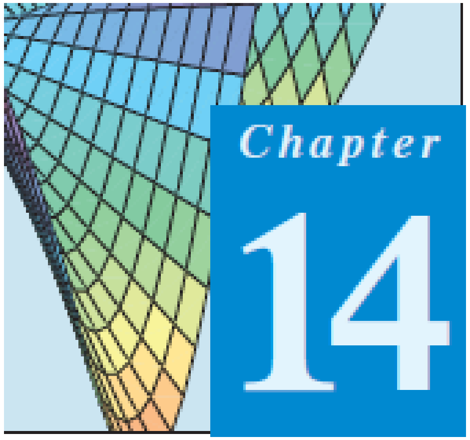


Chapter 14

PARTIAL DERIVATIVES

14.1 Functions of Several Variables

- A real-world phenomenon usually depends on two or more **independent** variables.
- We need to extend the basic ideas of functions of a single variable to functions of several variables.

Many functions depend on more than one independent variable. The function $V = \pi r^2 h$ calculates the volume of a right circular cylinder from its radius and height. The function $f(x, y) = x^2 + y^2$ calculates the height of the paraboloid $z = x^2 + y^2$ above the point

Real-valued functions of several independent real variables are defined much the way you would imagine from the single-variable case. The domains are sets of ordered pairs (triples, quadruples, n-tuples) of real numbers, and the ranges are sets of real numbers of the kind we have worked with all along.

Evaluating a Function

The value of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at the point $(3, 0, 4)$ is

$$f(3, 0, 4) = \sqrt{(3)^2 + (0)^2 + (4)^2} = \sqrt{25} = 5.$$

DEFINITIONS Function of n Independent Variables

Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A **real-valued function** f on D is a rule that assigns a unique (single) real number

$$w = f(x_1, x_2, \ldots, x_n)$$

to each element in D. The set D is the function's **domain**. The set of w-values taken on by f is the function's **range**. The symbol w is the **dependent variable** of f, and f is said to be a function of the n **independent variables** x_1 to x_n. We also call the x_j's the function's **input variables** and call w the function's **output variable**.

Domains and Ranges

- Avoid complex numbers or division by zero

(a) Functions of Two Variables

Function	Domain	Range
$w = \sqrt{y - x^2}$	$y \geq x^2$	$[0, \infty)$
$w = \frac{1}{xy}$	$xy \neq 0$	$(-\infty, 0) \cup (0, \infty)$
$w = \sin xy$	Entire plane	$[-1, 1]$

(b) Functions of Three Variables

Function	Domain	Range
$w = \sqrt{x^2 + y^2 + z^2}$	Entire space	$[0, \infty)$
$w = \frac{1}{x^2 + y^2 + z^2}$	$(x, y, z) \neq (0, 0, 0)$	$(0, \infty)$
$w = xy \ln z$	Half-space $z > 0$	$(-\infty, \infty)$

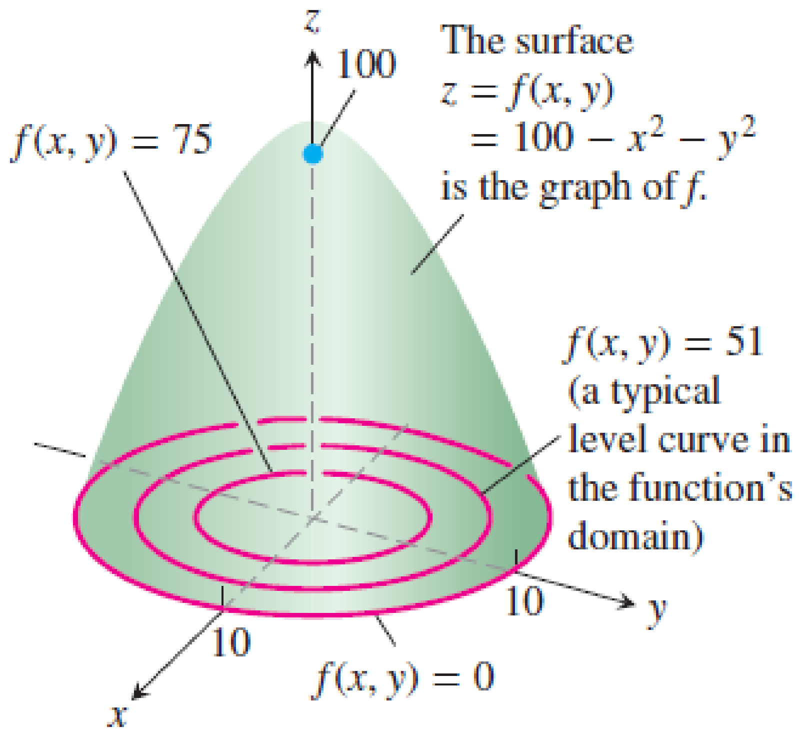
Graphs, Level Curves, and Contours of Functions of Two Variables

DEFINITIONS Level Curve, Graph, Surface

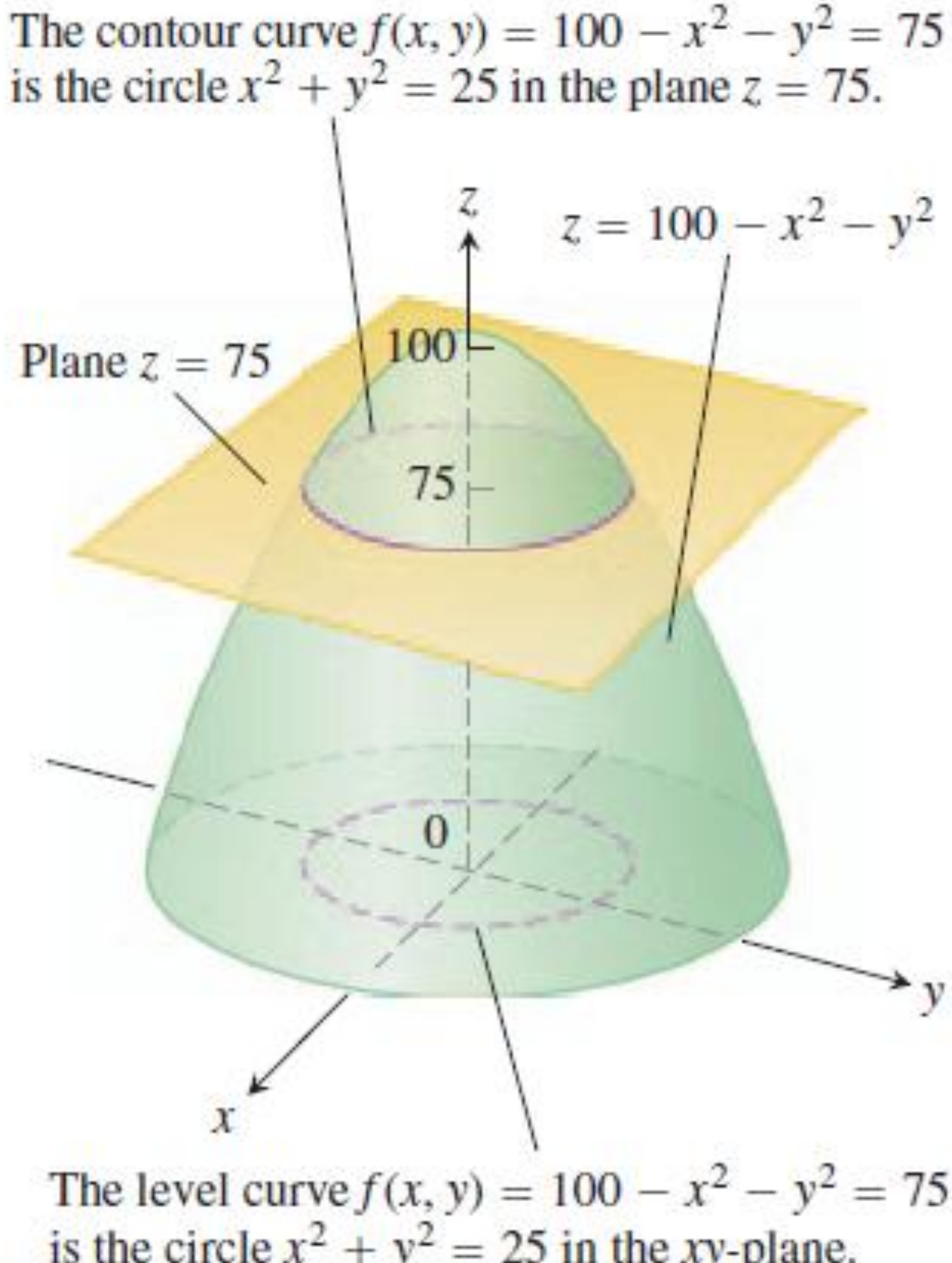
The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a **level curve** of f. The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f, is called the **graph** of f. The graph of f is also called the **surface** $z = f(x, y)$.

Graphing a Function of Two Variables

Graph $f(x, y) = 100 - x^2 - y^2$ and plot the level curves $f(x, y) = 0$, $f(x, y) = 51$, and $f(x, y) = 75$ in the domain of f in the plane.



The graph and selected level curves of the function $f(x, y) = 100 - x^2 - y^2$


The contour curve $f(x, y) = 100 - x^2 - y^2 = 75$ is the circle $x^2 + y^2 = 25$ in the plane $z = 75$.
z
$z = 100 - x^2 - y^2$
Plane $z = 75$
100
75
0
y
x
The level curve $f(x, y) = 100 - x^2 - y^2 = 75$ is the circle $x^2 + y^2 = 25$ in the xy-plane.

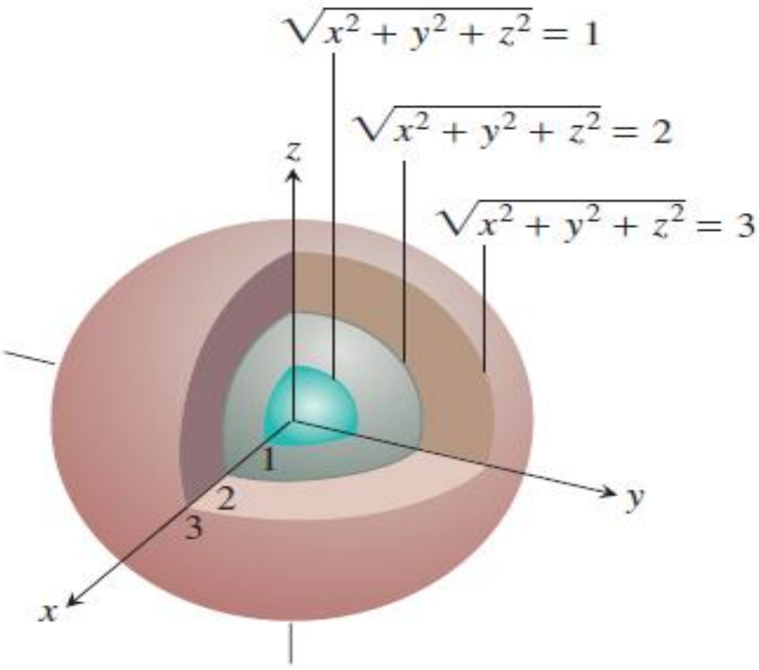
Functions of Three Variables

DEFINITION **Level Surface**

The set of points (x, y, z) in space where a function of three independent variables has a constant value $f(x, y, z) = c$ is called a **level surface** of f.

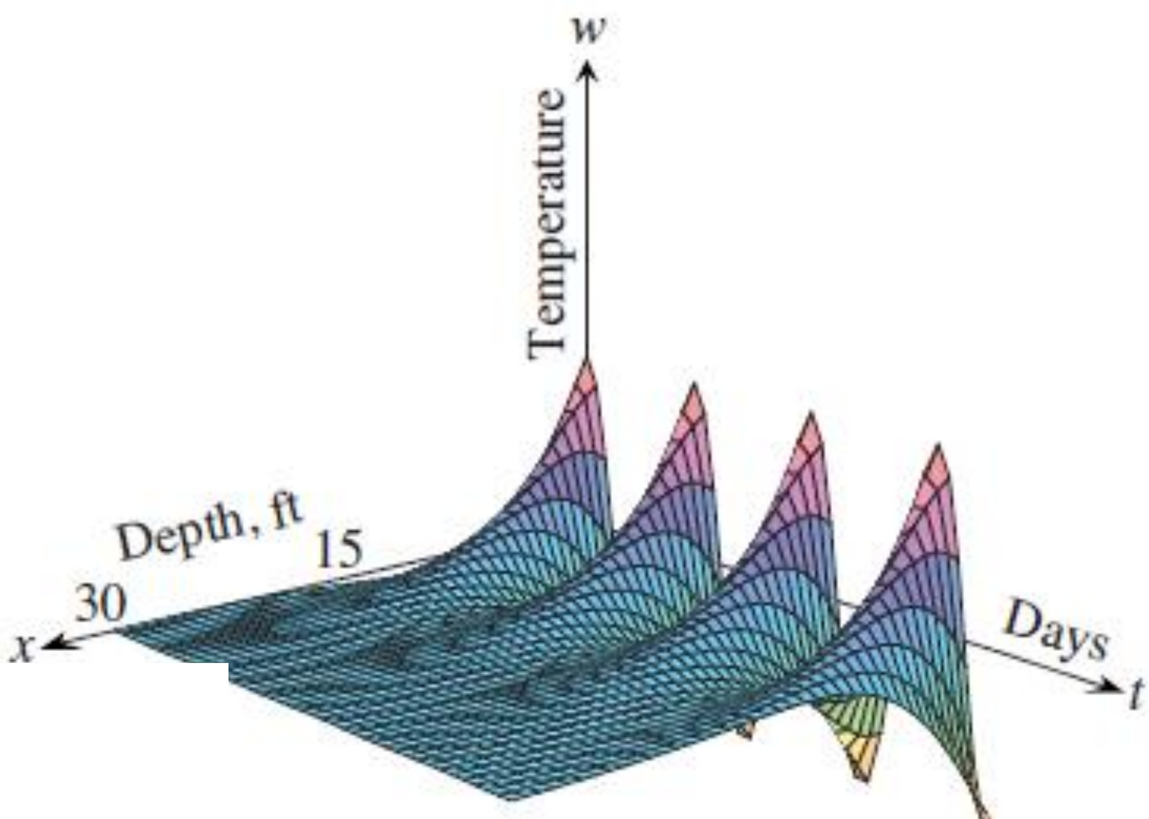
Describe the level surfaces of the function

