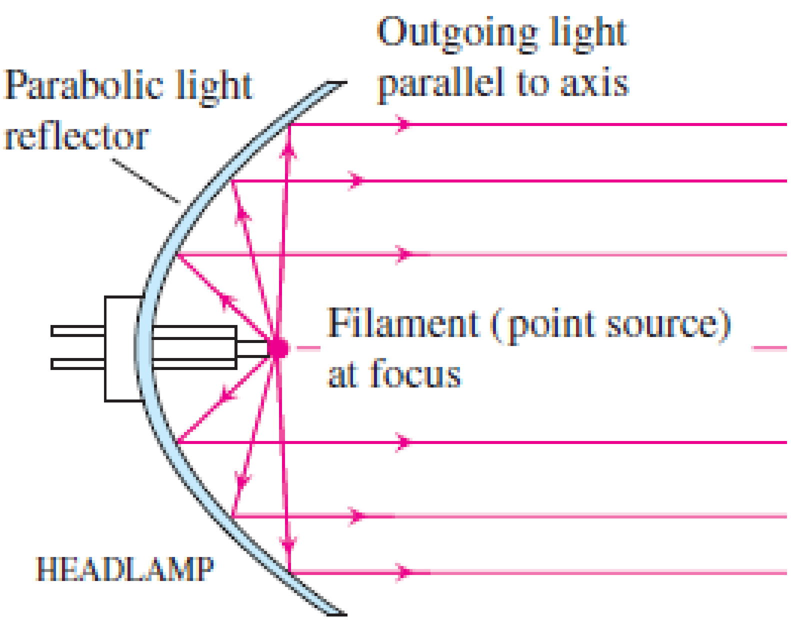
$a = 4$ and $b = 3$. Since $c^2 = 16 + 9 = 25$

