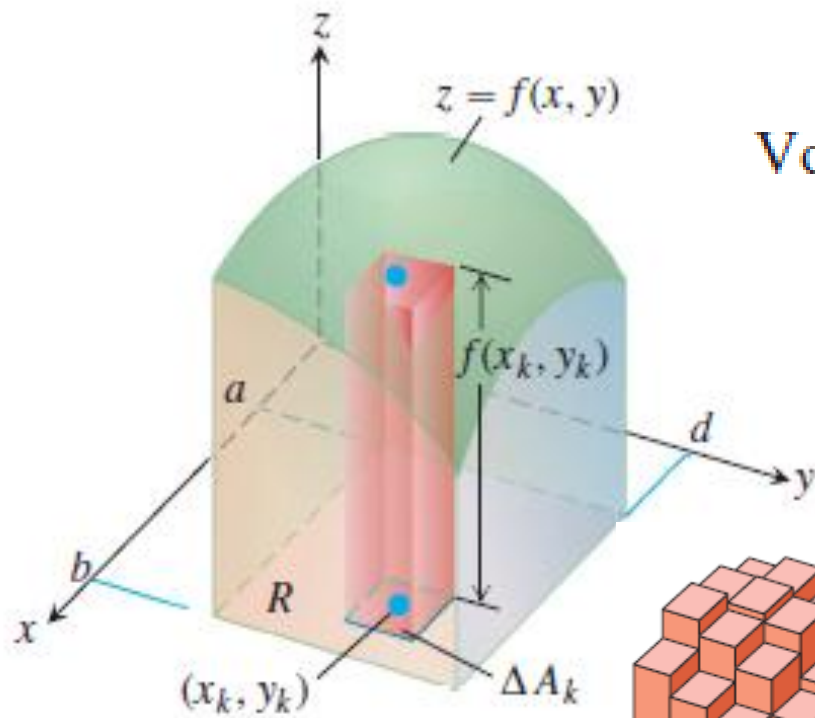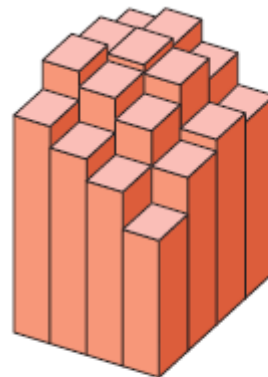# MULTIPLE INTEGRALS

Chapter 15

integral of a function of two variables *f(x, y)* over a region in the plane and the integral of a function of three variables *f(x, y, z)* *over a* region in space. These integrals are called *multiple integrals*
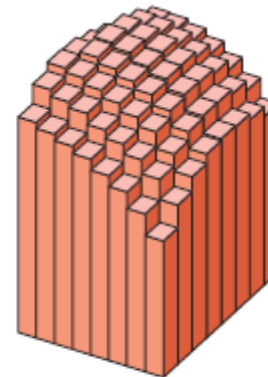
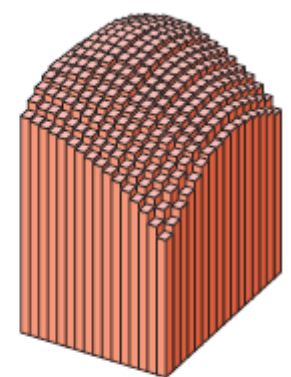$$\text{Volume} = \lim_{n \to \infty} S_n = \iint_R f(x, y) \, dA,$$

(a) $n = 16$ (b) $n = 64$ (c) $n = 256$

FIGURE 15.3   As n increases, the Riemann sum approximations approach the total
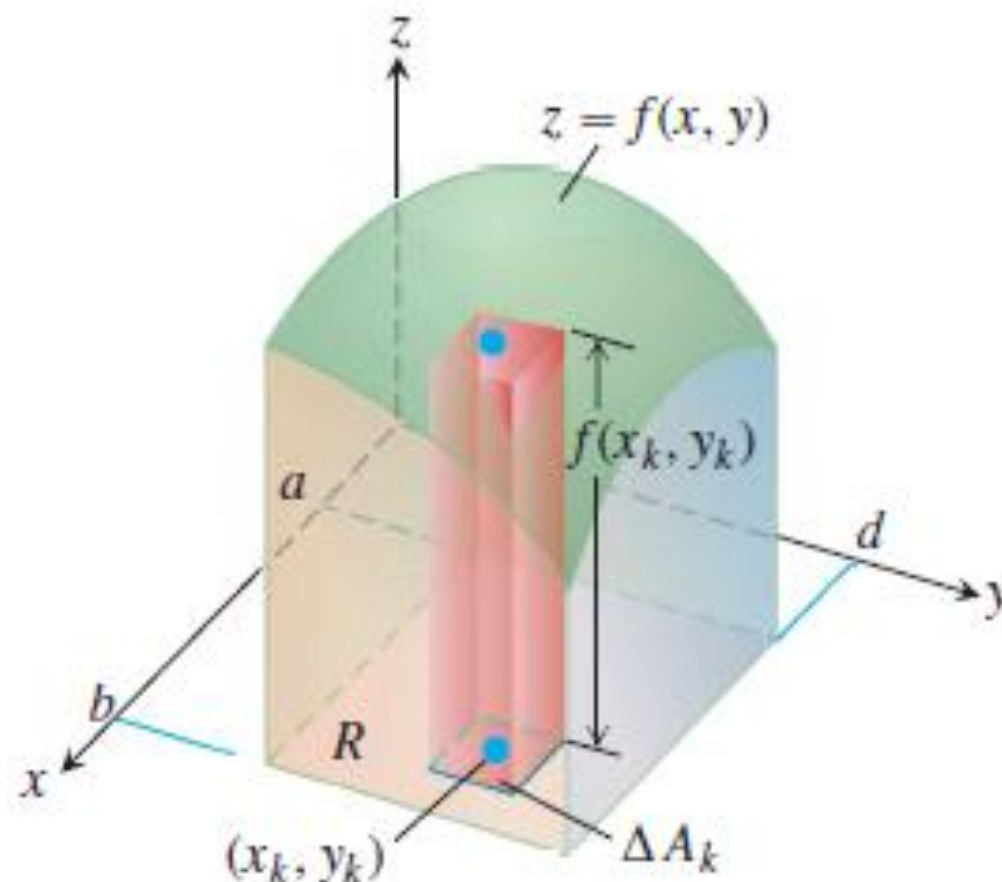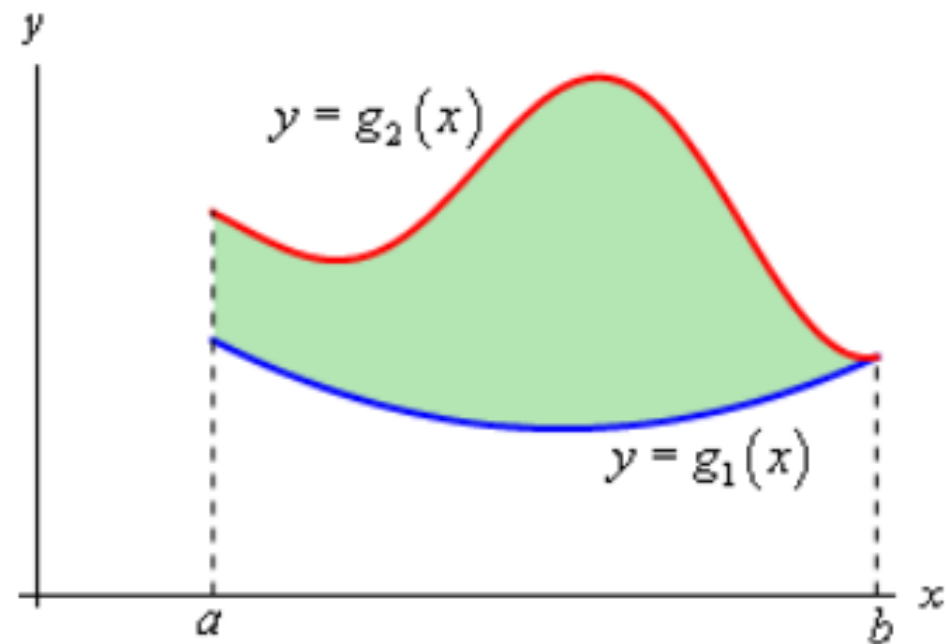
THEOREM 1   Fubini's Theorem (First Form)