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$$

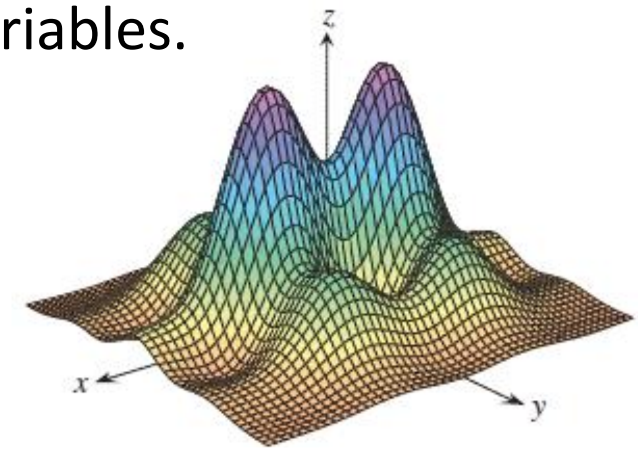


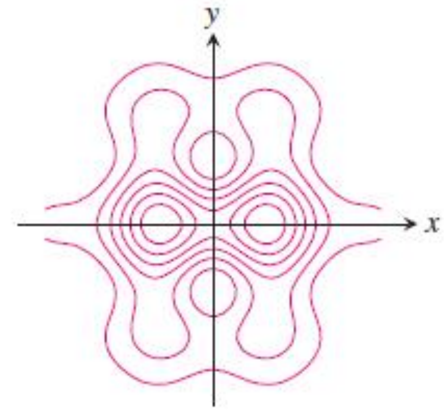
Modeling Temperature Beneath Earth's Surface

$$w = \cos(1.7 \times 10^{-2}t - 0.2x)e^{-0.2x}.$$

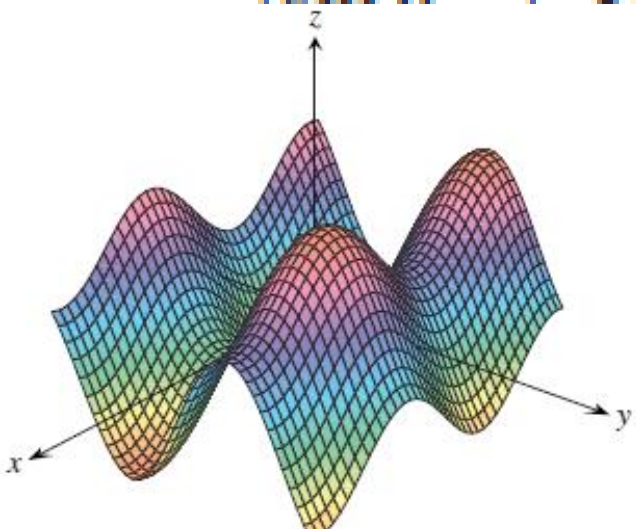


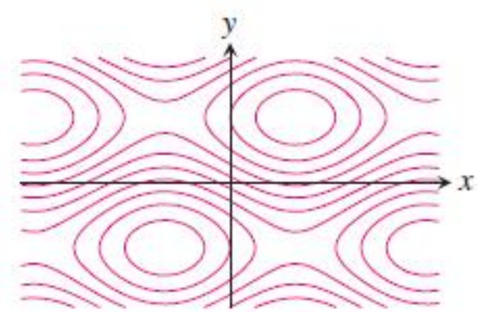
Computer-generated graphs and level surfaces of typical functions of two variables.



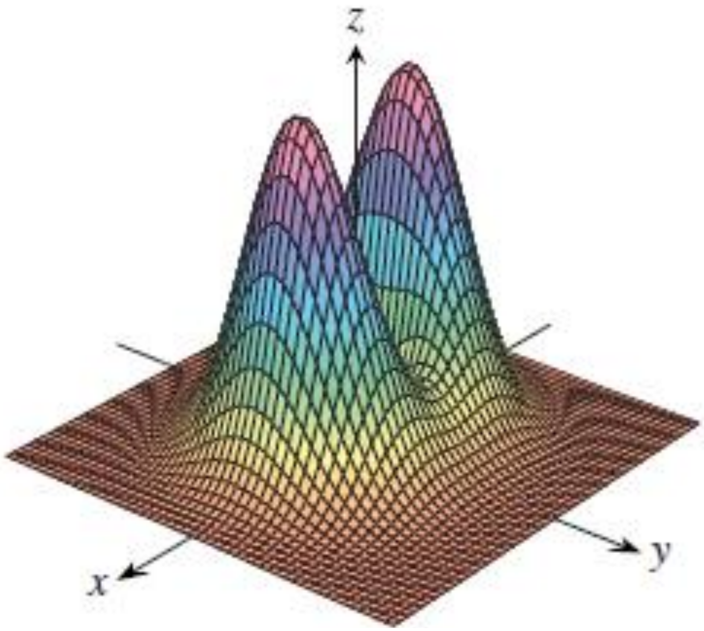


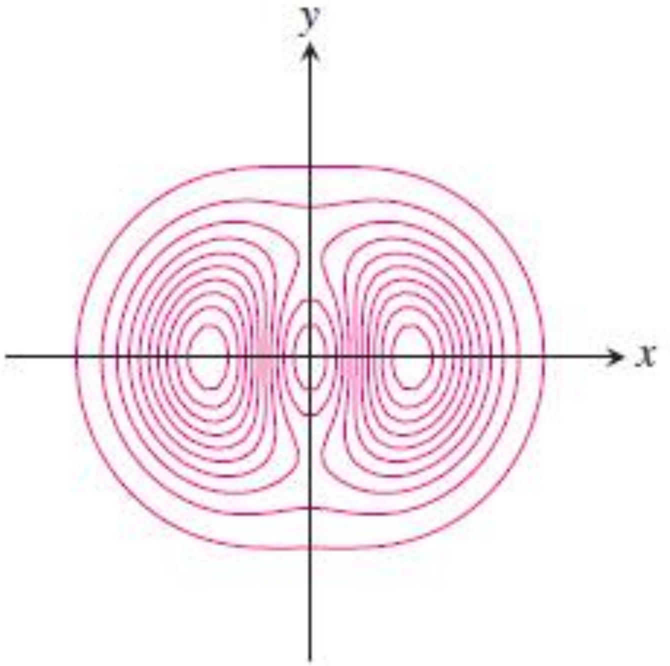
$$z = e^{-(x^2 + y^2)/8}(\sin x^2 + \cos y^2)$$



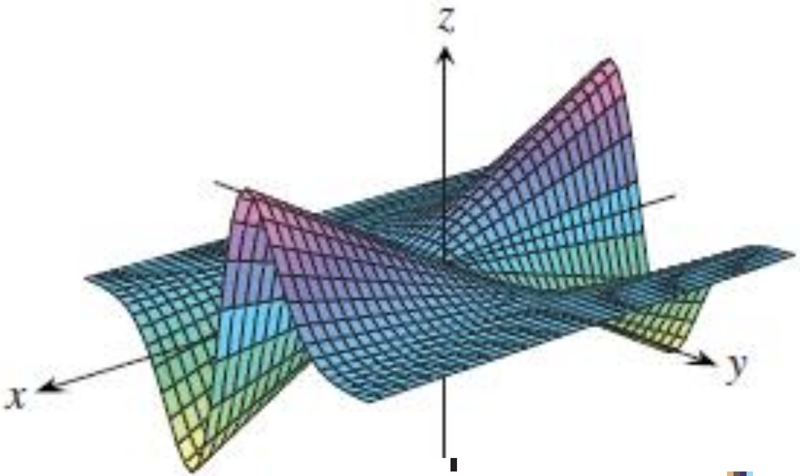


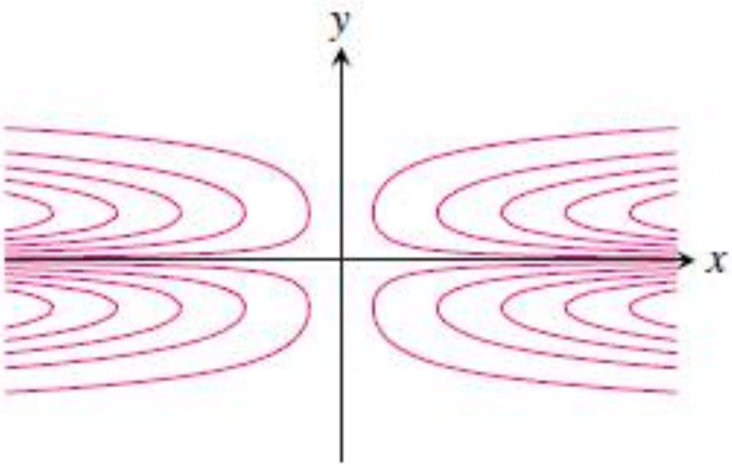
$$z = \sin x + 2 \sin y$$



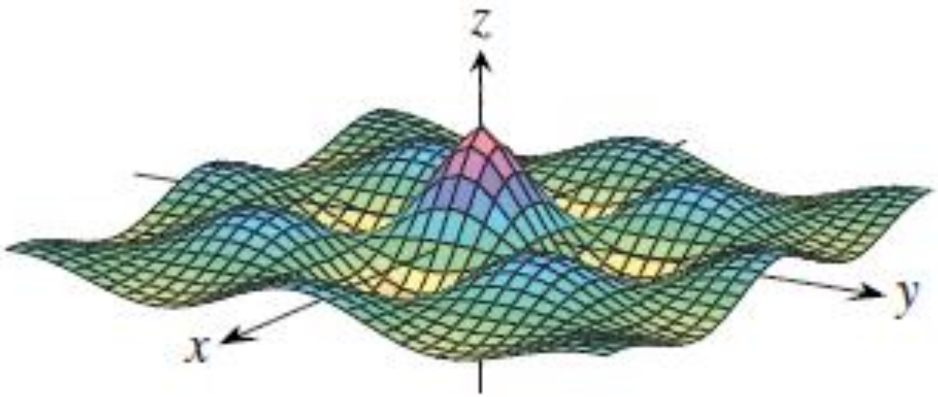


$$z = (4x^2 + y^2)e^{-x^2-y^2}$$

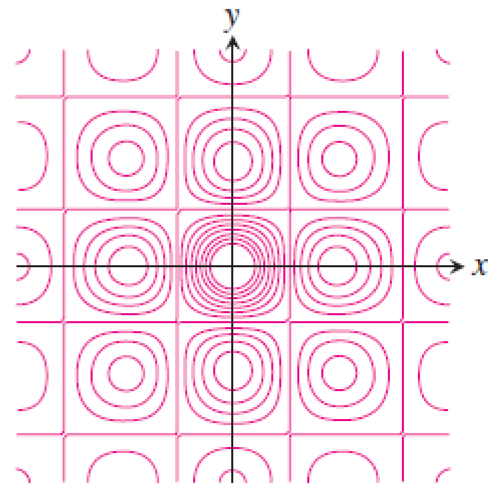




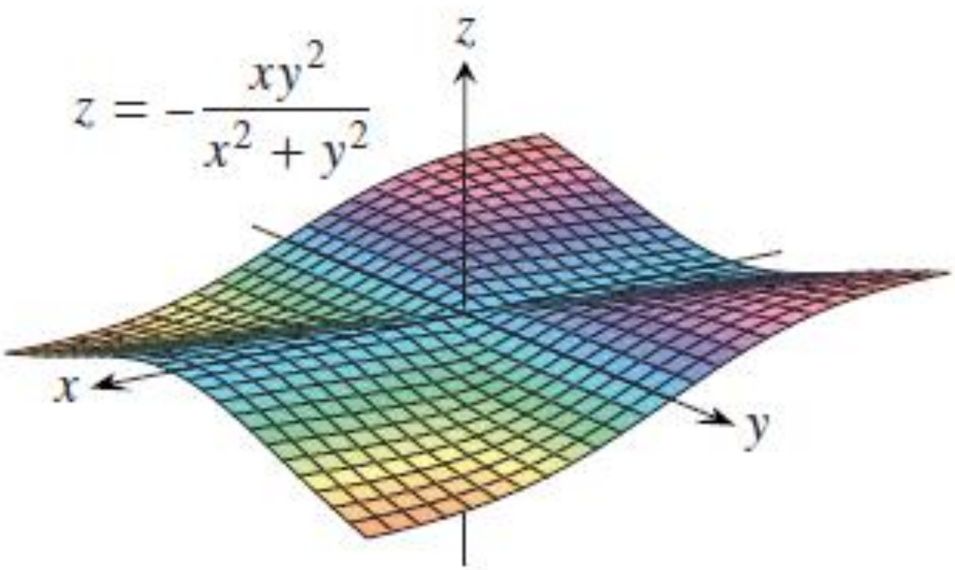
$$z = xye^{-y^2}$$

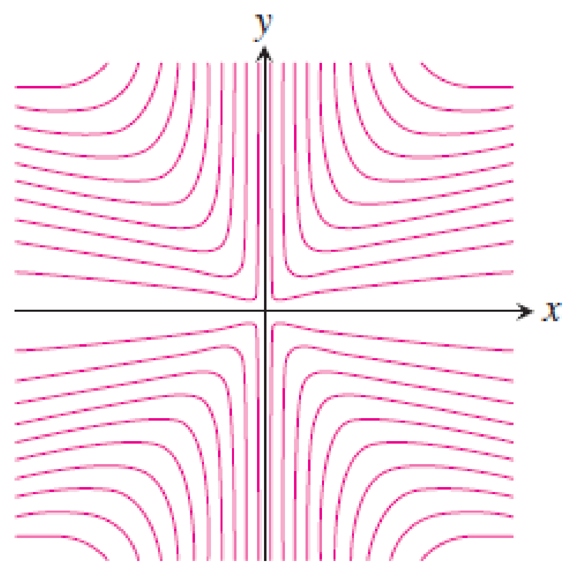


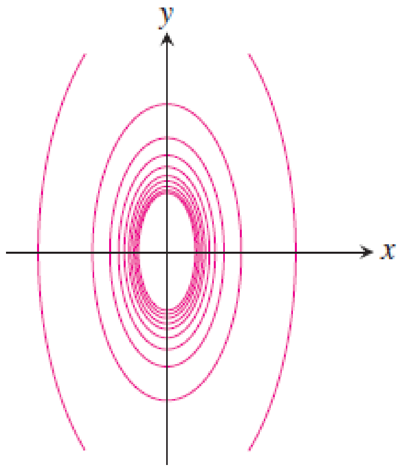
$z = (\cos x)(\cos y)\, e^{-\sqrt{x^2 + y^2}/4}$