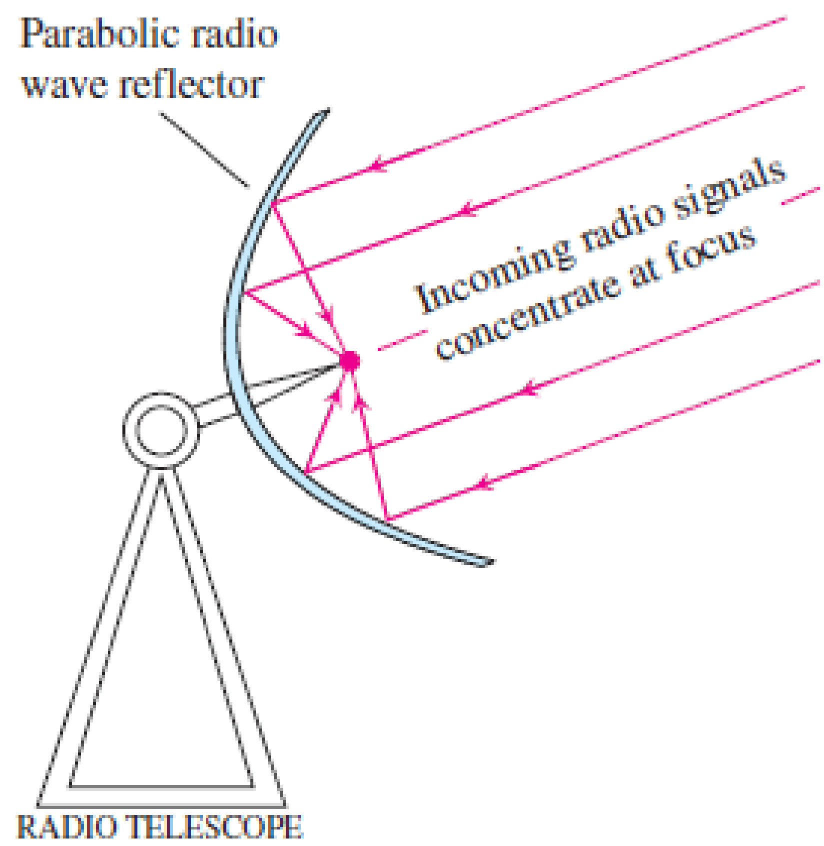


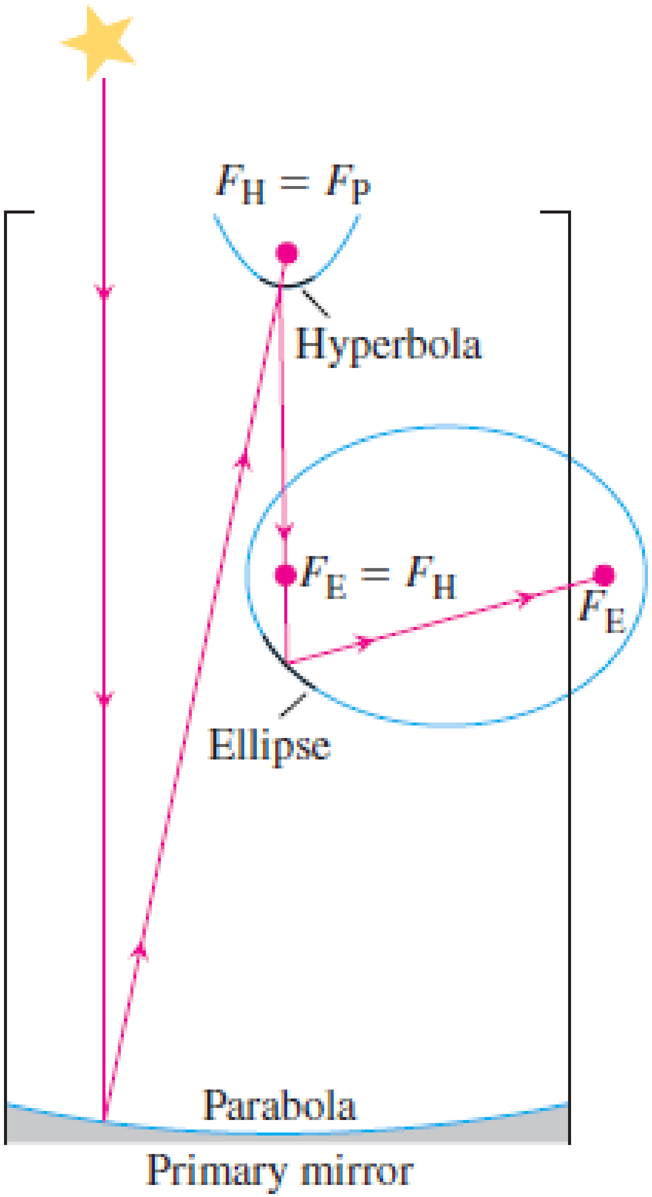
Reflective Properties

applications of parabolas involve their use as reflectors of light and radio waves.

Rays originating at a parabola’s focus are reflected out of the parabola parallel to the parabola’s axis







Schematic drawing of a reflecting telescope.

If an ellipse is revolved about its major axis to generate a surface (the surface is called an *ellipsoid*) and the interior is silvered to produce a mirror, light from one focus will be reflected to the other focus

Ellipsoids reflect sound the same way, and this property is used to construct *whispering galleries*, rooms in which a person standing at one focus can hear a whisper from the other focus.

Find the center, foci, vertices, asymptotes, and radius, as appropriate, of the conic sections

57. $x^2 + 4x + y^2 = 12$

58. $2x^2 + 2y^2 - 28x + 12y + 114 = 0$

59. $x^2 + 2x + 4y - 3 = 0$

61. $x^2 + 5y^2 + 4x = 1$

63. $x^2 + 2y^2 - 2x - 4y = -1$

62. $9x^2 + 6y^2 + 36y = 0 \Rightarrow 9x^2 + 6\left(y^2 + 6y + 9\right) = 54 \Rightarrow$