If $f(x, y)$ is continuous throughout the rectangular region $R: a \leq x \leq b$, $c \leq y \leq d$, then
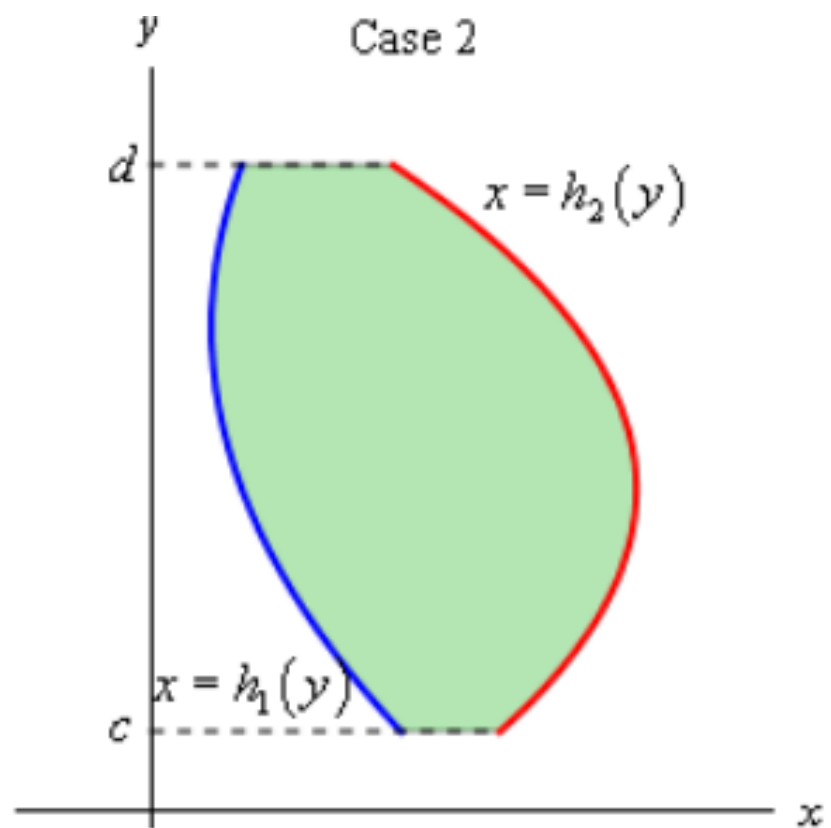
$$\iint\limits_{R} f(x, y)\, dA = \int_{c}^{d} \int_{a}^{b} f(x, y)\, dx\, dy = \int_{a}^{b} \int_{c}^{d} f(x, y)\, dy\, dx.$$

Case 1

$y = g_2(x)$

$y = g_1(x)$

a

b

x

y

Case 2

d

$x = h_2(y)$

$x = h_1(y)$

c

x

y
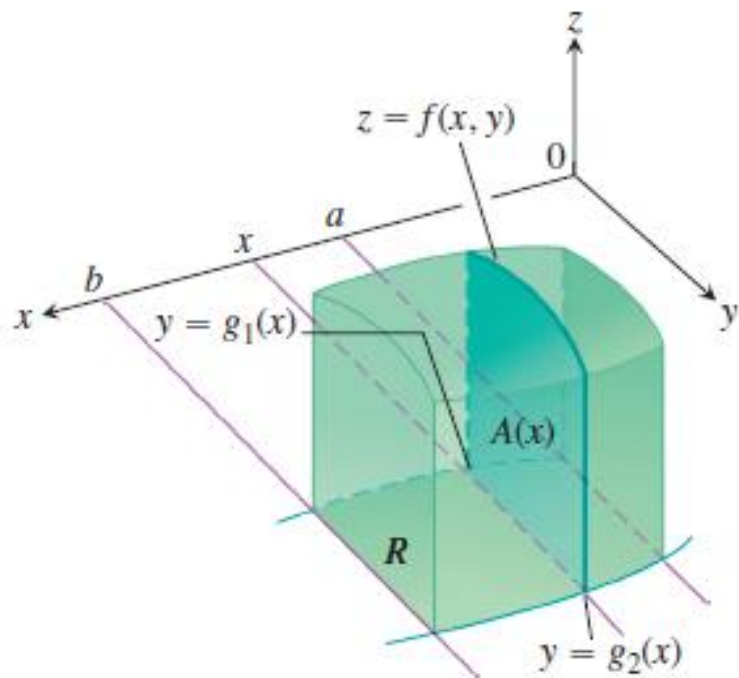
FIGURE 15.9 The area of the vertical slice shown here is