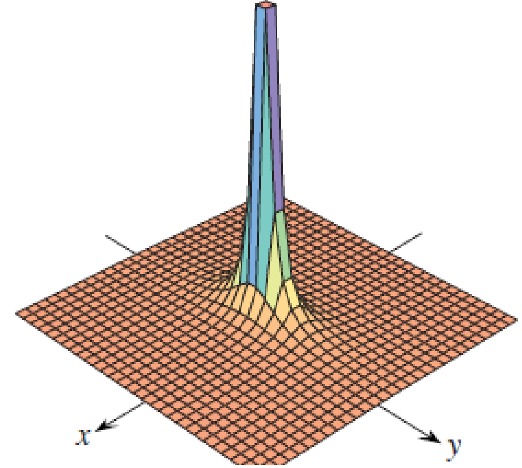
$z = -\dfrac{xy^2}{x^2 + y^2}$



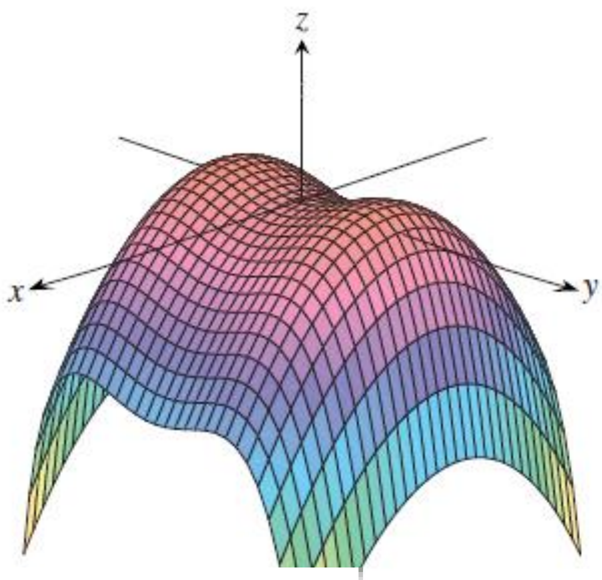




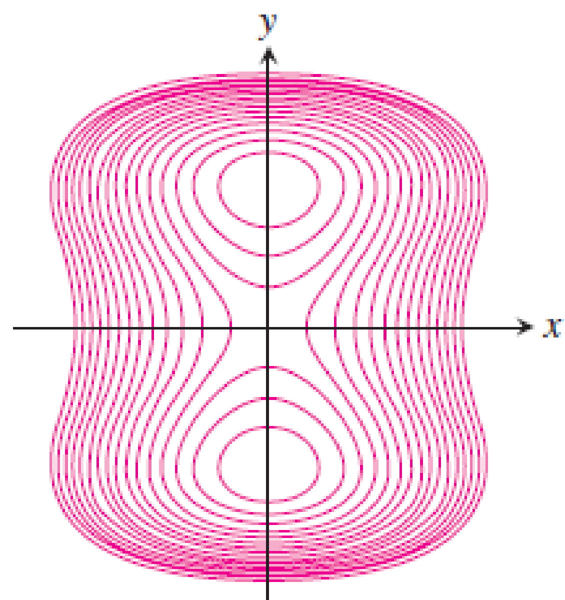
c.



$$z = \frac{1}{4x^2 + y^2}$$



$$z = y^2 - y^4 - x^2$$



14.2 Limits and Continuity in Higher Dimensions

THEOREM 1 Properties of Limits of Functions of Two Variables

The following rules hold if L, M, and k are real numbers and

$$\lim_{(x, y)\to(x_0, y_0)} f(x, y) = L \qquad \text{and} \qquad \lim_{(x, y)\to(x_0, y_0)} g(x, y) = M.$$

1. *Sum Rule:* $\displaystyle\lim_{(x, y)\to(x_0, y_0)} (f(x, y) + g(x, y)) = L + M$
2. *Difference Rule:* $\displaystyle\lim_{(x, y)\to(x_0, y_0)} (f(x, y) - g(x, y)) = L - M$
3. *Product Rule:* $\displaystyle\lim_{(x, y)\to(x_0, y_0)} (f(x, y) \cdot g(x, y)) = L \cdot M$
4. *Constant Multiple Rule:* $\displaystyle\lim_{(x, y)\to(x_0, y_0)} (kf(x, y)) = kL \qquad$ (any number k)
5. *Quotient Rule:* $\displaystyle\lim_{(x, y)\to(x_0, y_0)} \frac{f(x, y)}{g(x, y)} = \frac{L}{M} \qquad M \neq 0$
6. *Power Rule:* If r and s are integers with no common factors, and $s \neq 0$, then

$$\lim_{(x, y)\to(x_0, y_0)} (f(x, y))^{r/s} = L^{r/s}$$

provided $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

Continuity

As with functions of a single variable, continuity is defined in terms of limits.

DEFINITION Continuous Function of Two Variables

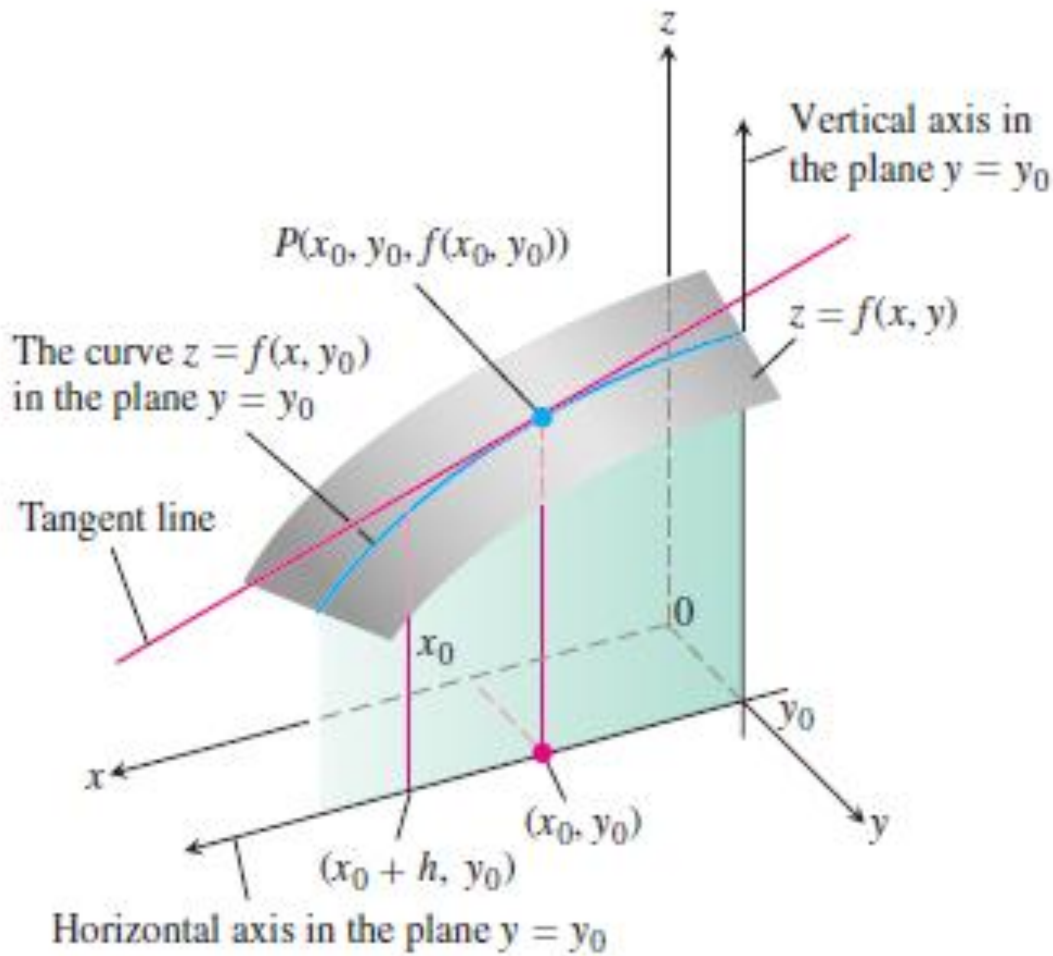
A function $f(x, y)$ is **continuous at the point** (x_0, y_0) if

1. f is defined at (x_0, y_0),
2. $\lim_{(x, y)\to(x_0, y_0)} f(x, y)$ exists,
3. $\lim_{(x, y)\to(x_0, y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

14.3 Partial Derivatives

- The calculus of several variables is basically single-variable calculus applied one at a time.
- Hold all but one of the independent variables constant and differentiate with respect to that one variable, we get a **"partial" derivative**.
- To distinguish partial derivatives from ordinary derivatives we use the symbol ∂ rather than the ***d*** *previously used*



DEFINITION Partial Derivative with Respect to x

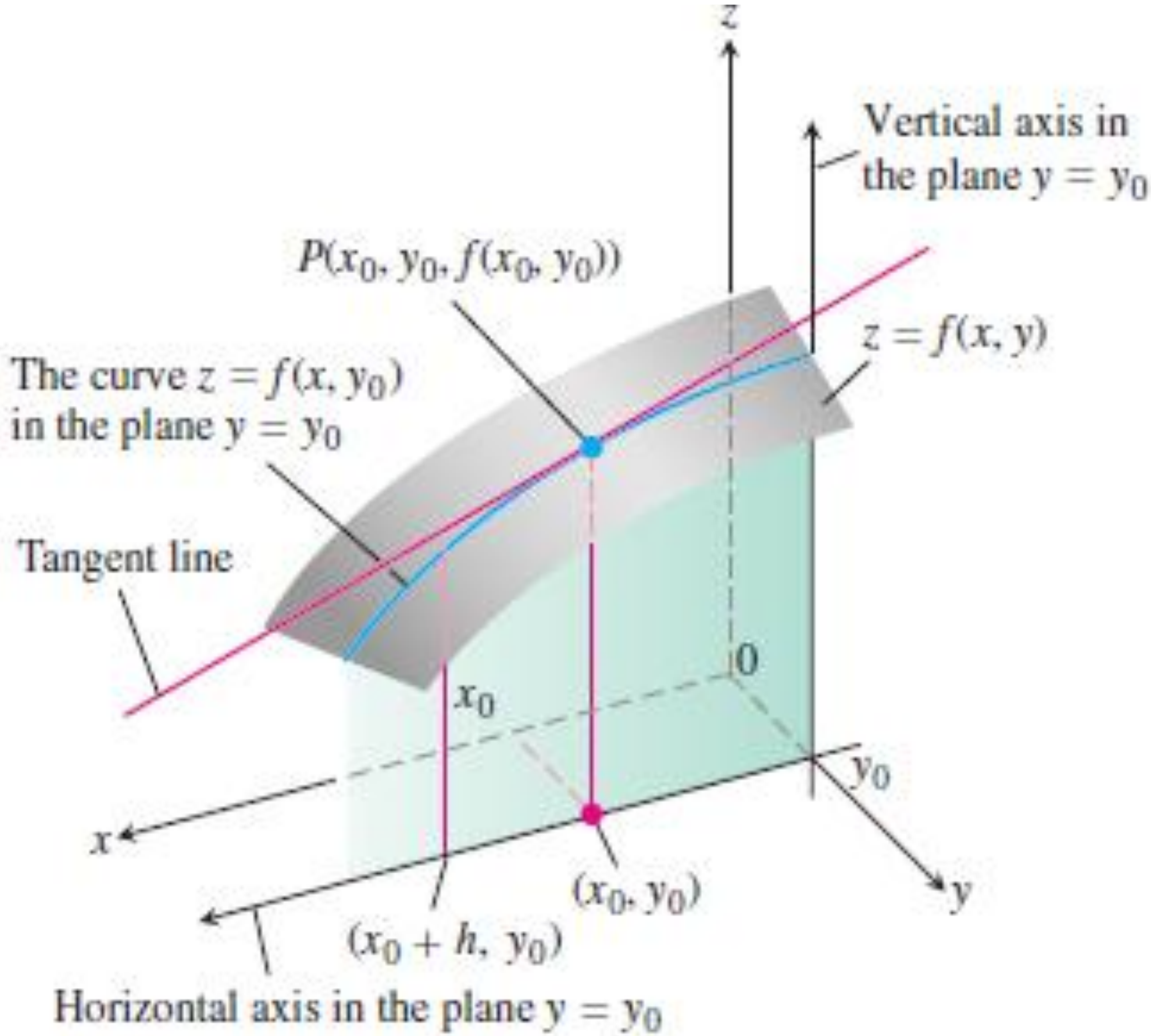
The **partial derivative of $f(x, y)$ with respect to x** at the point (x_0, y_0) is