$9x^2 + 6(y+3)^2 = 54 \Rightarrow \frac{x^2}{6} + \frac{(y+3)^2}{9} = 1$; this is an ellipse:

the center is $(0, -3)$, the vertices are $(0, 0)$ and $(0, -6)$;

$c = \sqrt{a^2 - b^2} = \sqrt{9-6} = \sqrt{3} \Rightarrow$ the foci are $\left(0, -3 \pm \sqrt{3}\right)$

Sketch the regions in the xy-plane whose coordinates satisfy the inequalities or pairs of inequalities

69. $9x^2 + 16y^2 \leq 144$

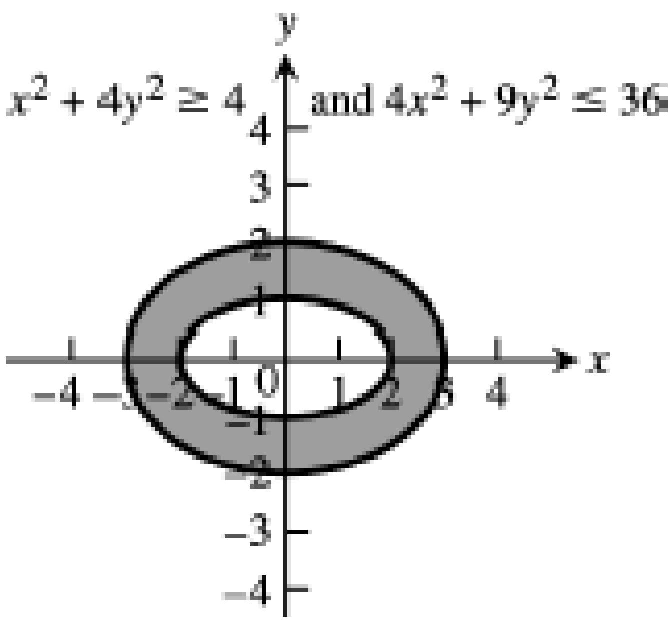
70. $x^2 + y^2 \geq 1$ and $4x^2 + y^2 \leq 4$

71. $x^2 + 4y^2 \geq 4$ and $4x^2 + 9y^2 \leq 36$

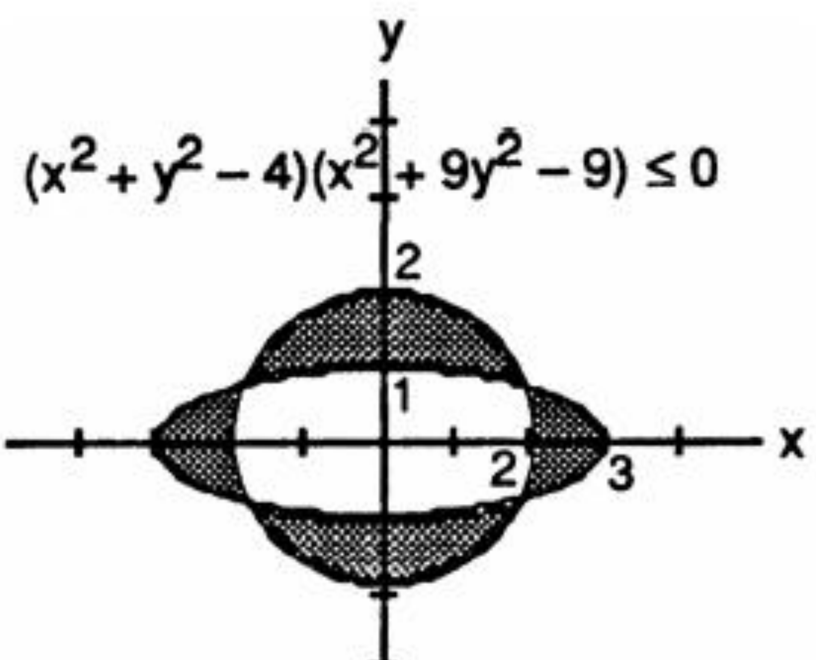
72. $(x^2 + y^2 - 4)(x^2 + 9y^2 - 9) \leq 0$