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy.$$

FIGURE 15.10 The volume of the solid shown here is

$$\int_c^d A(y)\, dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

$$V = \int_a^b A(x)\, dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx. \qquad = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$

THEOREM 2   Fubini's Theorem (Stronger Form)

Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$
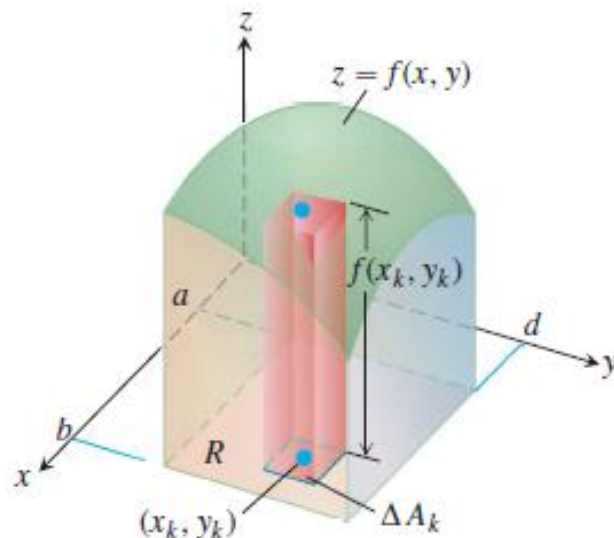
$$\int_{y=1}^{2} \int_{x=0}^{3} (1 + 8xy) \, dx \, dy$$

$$\text{integral} \ = \ \int_{y=1}^{2} \left(\underbrace{\int_{x=0}^{3} (1 + 8xy) \, dx}_{\text{work out treating } y \text{ as constant}} \right) dy$$

$$= \ \int_{y=1}^{2} \left[x + \frac{8x^2 y}{2} \right]_{x=0}^{3} dy$$

$$= \ \int_{y=1}^{2} (3 + 36y) \, dy$$

$$= \ \left[3y + \frac{36y^2}{2} \right]_{y=1}^{2}$$

$$= \ (6 + 72) - (3 + 18)$$

$$= \ 57$$

Find the volume of the solid bounded above by the plane $z = 4 - x - y$ and below by the rectangle $R = \{(x, y) : \quad 0 \le x \le 1 \quad 0 \le y \le 2\}$.

$$
\begin{aligned}
V &= \int_0^2 \int_0^1 (4 - x - y) \, dx \, dy \\
&= \int_0^2 \left[4x - \tfrac{1}{2}x^2 - yx \right]_{x=0}^{1} dy = \int_0^2 (4 - \tfrac{1}{2} - y) \, dy \\
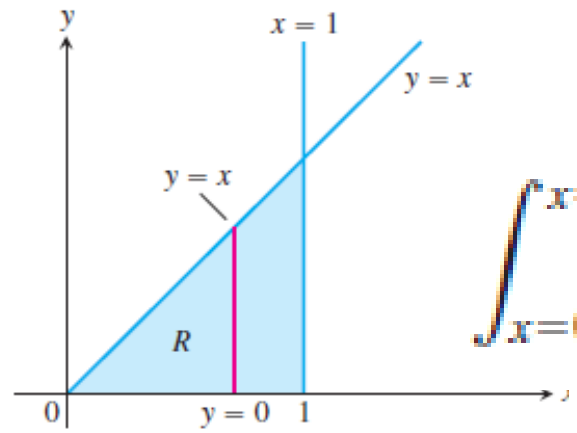&= \left[\frac{7y}{2} - \frac{y^2}{2} \right]_{y=0}^{2} = (7 - 2) - (0) = 5
\end{aligned}
$$

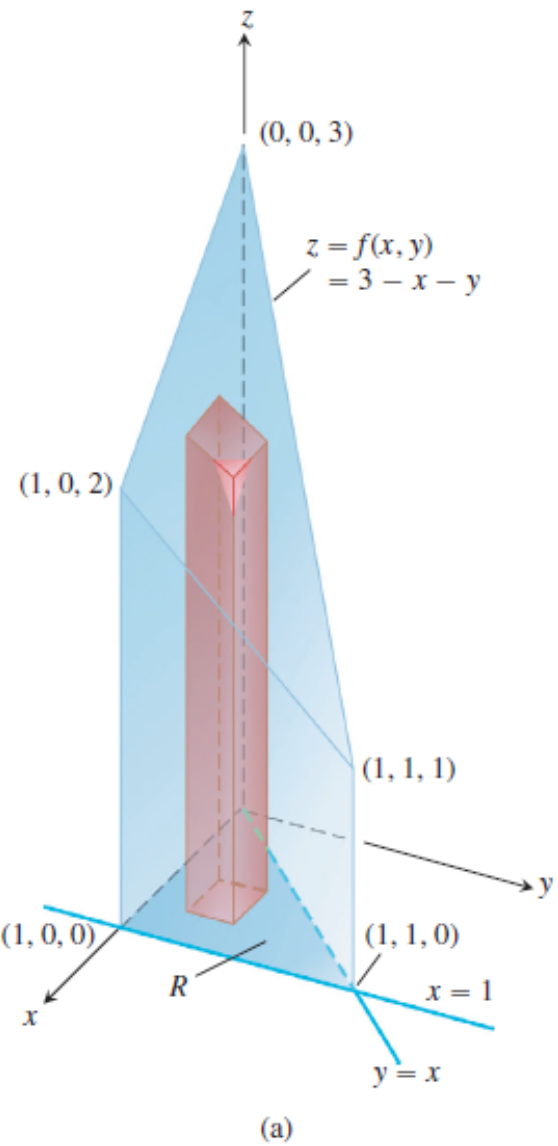Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane

$$z = f(x, y) = 3 - x - y.$$

$$V = \int_0^1 \int_0^x (3 - x - y)\, dy\, dx = \int_0^1 \left[3y - xy - \frac{y^2}{2} \right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \left(3x - \frac{3x^2}{2} \right) dx = \left[\frac{3x^2}{2} - \frac{x^3}{2} \right]_{x=0}^{x=1} = 1.$$

z

(0, 0, 3)

$z = f(x, y)$
$= 3 - x - y$

(1, 0, 2)

(1, 1, 1)

(1, 0, 0)

(1, 1, 0)