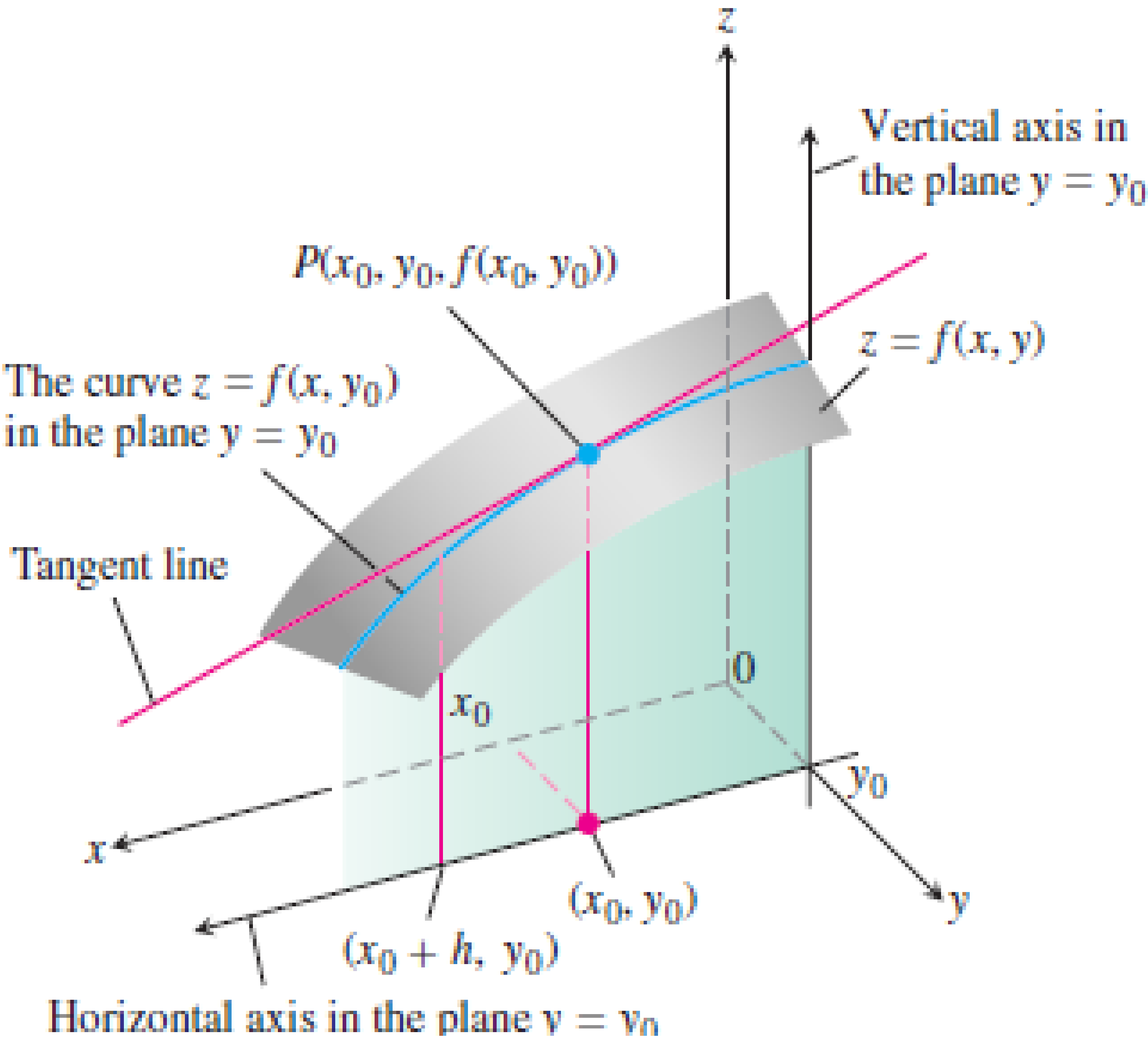
$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

provided the limit exists.

Partial Derivatives of a Function of Two Variables

If (x_0, y_0) is a point in the domain of a function $f(x, y)$, the vertical plane $y = y_0$ will cut the surface $z = f(x, y)$ in the curve $z = f(x, y_0)$

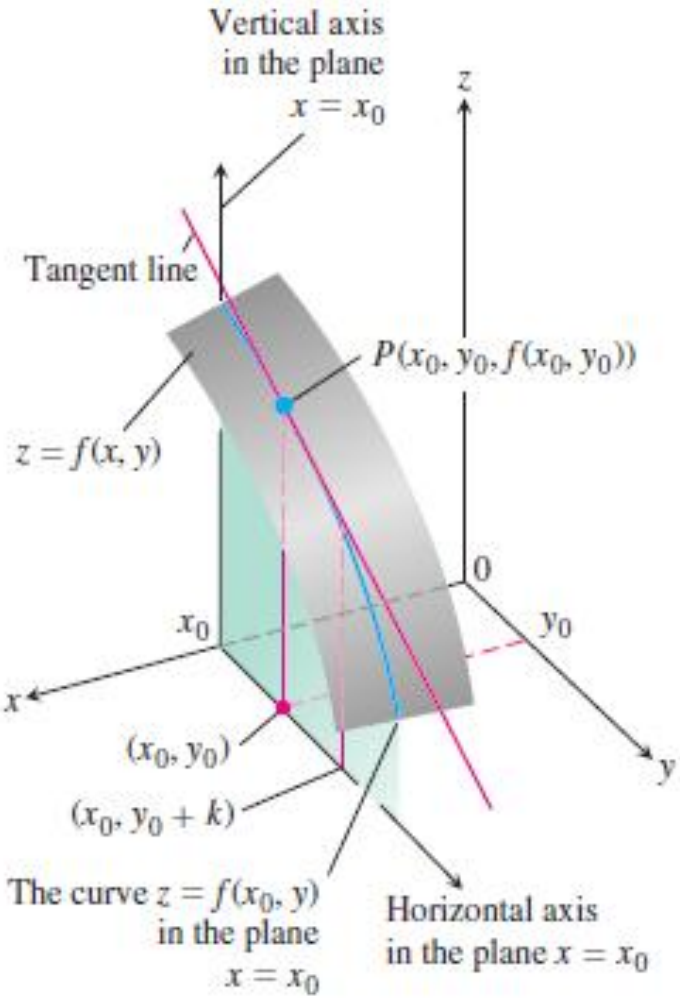



z
Vertical axis in
the plane $y = y_0$
$P(x_0, y_0, f(x_0, y_0))$
$z = f(x, y)$
The curve $z = f(x, y_0)$
in the plane $y = y_0$
Tangent line
0
x_0
y_0
x
y
(x_0, y_0)
$(x_0 + h,\ y_0)$
Horizontal axis in the plane $y = y_0$

DEFINITION Partial Derivative with Respect to y

The partial derivative of $f(x, y)$ with respect to y at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = \left.\frac{d}{dy} f(x_0, y)\right|_{y=y_0} = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

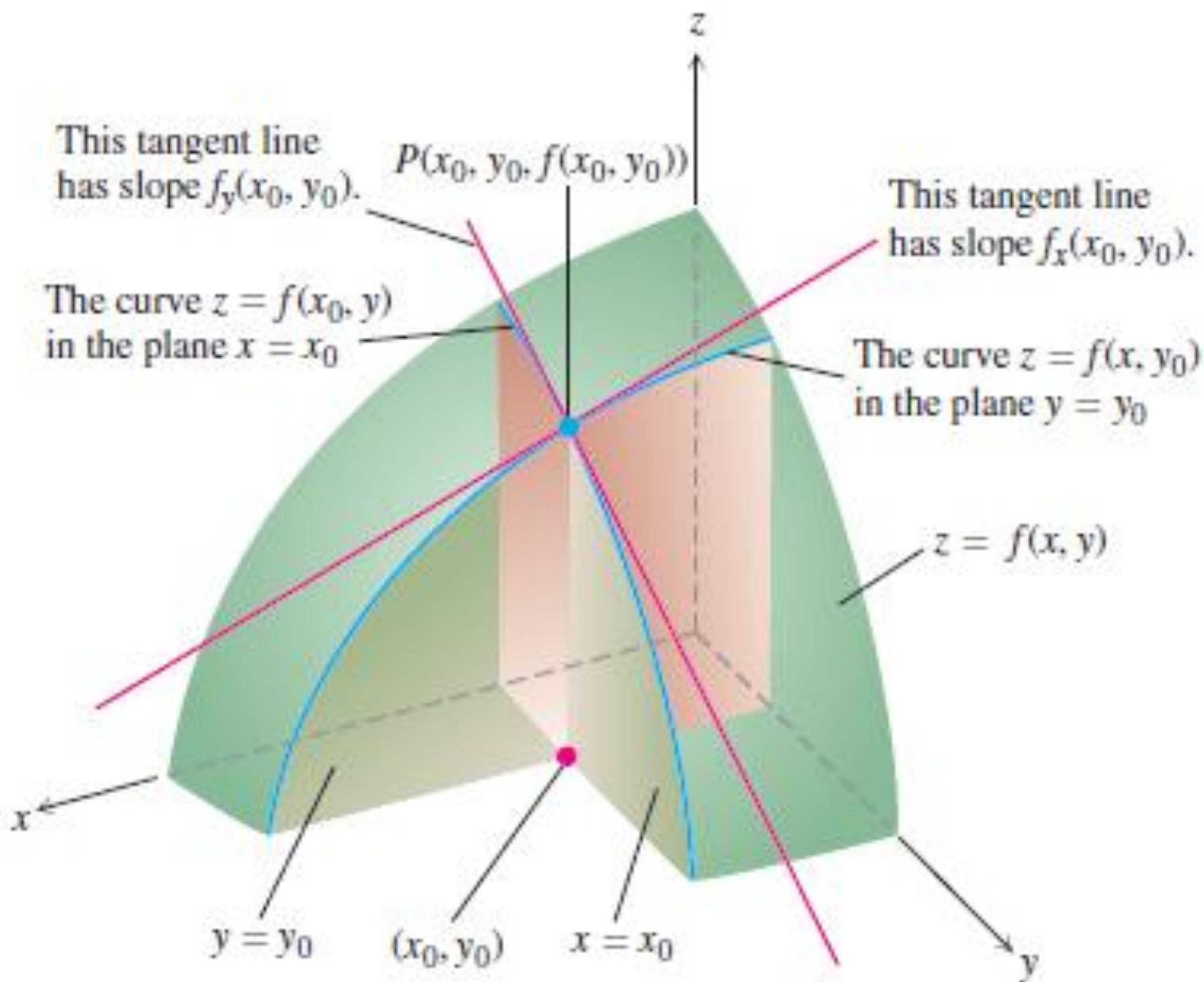


$$\frac{\partial f}{\partial y}(x_0, y_0), \qquad f_y(x_0, y_0),$$

$$\frac{\partial f}{\partial y}, \qquad f_y.$$

The notation for a partial derivative depends on what we want to emphasize:

$\dfrac{\partial f}{\partial x}(x_0, y_0)$ or $f_x(x_0, y_0)$	"Partial derivative of f with respect to x at (x_0, y_0)" or "f sub x at (x_0, y_0)." Convenient for stressing the point (x_0, y_0).
$\left.\dfrac{\partial z}{\partial x}\right\|_{(x_0, y_0)}$	"Partial derivative of z with respect to x at (x_0, y_0)." Common in science and engineering when you are dealing with variables and do not mention the function explicitly.
$f_x, \dfrac{\partial f}{\partial x}, z_x$, or $\dfrac{\partial z}{\partial x}$	"Partial derivative of f (or z) with respect to x." Convenient when you regard the partial derivative as a function in its own right.



The tangent lines at the point $(x_0, y_0, f(x_0, y_0))$ determine a plane that, in this picture at least, appears to be tangent to the surface.

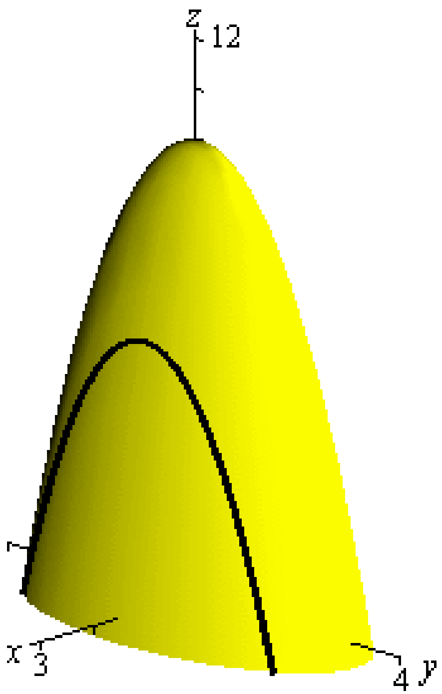
Let $f(x, y) = y^3 x^2$.

calculate $\dfrac{\partial f}{\partial x}(1,2)$ calculate $\dfrac{\partial f}{\partial y}(1,2)$

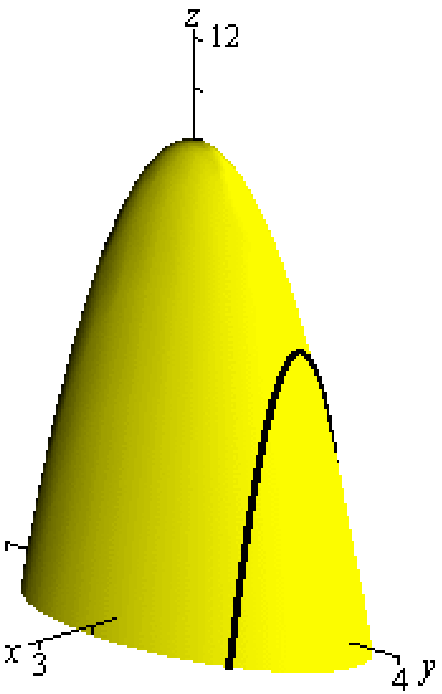
Find all of the first order partial derivatives for the following function

$$g(x, y, z) = \frac{x \sin(y)}{z^2}$$

Find the slopes of the traces to $z = 10 - 4x^2 - y^2$ at the point $(1, 2)$.