73. $4y^2 - x^2 \geq 4$

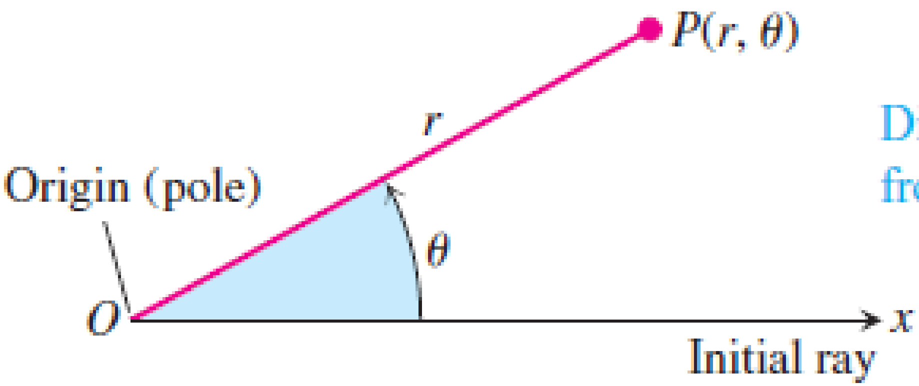
71.

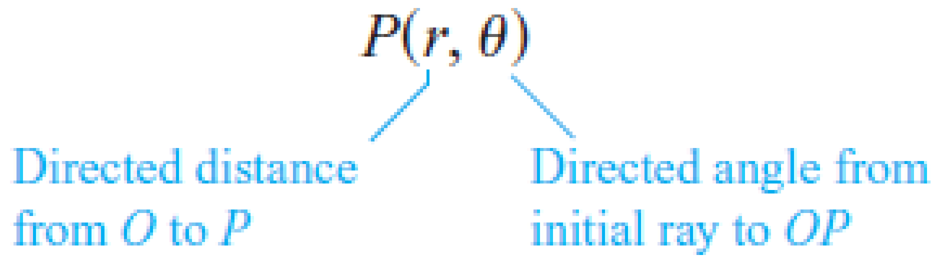


72.