R

x = 1

y = x

x

y

(a)

y

x = 1

y = x

y = x

R

0

y = 0   1

(b)

$$\int_{x=0}^{x=1} \int_{y=0}^{y=x} f(x, y) \, dy \, dx.$$

y

x = 1

y = x

x = y

x = 1

R

0

1

x

(c)
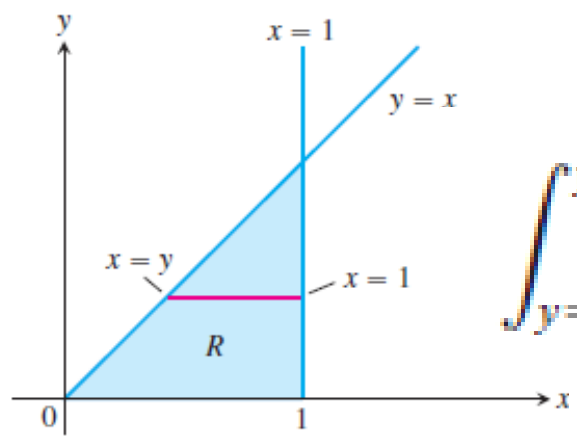
$$\int_{y=0}^{y=1} \int_{x=y}^{x=1} f(x, y) \, dx \, dy.$$

Properties of Double Integrals

If $f(x, y)$ and $g(x, y)$ are continuous, then

1. *Constant Multiple:* $\iint\limits_R cf(x, y)\, dA = c \iint\limits_R f(x, y)\, dA$ (any number c)

2. *Sum and Difference:*

$$\iint\limits_R (f(x, y) \pm g(x, y))\, dA = \iint\limits_R f(x, y)\, dA \pm \iint\limits_R g(x, y)\, dA$$

3. *Domination:*

   (a) $\iint\limits_R f(x, y)\, dA \geq 0$ if $f(x, y) \geq 0$ on R

   (b) $\iint\limits_R f(x, y)\, dA \geq \iint\limits_R g(x, y)\, dA$ if $f(x, y) \geq g(x, y)$ on R

4. *Additivity:* $\iint\limits_R f(x, y)\, dA = \iint\limits_{R_1} f(x, y)\, dA + \iint\limits_{R_2} f(x, y)\, dA$
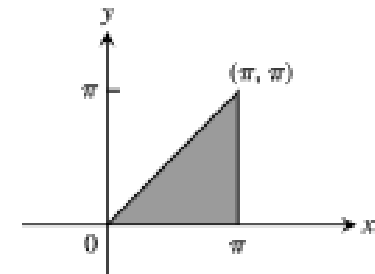
sketch the region of integration and evaluate

$$\int_{-1}^{0} \int_{-1}^{1} (x + y + 1)\, dx\, dy$$

$$\int_{-1}^{0} \int_{-1}^{1} (x+y+1)\, dx\, dy = \int_{-1}^{0} \left[\frac{x^2}{2} + yx + x \right]_{-1}^{1} dy$$

$$= \int_{-1}^{0} (2y + 2)\, dy = [y^2 + 2y]_{-1}^{0} = 1$$
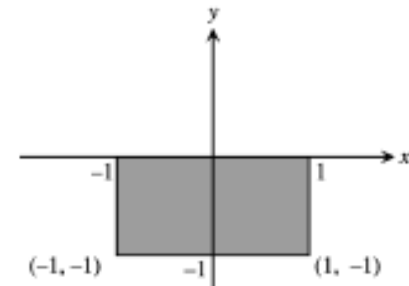
$$\int_{0}^{\pi} \int_{0}^{x} x \sin y \, dy \, dx$$

$$\int_{0}^{\pi} \int_{0}^{x} (x \sin y)\, dy\, dx = \int_{0}^{\pi} [-x \cos y]_{0}^{x} \, dx$$

$$= \int_{0}^{\pi} (x - x \cos x)\, dx = \left[\frac{x^2}{2} - (\cos x + x \sin x) \right]_{0}^{\pi}$$

$$= \frac{\pi^2}{2} + 2$$
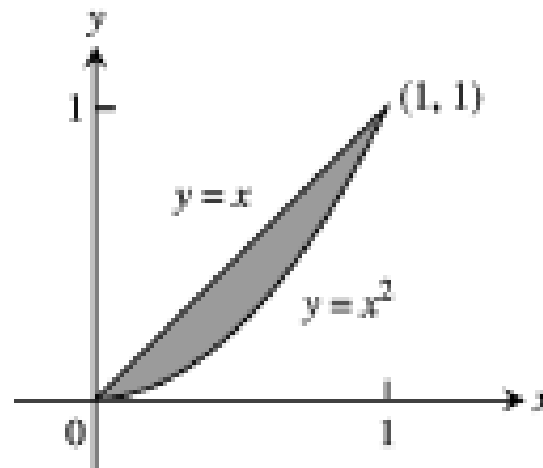
sketch the region of integration and write an equivalent double integral with the order of integration reversed

$$\int_0^2 \int_{y-2}^0 dx\, dy$$

$y = x + 2$
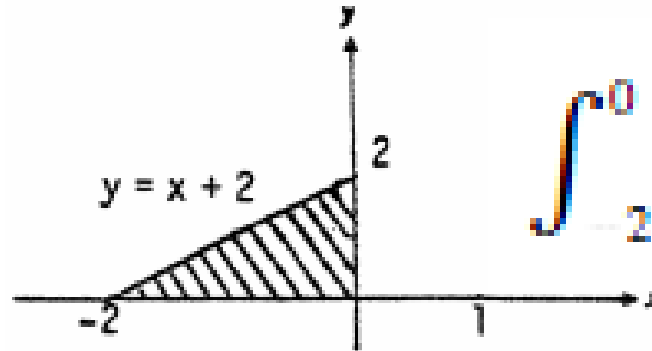
$$\int_{-2}^0 \int_0^{x+2} dy\, dx$$

$$\int_0^1 \int_y^{\sqrt{y}} dx\, dy$$

$(1, 1)$

$y = x$

$y = x^2$

$$\int_0^1 \int_{x^2}^x dy\, dx$$

$$\int_0^{\ln 2} \int_{e^x}^{2} dx\, dy$$
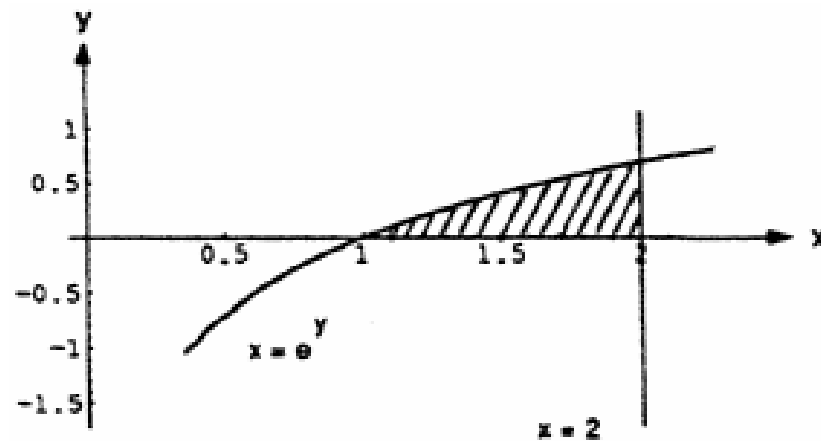
$$\int_1^2 \int_0^{\ln x} dy\, dx$$