Trace for $x = 1$



Trace for $y = 2$

Next we'll need the two partial derivatives so we can get the slopes.

$$f_x(x, y) = -8x \qquad f_y(x, y) = -2y$$

To get the slopes all we need to do is evaluate the partial derivatives at the point in question

$$f_x(1, 2) = -8 \qquad f_y(1, 2) = -4$$

Find $\partial z/\partial x$ if the equation

$$yz - \ln z = x + y$$

defines z as a function of the two independent variables x and y and the partial derivative exists.

$$\frac{\partial}{\partial x}(yz) - \frac{\partial}{\partial x}\ln z = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x}$$

$$y\frac{\partial z}{\partial x} - \frac{1}{z}\frac{\partial z}{\partial x} = 1 + 0$$

With y constant, $\frac{\partial}{\partial x}(yz) = y\frac{\partial z}{\partial x}$.

$$\left(y - \frac{1}{z}\right)\frac{\partial z}{\partial x} = 1$$

$$\frac{\partial z}{\partial x} = \frac{z}{yz - 1}.$$

Find $\dfrac{dy}{dx}$ for $3y^4 + x^7 = 5x$. implicitly

Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for each of the following functions

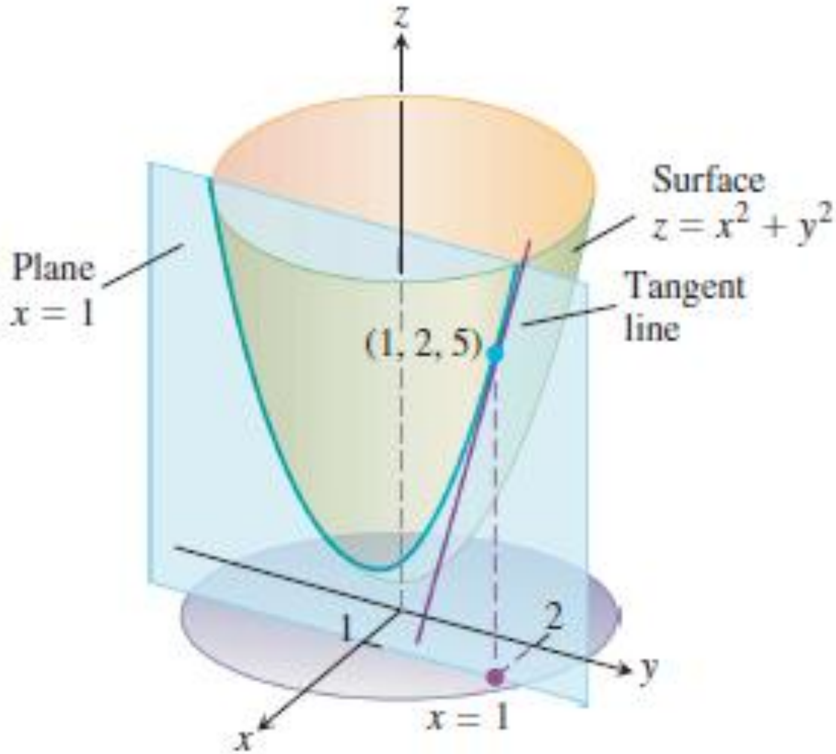
- $x^3 z^2 - 5xy^5 z = x^2 + y^3$
- $x^2 \sin(2y - 5z) = 1 + y\cos(6zx)$

EXAMPLE 5 Finding the Slope of a Surface in the y-Direction

The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at (1, 2, 5)

Solution The slope is the value of the partial derivative $\partial z/\partial y$ at (1, 2):

$$\left.\frac{\partial z}{\partial y}\right|_{(1,2)} = \left.\frac{\partial}{\partial y}(x^2 + y^2)\right|_{(1,2)} = 2y\Big|_{(1,2)} = 2(2) = 4.$$



$$\left.\frac{dz}{dy}\right|_{y=2} = \left.\frac{d}{dy}(1 + y^2)\right|_{y=2} = 2y\Big|_{y=2} = 4.$$

Finding Second-Order Partial Derivatives

If $f(x, y) = x \cos y + ye^x$, find

$$\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}.$$

Solution

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x \cos y + ye^x)$$

$$= \cos y + ye^x$$

So

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = ye^x.$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x \cos y + ye^x)$$

$$= -x \sin y + e^x$$

So

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -\sin y + e^x$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x \cos y.$$

The Mixed Derivative Theorem

You may have noticed that the "mixed" second-order partial derivatives

$$\frac{\partial^2 f}{\partial y \partial x} \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}$$

in Example 9 were equal. This was not a coincidence. They must be equal whenever f, f_x, f_y, f_{xy}, and f_{yx} are continuous, as stated in the following theorem.

THEOREM 2 The Mixed Derivative Theorem

If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Choosing the Order of Differentiation

Find $\partial^2 w/\partial x \partial y$ if

$$w = xy + \frac{e^y}{y^2 + 1}.$$

Solution The symbol $\partial^2 w/\partial x \partial y$ tells us to differentiate first with respect to y and then with respect to x. If we postpone the differentiation with respect to y and differentiate first with respect to x, however, we get the answer more quickly. In two steps,

$$\frac{\partial w}{\partial x} = y \qquad \text{and} \qquad \frac{\partial^2 w}{\partial y \partial x} = 1.$$

If we differentiate first with respect to y, we obtain $\partial^2 w/\partial x \partial y = 1$ as well. ■

14.4 The Chain Rule

Functions of Two Variables

The Chain Rule formula for a function $w = f(x, y)$ when $x = x(t)$ and $y = y(t)$ are both differentiable functions of t is given in the following theorem.

THEOREM 5 Chain Rule for Functions of Two Independent Variables

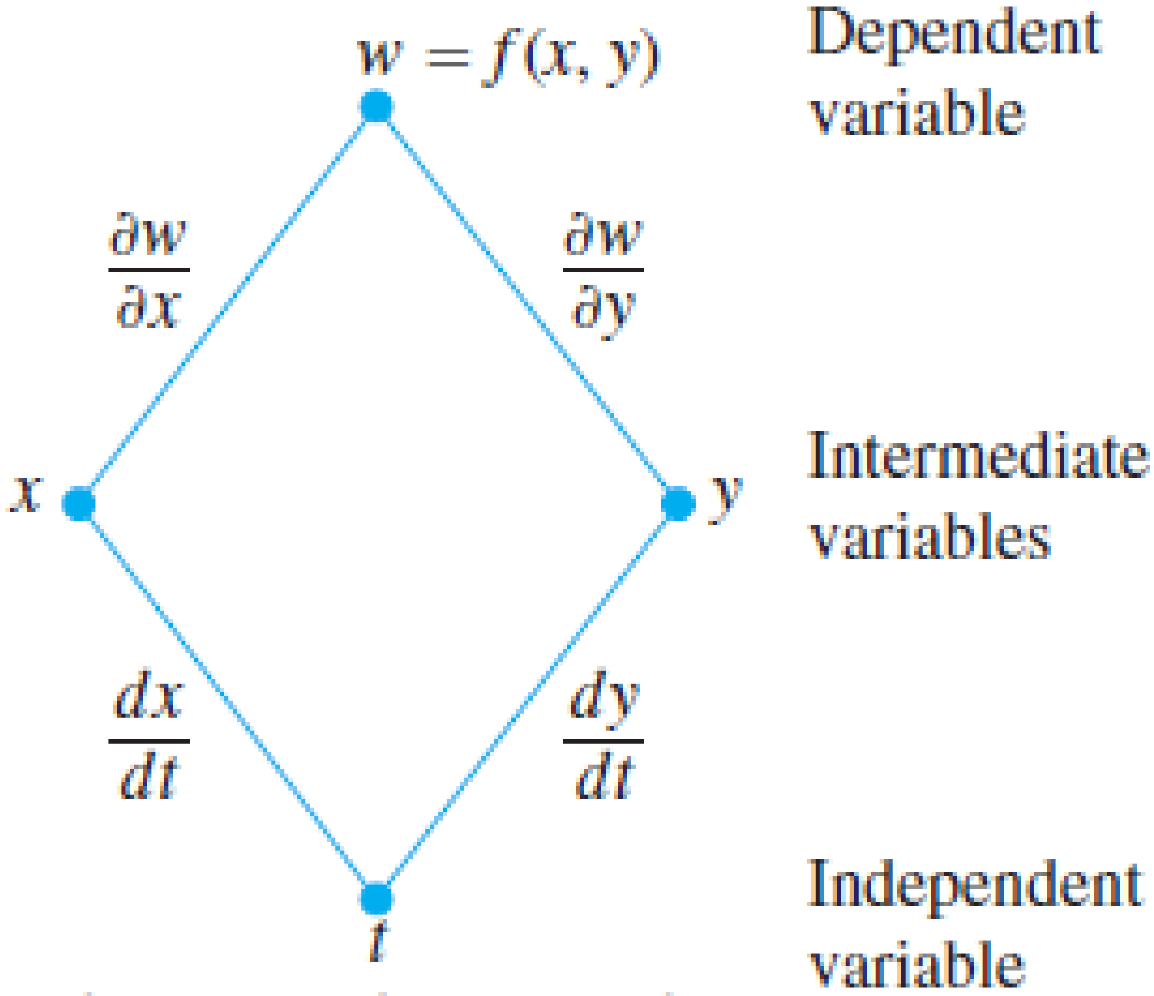
If $w = f(x, y)$ has continuous partial derivatives f_x and f_y and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then the composite $w = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{df}{dt} = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t),$$

or

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Chain Rule

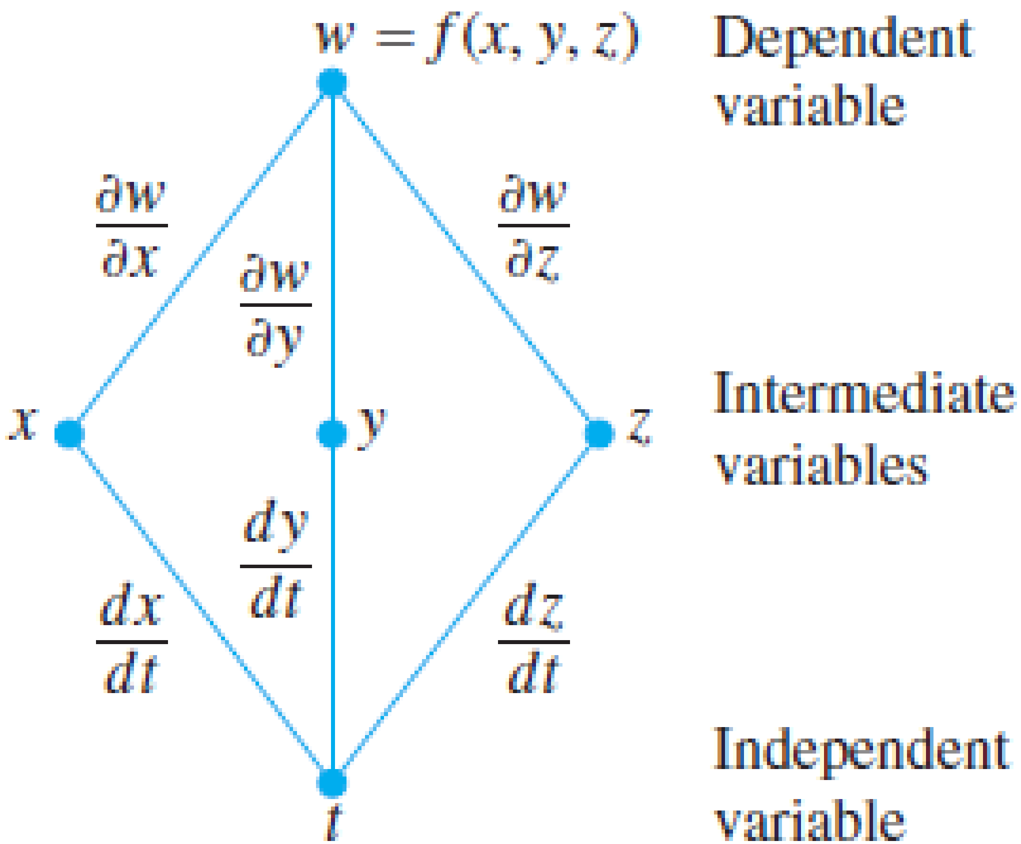


$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

THEOREM 6 Chain Rule for Functions of Three Independent Variables

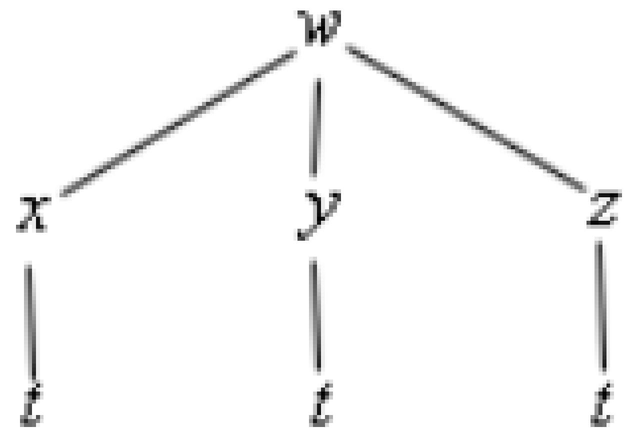
If $w = f(x, y, z)$ is differentiable and x, y, and z are differentiable functions of t, then w is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$



Use a tree diagram to write down the chain rule for the given derivatives.

$\frac{dw}{dt}$ for $w = f(x,y,z)$, $x = g_1(t)$, $y = g_2(t)$, and $z = g_3(t)$

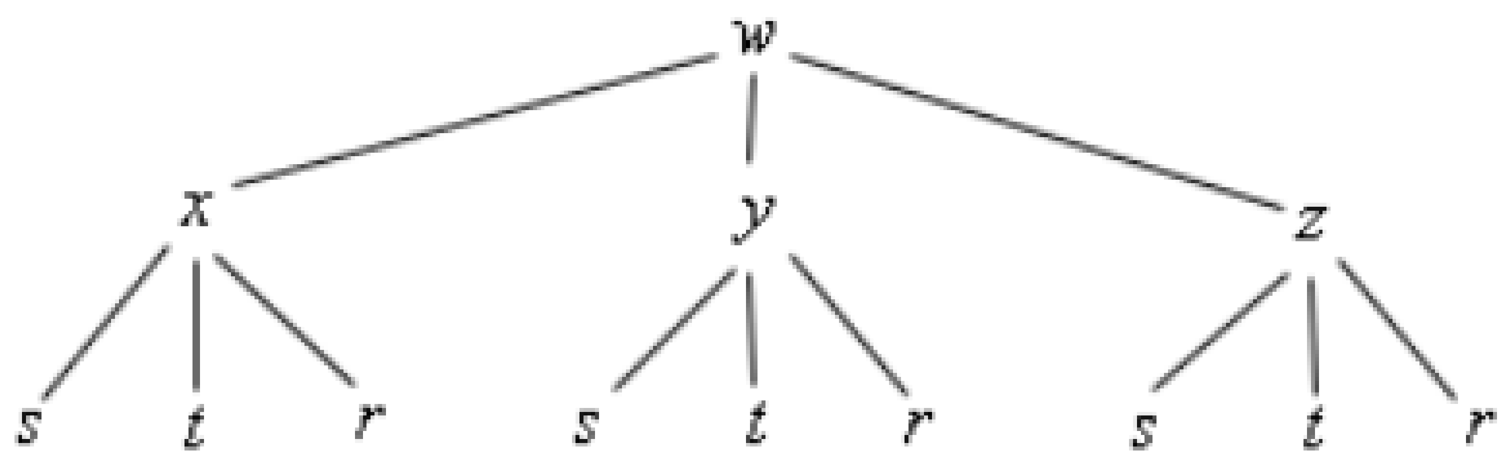


chain rule for this case should be,

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

Use a tree diagram to write down the chain rule for the given derivatives.