10.5 Polar Coordinates



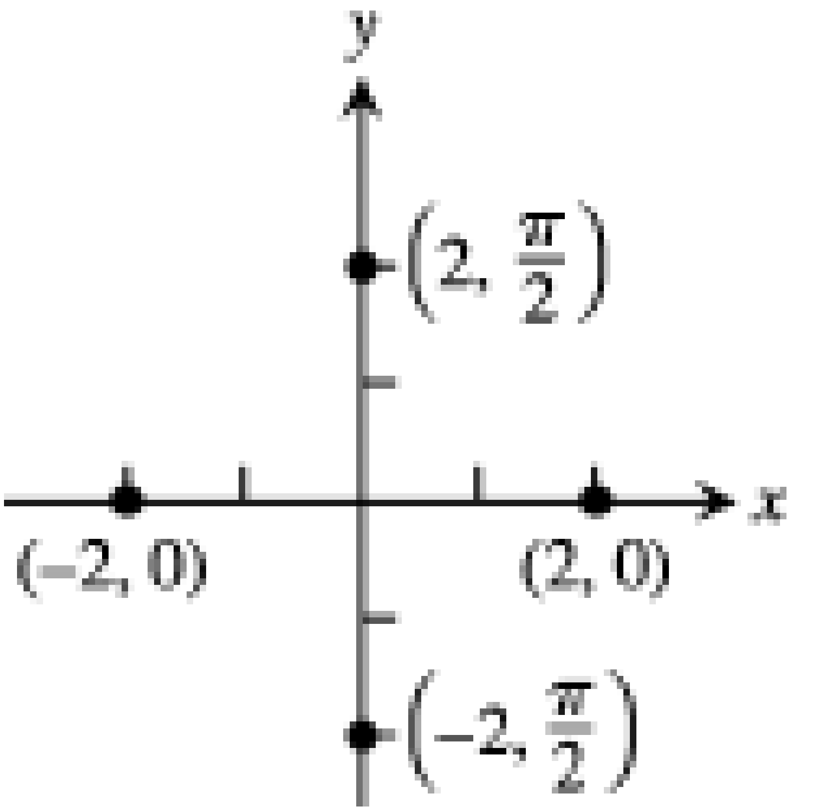


To define polar coordinates, we first fix an **origin** O (called the **pole**) and an **initial ray** from O Then each point P can be located by assigning to it a **polar coordinate pair** (r, θ) in which r gives the directed distance from O to P and θ gives the directed angle from the initial ray to ray OP.

Plot the following points (given in polar coordinates). Then find all the polar coordinates of each point

a. $(2, \pi/2)$ b. $(2, 0)$

c. $(-2, \pi/2)$ d. $(-2, 0)$



(a) $\left(2, \frac{\pi}{2} + 2n\pi\right)$ and $\left(-2, \frac{\pi}{2} + (2n+1)\pi\right)$, n an integer

(b) $(2, 2n\pi)$ and $(-2, (2n+1)\pi)$, n an integer

(c) $\left(2, \frac{3\pi}{2} + 2n\pi\right)$ and $\left(-2, \frac{3\pi}{2} + (2n+1)\pi\right)$, n an integer

(d) $(2, (2n+1)\pi)$ and $(-2, 2n\pi)$, n an integer

Find the Cartesian coordinates of the following points (given in polar coordinates).

a. $\left(\sqrt{2}, \pi/4\right)$ $(1, 1)$

c. $(0, \pi/2)$ $(1, 0)$

e. $(-3, 5\pi/6)$ $(0, 0)$

g. $(-1, 7\pi)$ $(-1, -1)$

b. $(1, 0)$ $\left(\frac{3\sqrt{3}}{2}, -\frac{3}{2}\right)$

d. $\left(-\sqrt{2}, \pi/4\right)$ $(3, 4)$

f. $(5, \tan^{-1}(4/3))$ $(1, 0)$

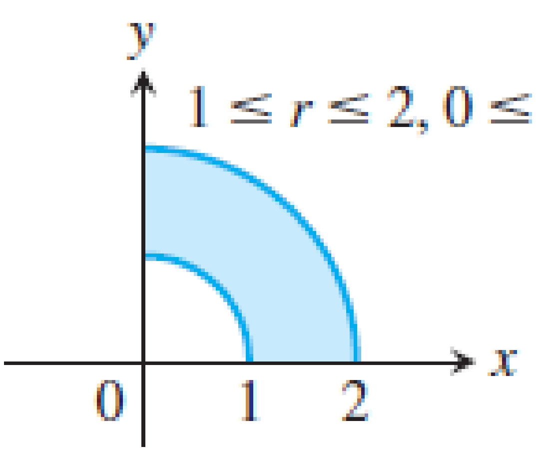
h. $\left(2\sqrt{3}, 2\pi/3\right)$ $\left(-\sqrt{3}, 3\right)$