$$\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3y\, dx\, dy$$
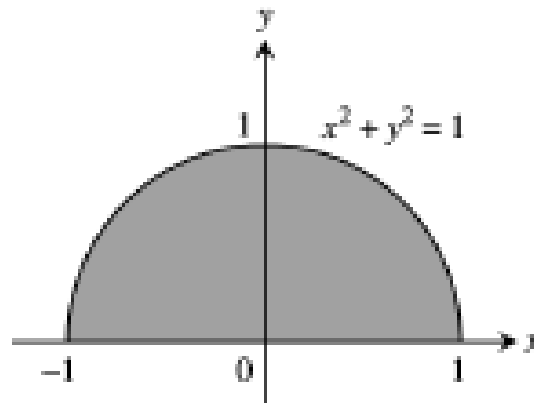
$$\int_{-1}^1 \int_0^{\sqrt{1-x^2}} 3y\, dy\, dx$$

Find the volume of the solid enclosed by the planes $4x + 2y + z = 10$, $y = 3x$, $z = 0$, $x = 0$.
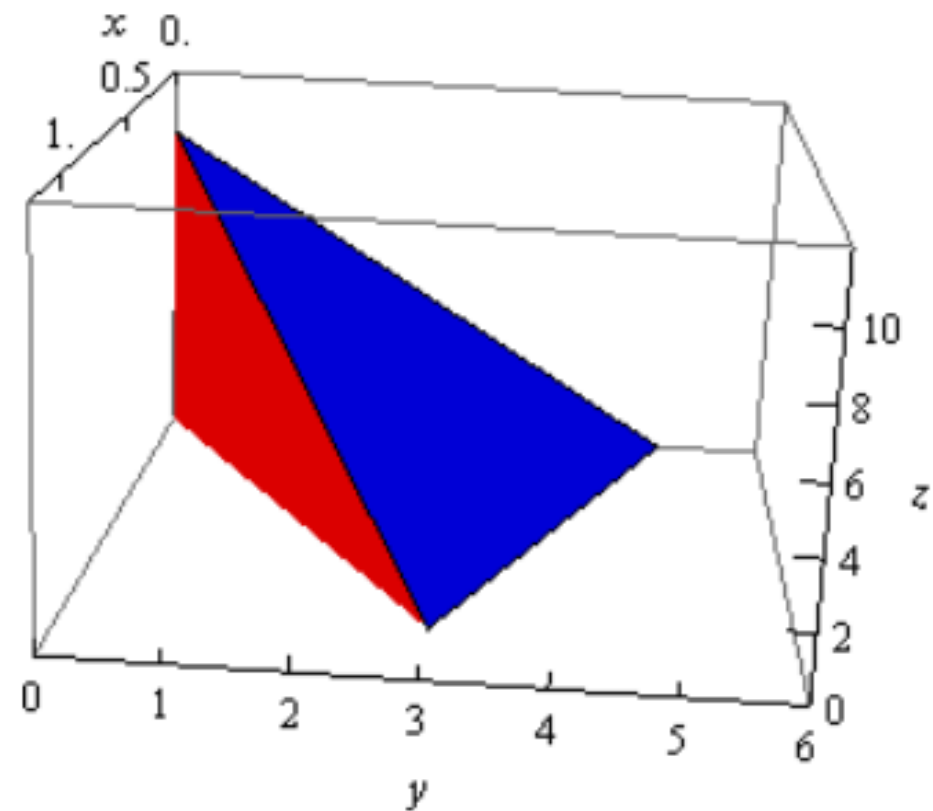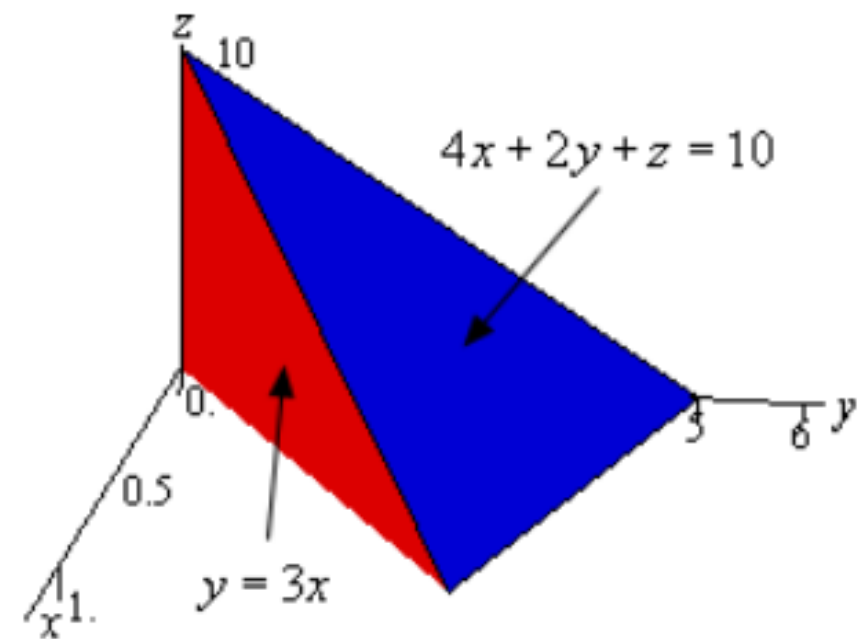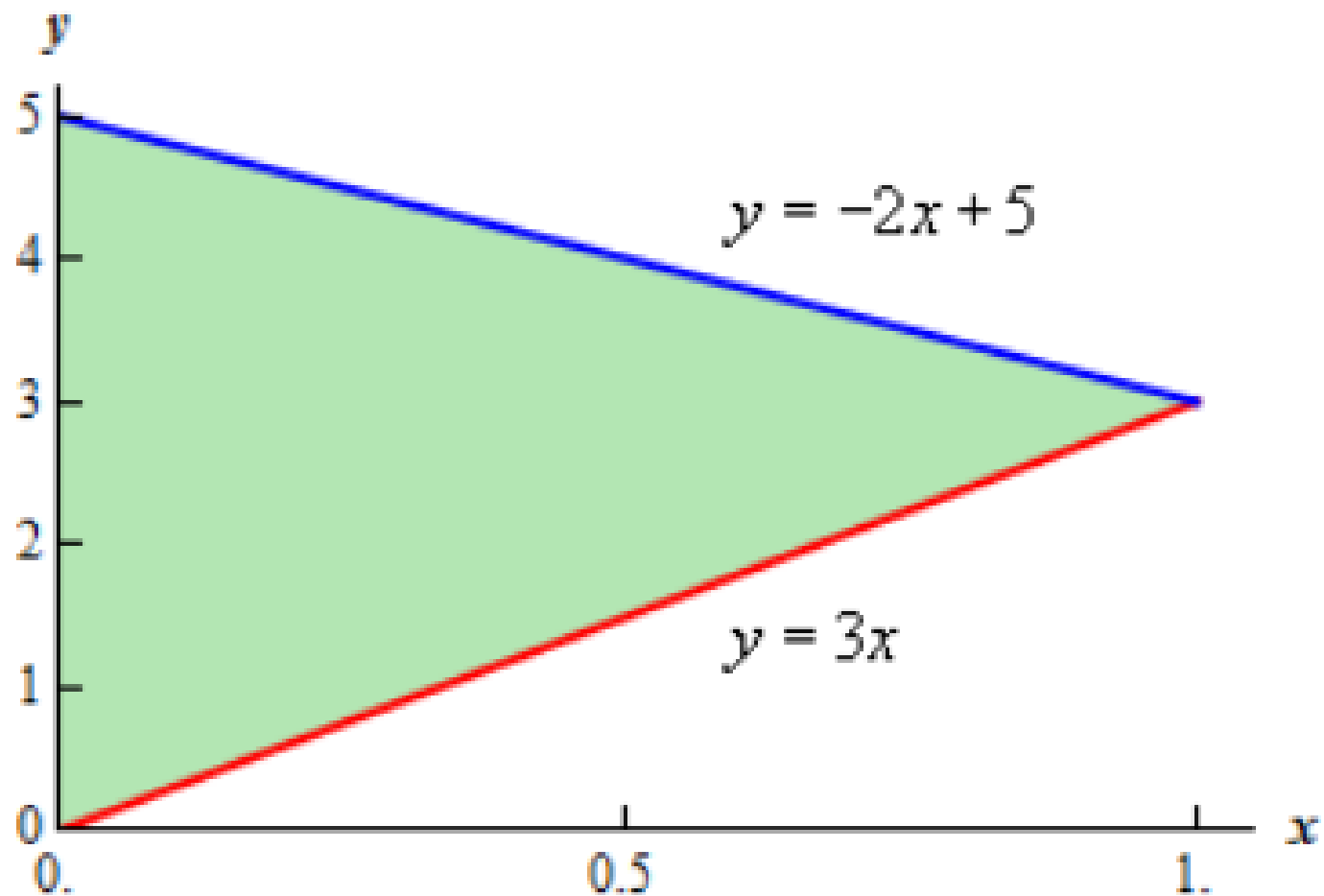
solid will sit on the xy-plane and here are the inequalities