$\frac{\partial w}{\partial r}$ for $w = f(x, y, z)$, $x = g_1(s, t, r)$, $y = g_2(s, t, r)$, and $z = g_3(s, t, r)$



derivative will be,

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}$$

Express $\partial w/\partial r$ and $\partial w/\partial s$ in terms of r and s if

$$w = x^2 + y^2, \qquad x = r - s, \qquad y = r + s$$

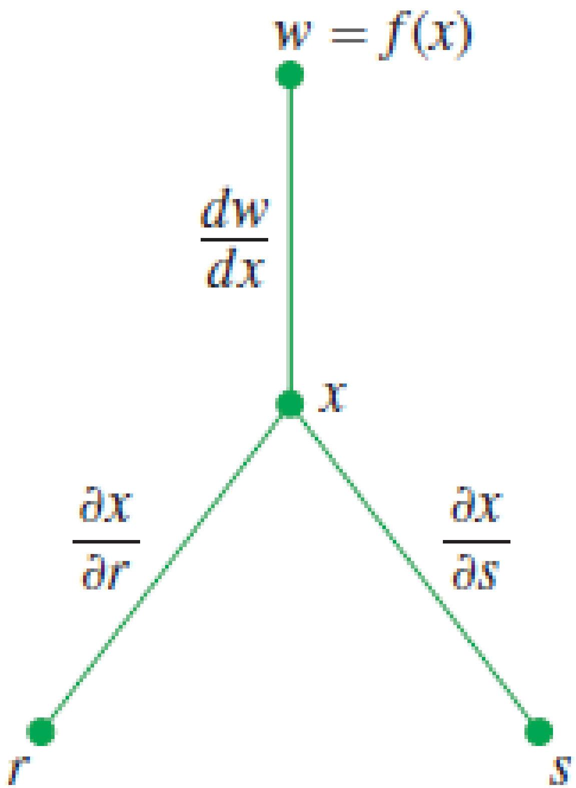
$$\begin{aligned} \frac{\partial w}{\partial r} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} \\ &= (2x)(1) + (2y)(1) \\ &= 2(r - s) + 2(r + s) \\ &= 4r \end{aligned}$$

$$\begin{aligned} \frac{\partial w}{\partial s} &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} \\ &= (2x)(-1) + (2y)(1) \\ &= -2(r - s) + 2(r + s) \\ &= 4s \end{aligned}$$

If $w = f(x)$ and $x = g(r, s)$, then

$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r} \quad \text{and} \quad \frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}$$

Chain Rule



$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}$$

THEOREM 8 A Formula for Implicit Differentiation
Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y) = 0$ defines y
as a differentiable function of x. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Find $\dfrac{dy}{dx}$ for $x\cos(3y) + x^3y^5 = 3x - \mathbf{e}^{xy}$.

$$x\cos(3y) + x^3y^5 - 3x + \mathbf{e}^{xy} = 0$$

$F(x, y)$ in our formula so all we need to do is use the formula

$$\frac{dy}{dx} = -\frac{\cos(3y) + 3x^2y^5 - 3 + y\mathbf{e}^{xy}}{-3x\sin(3y) + 5x^3y^4 + x\mathbf{e}^{xy}}$$

14.4 Chain Rule

The Chain Rule for functions of two or more variables

- Chain Rule has several forms.
- The form depends on how many variables are involved
- works like the Chain Rule in Section 3.5

THEOREM 5 Chain Rule for Functions of Two Independent Variables

If $w = f(x, y)$ has continuous partial derivatives f_x and f_y and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then the composite $w = f(x(t), y(t))$ is a differentiable function of t and

$$\frac{df}{dt} = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t),$$

or

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Example 1 $w = x^2 + y^2, \quad x = \cos t, \quad y = \sin t; \quad t = \pi$

(a) $\frac{\partial w}{\partial x} = 2x$, $\frac{\partial w}{\partial y} = 2y$, $\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = \cos t \Rightarrow \frac{dw}{dt} = -2x \sin t + 2y \cos t = -2 \cos t \sin t + 2 \sin t \cos t$
$= 0$; $w = x^2 + y^2 = \cos^2 t + \sin^2 t = 1 \Rightarrow \frac{dw}{dt} = 0$

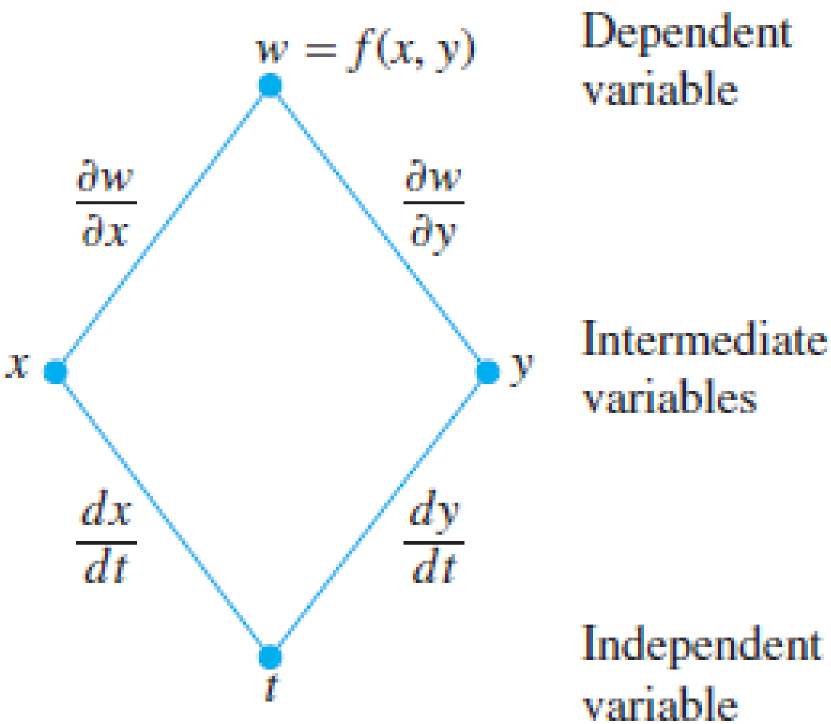
(b) $\frac{dw}{dt}(\pi) = 0$

THEOREM 6 Chain Rule for Functions of Three Independent Variables

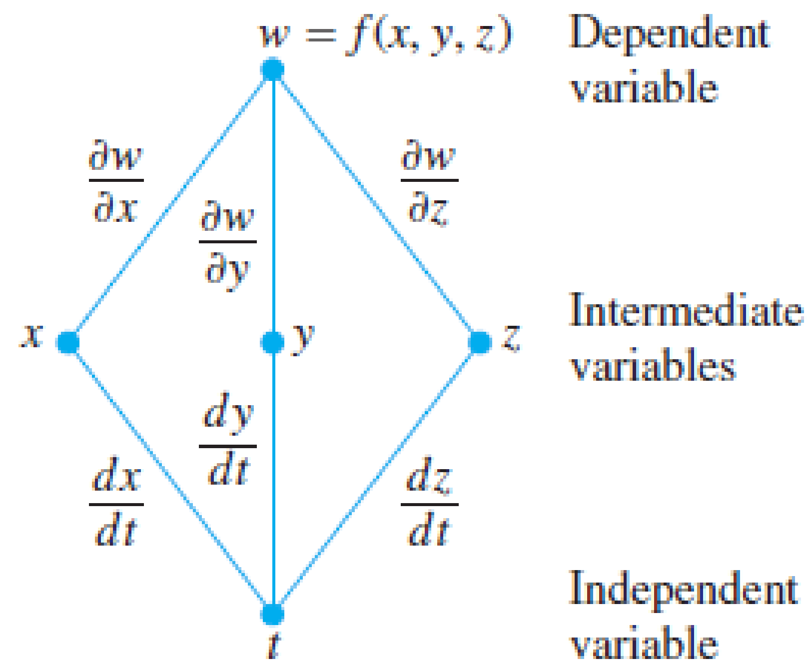
If $w = f(x, y, z)$ is differentiable and x, y, and z are differentiable functions of t, then w is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

Chain Rule diagrams



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$



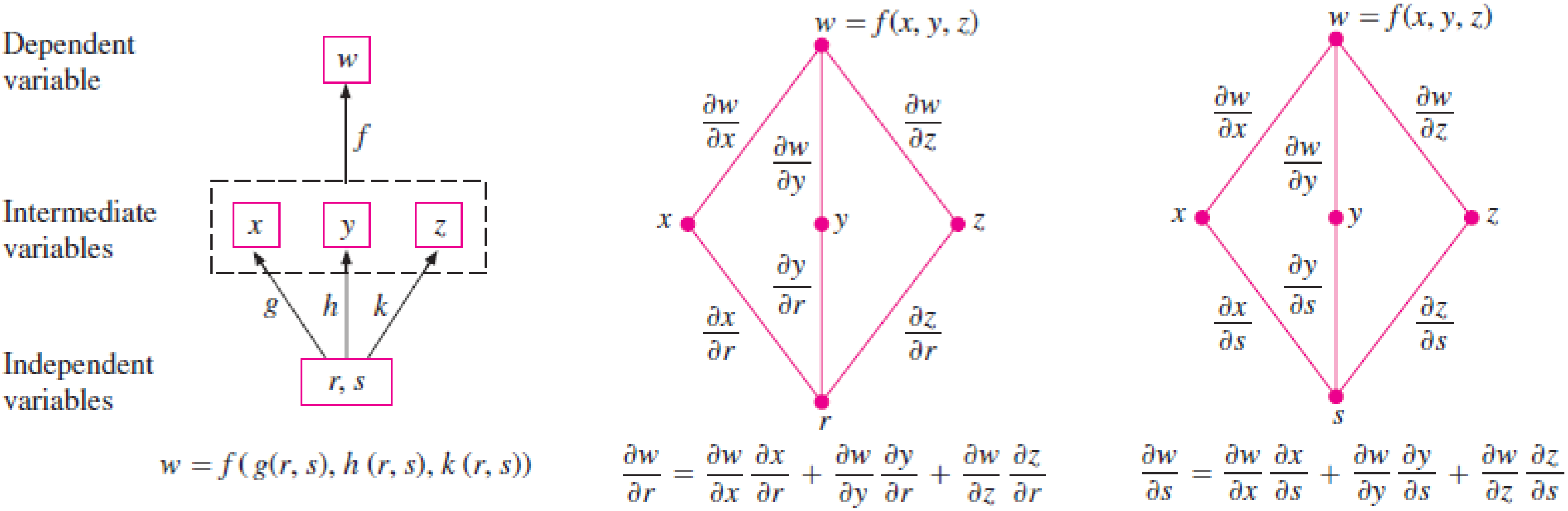
$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

THEOREM 7 Chain Rule for Two Independent Variables and Three Intermediate Variables

Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$. If all four functions are differentiable, then w has partial derivatives with respect to r and s, given by the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial v}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$



(a) (b) (c)

Example 2

$w = xy + yz + xz, \quad x = u + v, \quad y = u - v, \quad z = uv;$
$(u, v) = (1/2, 1)$

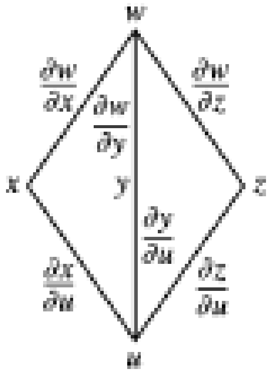
Example 3

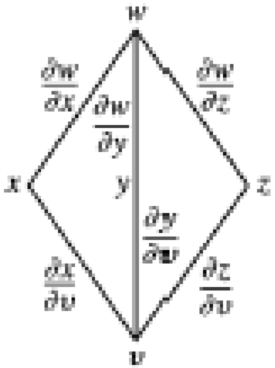
Draw a tree diagram and write a Chain Rule formula

$\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$ for $w = h(x, y, z), \quad x = f(u, v), \quad y = g(u, v),$
$z = k(u, v)$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial v}$$





THEOREM 8 A Formula for Implicit Differentiation

Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y) = 0$ defines y as a differentiable function of x. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

Example 4 $x^3 - 2y^2 + xy = 0, \quad (1, 1)$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

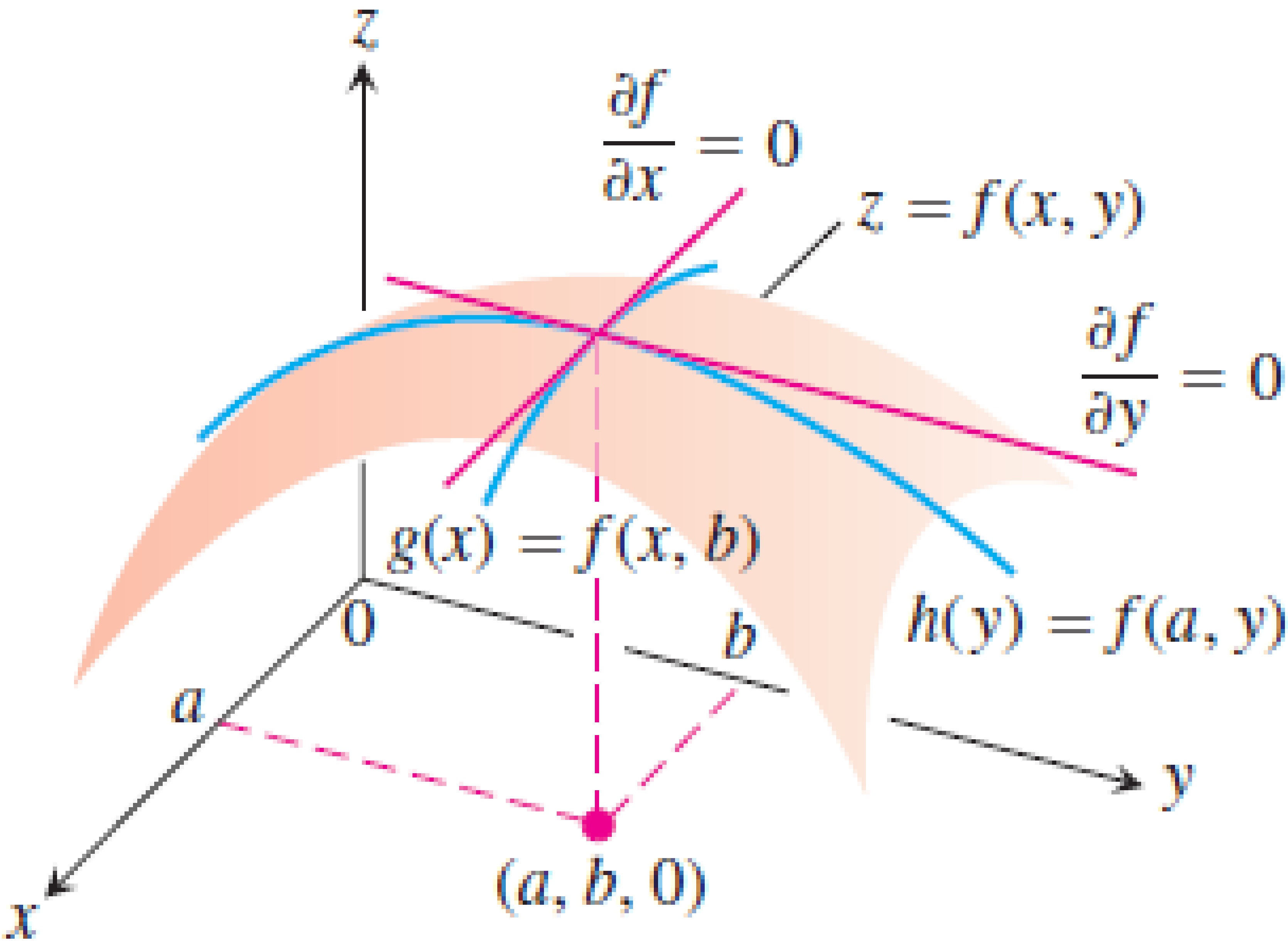
Use these equations to find the values of $\partial z/\partial x$ and $\partial z/\partial y$ at the points in .