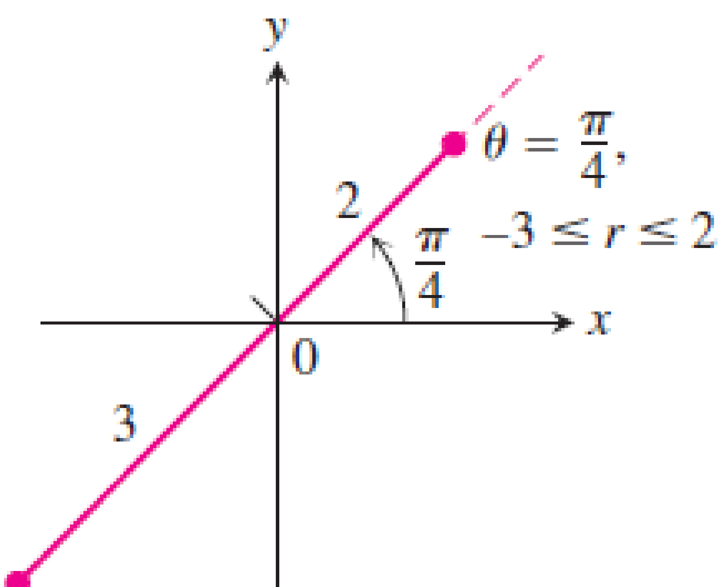
Equation	Graph
$r = a$	Circle radius $\lvert a \rvert$ centered at O
$\theta = \theta_0$	Line through O making an angle θ_0 with the initial ray

Graph the sets of points whose polar coordinates satisfy the following conditions.

(a) $1 \le r \le 2 \quad$ and $\quad 0 \le \theta \le \dfrac{\pi}{2}$

(b) $-3 \le r \le 2 \quad$ and $\quad \theta = \dfrac{\pi}{4}$



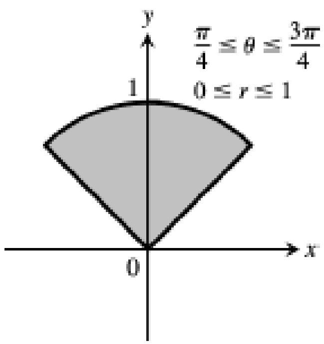


19. $\pi/4 \le \theta \le 3\pi/4, \quad 0 \le r \le 1$

20. $-\pi/4 \le \theta \le \pi/4, \quad -1 \le r \le 1$

21. $-\pi/2 \le \theta \le \pi/2, \quad 1 \le r \le 2$

19.



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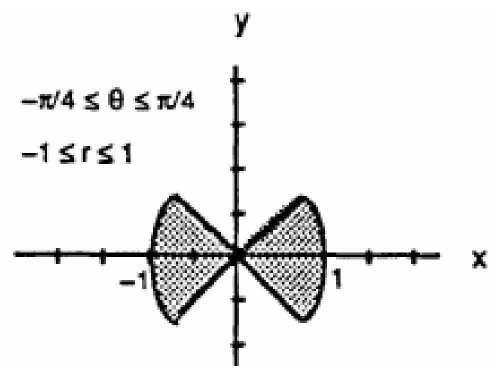