$$0 \le x \le 1$$

$$3x \le y \le -2x+5$$

$$V = \iint_D 10 - 4x - 2y \, dA$$

$$= \int_0^1 \int_{3x}^{-2x+5} 10 - 4x - 2y \, dy \, dx$$

$$= \int_0^1 \left(10y - 4xy - y^2\right)\Big|_{3x}^{-2x+5} dx$$

$$= \int_0^1 25x^2 - 50x + 25 \, dx$$

$$= \left(\frac{25}{3}x^3 - 25x^2 + 25x\right)\Big|_0^1 = \frac{25}{3}$$

41. Find the volume of the region bounded by the paraboloid $z = x^2 + y^2$ and below by the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy-plane.

41. $V = \int_0^1 \int_x^{2-x} (x^2 + y^2)\, dy\, dx = \int_0^1 \left[x^2 y + \frac{y^3}{3} \right]_x^{2-x} dx = \int_0^1 \left[2x^2 - \frac{7x^3}{3} + \frac{(2-x)^3}{3} \right] dx = \left[\frac{2x^3}{3} - \frac{7x^4}{12} - \frac{(2-x)^4}{12} \right]_0^1$

$= \left(\frac{2}{3} - \frac{7}{12} - \frac{1}{12} \right) - \left(0 - 0 - \frac{16}{12} \right) = \frac{4}{3}$

43. Find the volume of the solid whose base is the region in the xy-plane that is bounded by the parabola $y = 4 - x^2$ and the line $y = 3x$, while the top of the solid is bounded by the plane $z = x + 4$.

43. $V = \int_{-4}^1 \int_{3x}^{4-x^2} (x+4)\, dy\, dx = \int_{-4}^1 [xy + 4y]_{3x}^{4-x^2}\, dx = \int_{-4}^1 [x(4 - x^2) + 4(4 - x^2) - 3x^2 - 12x]\, dx$

$= \int_{-4}^1 (-x^3 - 7x^2 - 8x + 16)\, dx = \left[-\frac{1}{4}x^4 - \frac{7}{3}x^3 - 4x^2 + 16x \right]_{-4}^1 = \left(-\frac{1}{4} - \frac{7}{3} + 12 \right) - \left(\frac{64}{3} - 64 \right)$