Example 5 $z^3 - xy + yz + y^3 - 2 = 0, \quad (1, 1, 1)$

Example 6 $xe^y + ye^z + 2 \ln x - 2 - 3 \ln 2 = 0, \quad (1, \ln 2, \ln 3)$

14.7 Extreme Values and Saddle Points

- Continuous functions of two variables assume extreme values on closed, bounded domains
- we can narrow the search for extreme values by examining the first partial derivatives.
- extreme values only at domain boundary points or at interior domain points
- **<u>where both first partial derivatives are zero or</u>**
- **<u>where one or both of the first partial derivatives fails to exist.</u>**

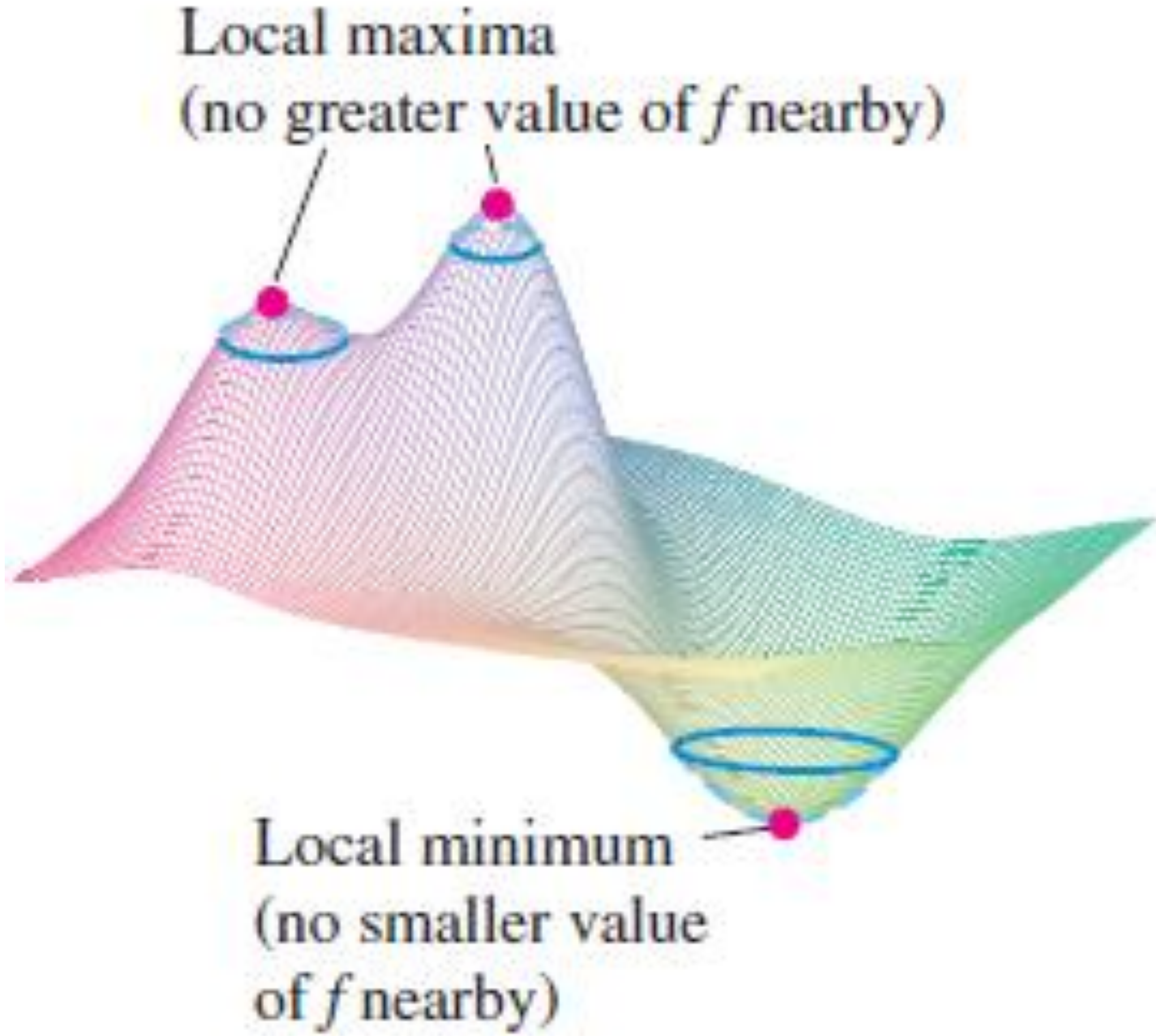

z
$\frac{\partial f}{\partial x} = 0$
$z = f(x, y)$
$\frac{\partial f}{\partial y} = 0$
$g(x) = f(x, b)$
$h(y) = f(a, y)$
0
b
a
y
$(a, b, 0)$
x

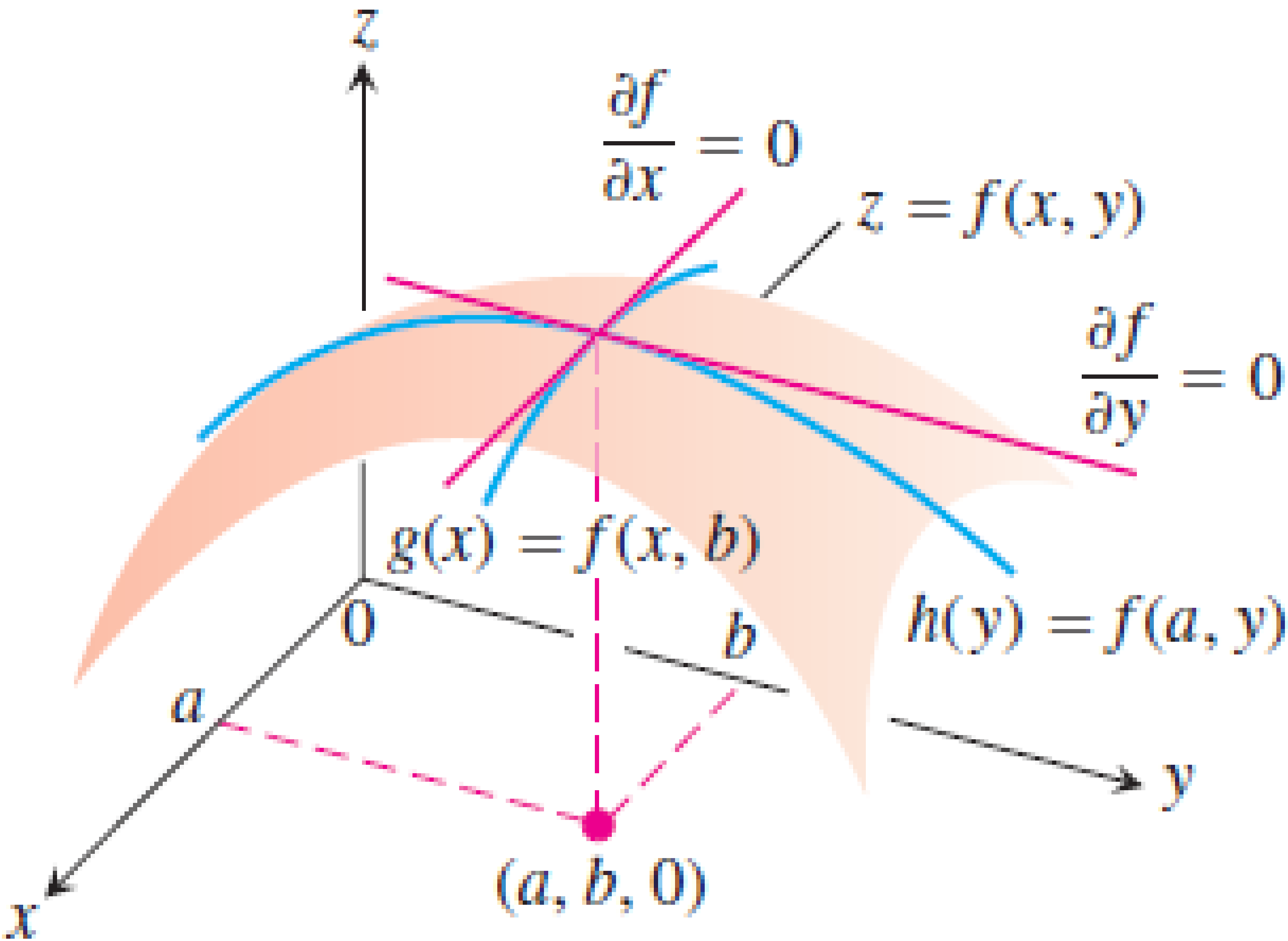
Derivative Tests for Local Extreme Values

DEFINITIONS Local Maximum, Local Minimum

Let $f(x, y)$ be defined on a region R containing the point (a, b). Then

1. $f(a, b)$ is a **local maximum** value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).
2. $f(a, b)$ is a **local minimum** value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).


Local maxima
(no greater value of f nearby)
Local minimum
(no smaller value
of f nearby)


z
$\frac{\partial f}{\partial x} = 0$
$z = f(x, y)$
$\frac{\partial f}{\partial y} = 0$
$g(x) = f(x, b)$
$h(y) = f(a, y)$
0
b
a
y
$(a, b, 0)$
x

THEOREM 10 First Derivative Test for Local Extreme Values

If $f(x, y)$ has a local maximum or minimum value at an interior point (a, b) of its domain and if the first partial derivatives exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

DEFINITION Critical Point

An interior point of the domain of a function $f(x, y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is a **critical point** of f.

DEFINITION Saddle Point

A differentiable function $f(x, y)$ has a **saddle point** at a critical point (a, b) if in every open disk centered at (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and domain points (x, y) where $f(x, y) < f(a, b)$. The corresponding point $(a, b, f(a, b))$ on the surface $z = f(x, y)$ is called a saddle point of

THEOREM 11 Second Derivative Test for Local Extreme Values

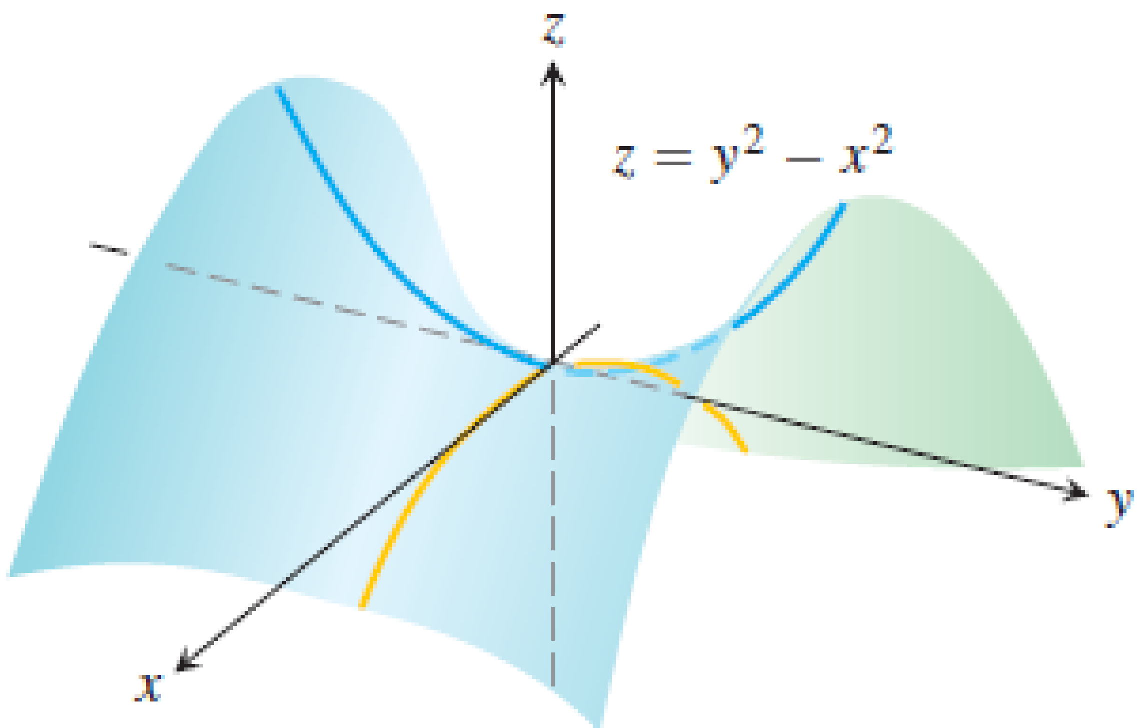
Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then

i. f has a **local maximum** at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).

ii. f has a **local minimum** at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).

iii. f has a **saddle point** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b).

iv. **The test is inconclusive** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b). In this case, we must find some other way to determine the behavior of f at (a, b).