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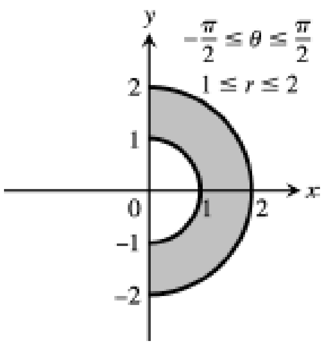
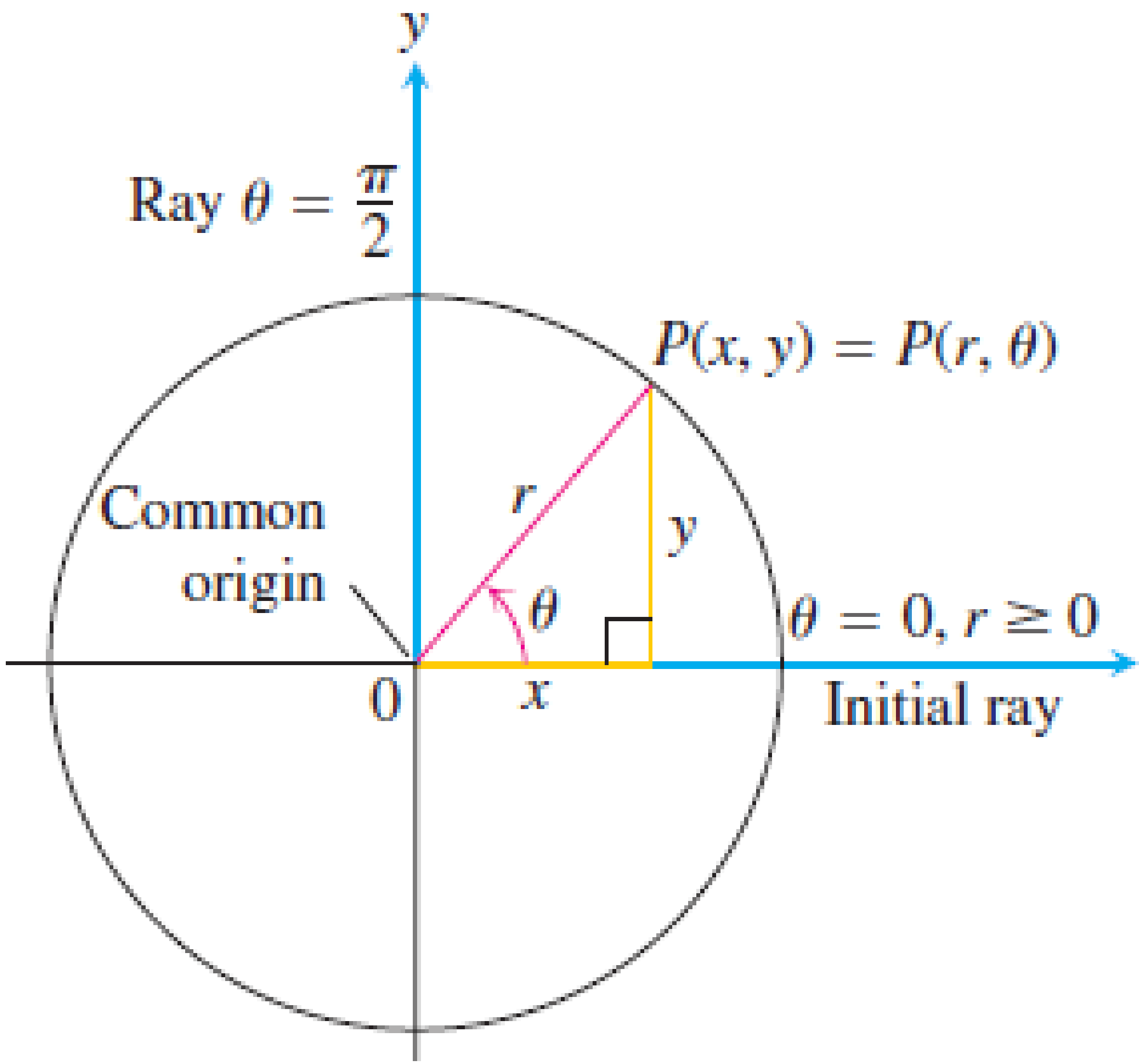


Equations Relating Polar and Cartesian Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad x^2 + y^2 = r^2$$



Polar equation	Cartesian equivalent
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$

Replace the Cartesian equations by equivalent polar equations

49. $x = 7$ **50.** $y = 1$ **51.** $x = y$

52. $x - y = 3$ **53.** $x^2 + y^2 = 4$ **54.** $x^2 - y^2 = 1$

49. $x = 7 \Rightarrow r\cos\theta = 7$

50. $y = 1 \Rightarrow r\sin\theta = 1$

51. $x = y \Rightarrow r\cos\theta = r\sin\theta \Rightarrow \theta = \frac{\pi}{4}$

52. $x - y = 3 \Rightarrow r\cos\theta - r\sin\theta = 3$

53. $x^2 + y^2 = 4 \Rightarrow r^2 = 4 \Rightarrow r = 2$ or $r = -2$

54. $x^2 - y^2 = 1 \Rightarrow r^2\cos^2\theta - r^2\sin^2\theta = 1 \Rightarrow r^2(\cos^2\theta - \sin^2\theta) = 1 \Rightarrow r^2\cos 2\theta = 1$