$= \frac{157}{3} - \frac{1}{4} = \frac{625}{12}$

DEFINITION   Area

The **area** of a closed, bounded plane region R is
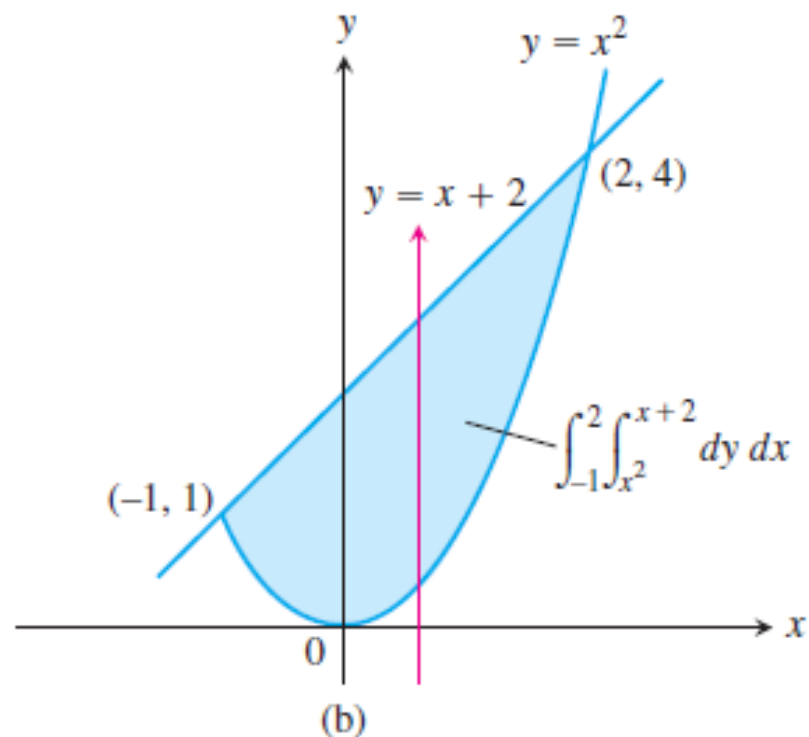
$$A = \iint\limits_{R} dA.$$

Find the area of the region R bounded by $y = x$ and $y = x^2$ in the first quadrant.
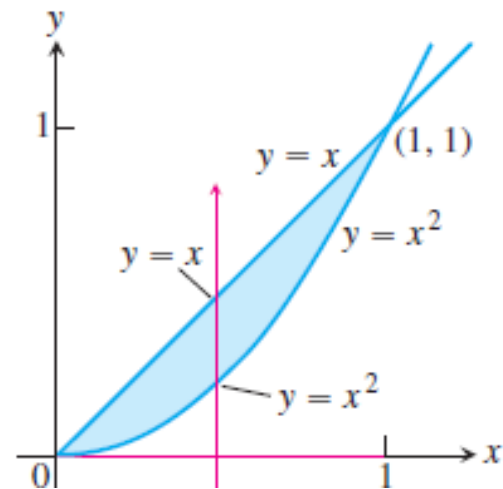
$$A = \int_0^1 \int_{x^2}^{x} dy\, dx = \int_0^1 \left[y \right]_{x^2}^{x} dx$$

$$= \int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}.$$

Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

$y = x^2$

$(2, 4)$

$y = x + 2$

$\int_1^4 \int_{y-2}^{\sqrt{y}} dx\, dy$

R_2

$(-1, 1)$

R_1

$\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} dx\, dy$

0

x

(a)

$y = x^2$

$(2, 4)$

$y = x + 2$

$\int_{-1}^2 \int_{x^2}^{x+2} dy\, dx$

$(-1, 1)$

0

x

(b)
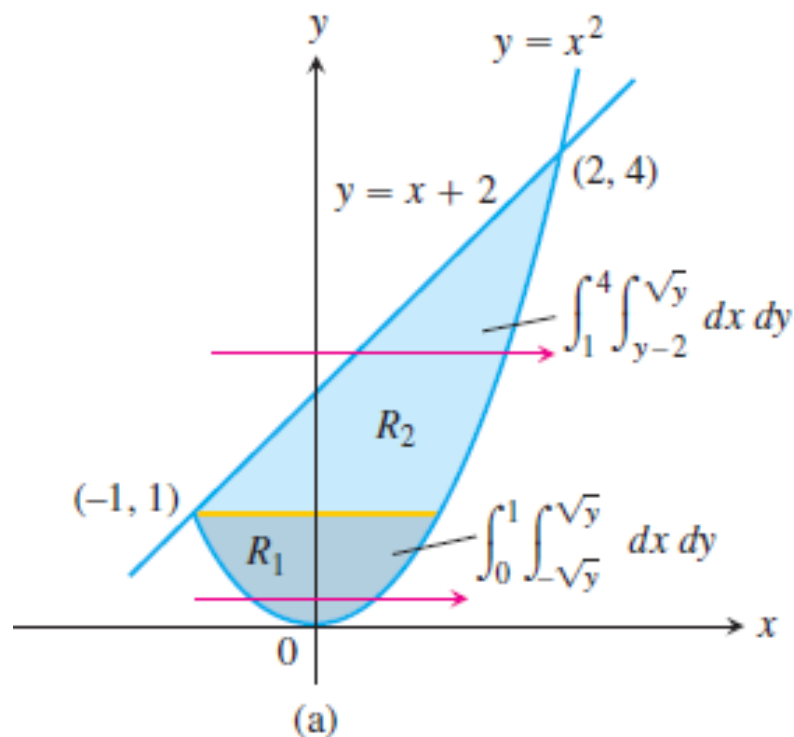
$$A = \int_{-1}^2 \Big[y\Big]_{x^2}^{x+2} dx = \int_{-1}^2 (x + 2 - x^2)\, dx = \left[\frac{x^2}{2} + 2x - \frac{x^3}{3}\right]_{-1}^2 = \frac{9}{2}.$$

$$\text{Average value of } f \text{ over } R = \frac{1}{\text{area of } R} \iint\limits_{R} f \, dA.$$

TABLE 15.1 Mass and first moment formulas for thin plates covering a region R in the xy-plane

---

Mass: $\quad M = \iint\limits_{R} \delta(x, y) \, dA \qquad \delta(x, y)$ is the density at (x, y)

First moments: $\quad M_x = \iint\limits_{R} y\delta(x, y) \, dA, \qquad M_y = \iint\limits_{R} x\delta(x, y) \, dA$

Center of mass: $\quad \bar{x} = \dfrac{M_y}{M}, \qquad \bar{y} = \dfrac{M_x}{M}$

Moments of Inertia

• A body's first moments (Table 15.1) tell us about balance and about the torque the body.

• If the body is a rotating shaft, however, we are more likely to be interested in how much energy is stored in the shaft or about how much energy it will take to accelerate the shaft to a particular angular velocity.