Summary of Max-Min Tests

The extreme values of $f(x, y)$ can occur only at

i. **boundary points** of the domain of f

ii. **critical points** (interior points where $f_x = f_y = 0$ or points where f_x or f_y fail to exist).

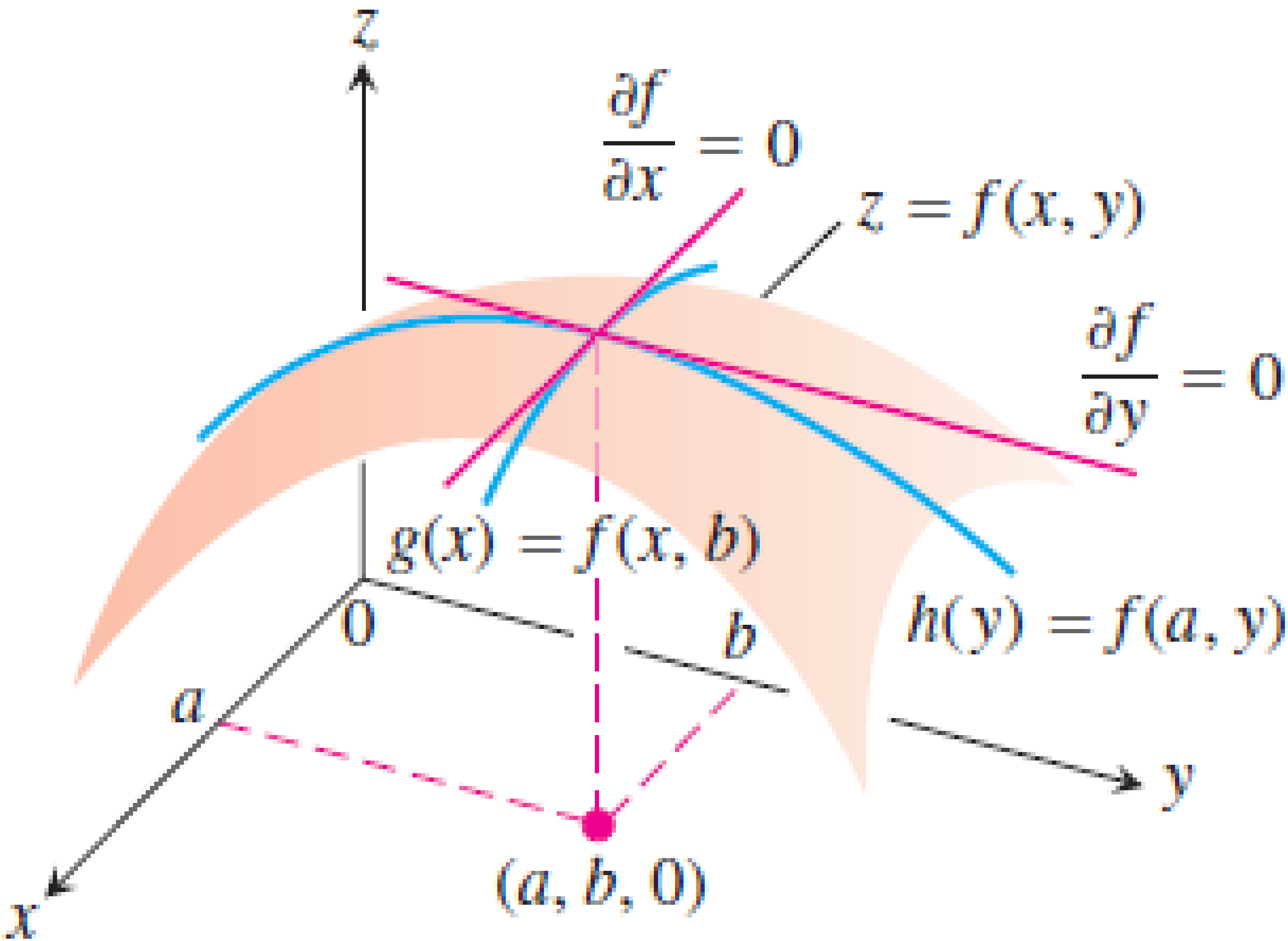
If the first- and second-order partial derivatives of f are continuous throughout a disk centered at a point (a, b) and $f_x(a, b) = f_y(a, b) = 0$, the nature of $f(a, b)$ can be tested with the **Second Derivative Test**:

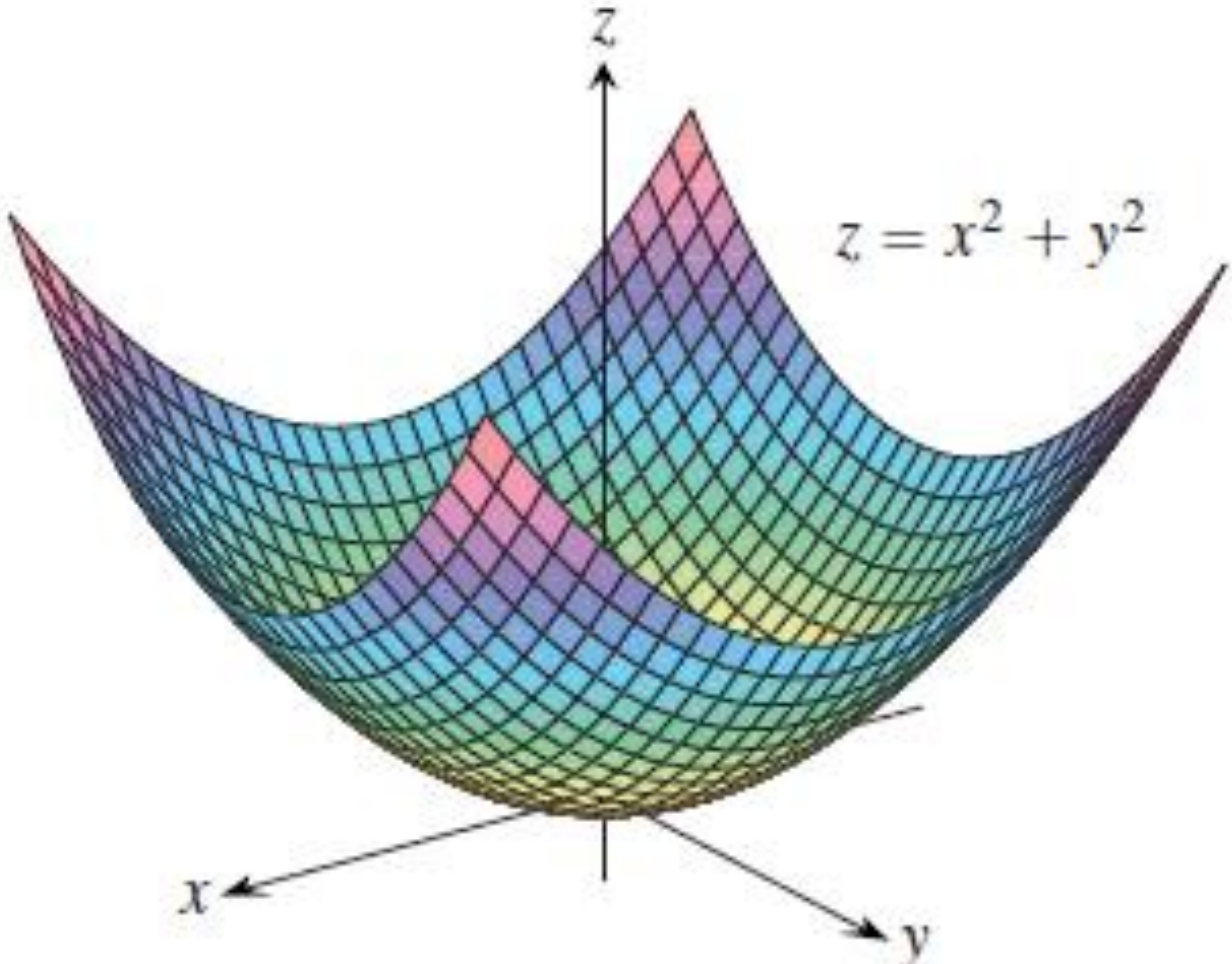
i. $f_{xx} < 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a, b) $\Rightarrow$ **local maximum**

ii. $f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$ at (a, b) $\Rightarrow$ **local minimum**

iii. $f_{xx} f_{yy} - f_{xy}^2 < 0$ at (a, b) $\Rightarrow$ **saddle point**

iv. $f_{xx} f_{yy} - f_{xy}^2 = 0$ at (a, b) $\Rightarrow$ **test is inconclusive.**


z
∂f/∂x = 0
z = f(x, y)
∂f/∂y = 0
g(x) = f(x, b)
h(y) = f(a, y)
0
b
a
y
(a, b, 0)
x


z
$z = x^2 + y^2$
x
y

Find all local maxima, local minima, and saddle points of the function $f(x,y) = x^4 + y^4 + 4xy$.

The critical points for this function are (0, 0), (1, -1), and (-1, 1).

$f_{xx} = 12x^2 \qquad f_{yy} = 12y^2 \qquad f_{xy} = 4$

$(0, 0) \quad f_{xx} = 0 \qquad f_{yy} = 0 \qquad f_{xx}f_{yy} - f_{xy}^2 = 0 - 16 < 0$

The point (0, 0) is a saddle point, and f (0, 0) = 0.

$(1, -1) \quad f_{xx} = 12 \quad f_{yy} = 12$

$f_{xx} > 0 \quad f_{xx}f_{yy} - f_{xy}^2 = (12)(12) - 16 > 0$

The point (1, -1) is a local minimum, and f (1, -1) = -2.

$(-1, 1) \quad f_{xx} = 12 \quad f_{yy} = 12$

$f_{xx} > 0 \quad f_{xx}f_{yy} - f_{xy}^2 = (12)(12) - 16 > 0$

The point (-1, 1) is a local minimum, and f (-1, 1) = -2.

19. $f(x, y) = 6x^2 - 2x^3 + 3y^2 + 6xy$

$f_x(x, y) = 12x - 6x^2 + 6y = 0$ and $f_y(x, y) = 6y + 6x = 0 \Rightarrow x = 0$ and $y = 0$, or $x = 1$ and $y = -1 \Rightarrow$ critical points are $(0, 0)$ and $(1, -1)$; for $(0, 0)$: $f_{xx}(0, 0) = 12 - 12x|_{(0,0)} = 12$, $f_{yy}(0, 0) = 6$, $f_{xy}(0, 0) = 6 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(0, 0) = 0$; for $(1, -1)$: $f_{xx}(1, -1) = 0$, $f_{yy}(1, -1) = 6$, $f_{xy}(1, -1) = 6 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point

23. $f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$

3. $f_x(x, y) = 3x^2 + 6x = 0 \Rightarrow x = 0$ or $x = -2$; $f_y(x, y) = 3y^2 - 6y = 0 \Rightarrow y = 0$ or $y = 2 \Rightarrow$ the critical points $(0, 0)$, $(0, 2)$, $(-2, 0)$, and $(-2, 2)$; for $(0, 0)$: $f_{xx}(0, 0) = 6x + 6|_{(0,0)} = 6$, $f_{yy}(0, 0) = 6y - 6|_{(0,0)} = -6$, $f_{xy}(0, 0) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point; for $(0, 2)$: $f_{xx}(0, 2) = 6$, $f_{yy}(0, 2) = 6$, $f_{xy}(0, 2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} > 0 \Rightarrow$ local minimum of $f(0, 2) = -12$; for $(-2, 0)$: $f_{xx}(-2, 0) = -6$, $f_{yy}(-2, 0) = -6$, $f_{xy}(-2, 0) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f(-2, 0) = -4$; for $(-2, 2)$: $f_{xx}(-2, 2) = -6$, $f_{yy}(-2, 2) = 6$, $f_{xy}(-2, 2) = 0 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -36 < 0 \Rightarrow$ saddle point

3. $f(x, y) = 2xy - 5x^2 - 2y^2 + 4x + 4y - 4$

4. $f(x, y) = 2xy - 5x^2 - 2y^2 + 4x - 4$

5. $f(x, y) = x^2 + xy + 3x + 2y + 5$

3. $f_x(x, y) = 2y - 10x + 4 = 0$ and $f_y(x, y) = 2x - 4y + 4 = 0 \Rightarrow x = \frac{2}{3}$ and $y = \frac{4}{3} \Rightarrow$ critical point is $\left(\frac{2}{3}, \frac{4}{3}\right)$; $f_{xx}\left(\frac{2}{3}, \frac{4}{3}\right) = -10$, $f_{yy}\left(\frac{2}{3}, \frac{4}{3}\right) = -4$, $f_{xy}\left(\frac{2}{3}, \frac{4}{3}\right) = 2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f\left(\frac{2}{3}, \frac{4}{3}\right) = 0$

4. $f_x(x, y) = 2y - 10x + 4 = 0$ and $f_y(x, y) = 2x - 4y = 0 \Rightarrow x = \frac{4}{9}$ and $y = \frac{2}{9} \Rightarrow$ critical point is $\left(\frac{4}{9}, \frac{2}{9}\right)$; $f_{xx}\left(\frac{4}{9}, \frac{2}{9}\right) = -10$, $f_{yy}\left(\frac{4}{9}, \frac{2}{9}\right) = -4$, $f_{xy}\left(\frac{4}{9}, \frac{2}{9}\right) = 2 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = 36 > 0$ and $f_{xx} < 0 \Rightarrow$ local maximum of $f\left(\frac{4}{9}, \frac{2}{9}\right) = -\frac{28}{9}$

5. $f_x(x, y) = 2x + y + 3 = 0$ and $f_y(x, y) = x + 2 = 0 \Rightarrow x = -2$ and $y = 1 \Rightarrow$ critical point is $(-2, 1)$; $f_{xx}(-2, 1) = 2$, $f_{yy}(-2, 1) = 0$, $f_{xy}(-2, 1) = 1 \Rightarrow f_{xx}f_{yy} - f_{xy}^2 = -1 < 0 \Rightarrow$ saddle point