• This is where the second moment or moment of inertia comes in.

$$\text{KE}_{\text{shaft}} = \int \frac{1}{2}\omega^2 r^2 \, dm = \frac{1}{2}\omega^2 \int r^2 \, dm. \tag{4}$$
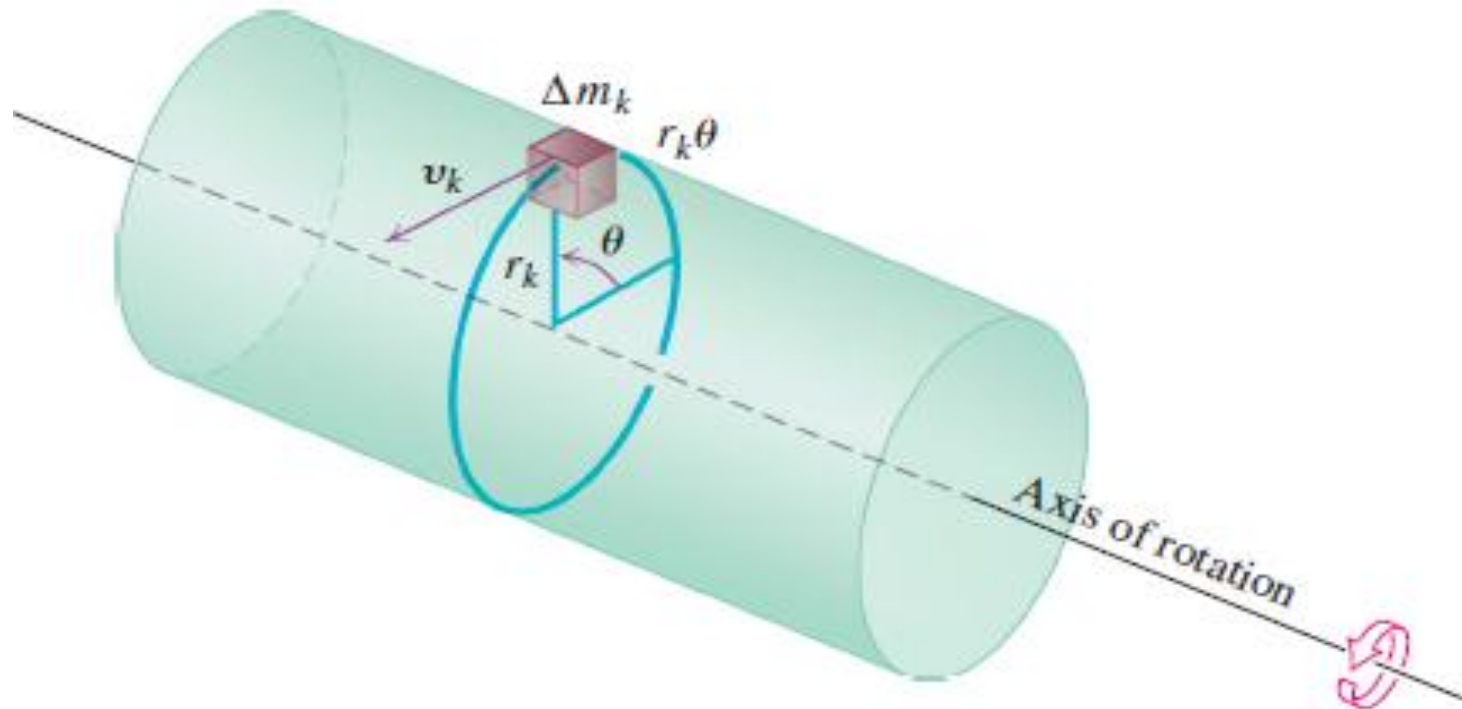
The factor

$$I = \int r^2 \, dm$$

is the *moment of inertia* of the shaft about its axis of rotation, and we see from Equation (4) that the shaft's kinetic energy is

$$\text{KE}_{\text{shaft}} = \frac{1}{2}I\omega^2.$$

Moments of Inertia

$$v_k = \frac{d}{dt}(r_k\theta) = r_k\frac{d\theta}{dt} = r_k\omega.$$

TABLE 15.2 Second moment formulas for thin plates in the xy-plane

Moments of inertia (second moments):

About the x-axis:

$$I_x = \iint y^2 \delta(x, y)\, dA$$

About the y-axis:

$$I_y = \iint x^2 \delta(x, y)\, dA$$

About a line L:

$$I_L = \iint r^2(x, y)\delta(x, y)\, dA,$$

where $r(x, y) = $ distance from (x, y) to L

About the origin
(polar moment):

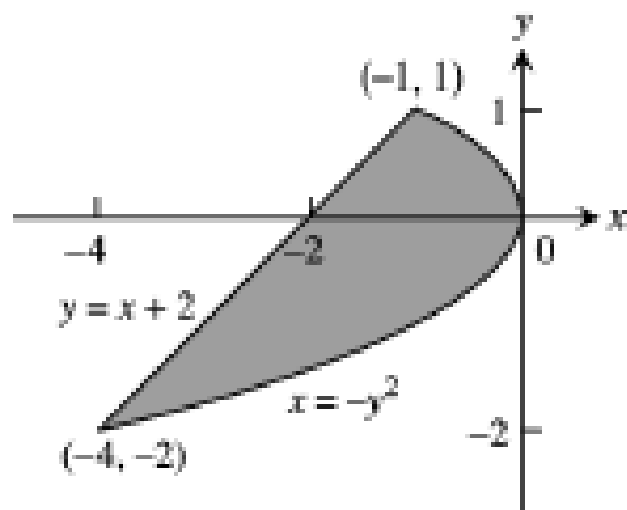$$I_0 = \iint (x^2 + y^2)\delta(x, y)\, dA = I_x + I_y$$

Radii of gyration: About the x-axis: $\quad R_x = \sqrt{I_x/M}$

About the y-axis: $\quad R_y = \sqrt{I_y/M}$

About the origin: $\quad R_0 = \sqrt{I_0/M}$

In Exercises 1–8, sketch the region bounded by the given lines and curves. Then express the region's area as an iterated double integral and evaluate the integral.

3. The parabola $x = -y^2$ and the line $y = x + 2$

$$\int_{-2}^{1}\int_{y-2}^{-y^2} dx\,dy = \int_{-2}^{1}(-y^2 - y + 2)\,dy$$

$$= \left[-\frac{y^3}{3} - \frac{y^2}{2} + 2y\right]_{-2}^{1}$$

$$= \left(-\frac{1}{3} - \frac{1}{2} + 2\right) - \left(\frac{8}{3} - 2 - 4\right) = \frac{9}{2}$$

36. Center of mass, moment of inertia, and radius of gyration

Find the center of mass and the moment of inertia and radius of gyration about the y-axis of a thin plate bounded by the line $y = 1$ and the parabola $y = x^2$ if the density is $\delta(x, y) = y + 1$.

$M = \int_{-1}^{1} \int_{x^2}^{1} (y + 1)\, dy\, dx = -\int_{-1}^{1} \left(\frac{x^4}{2} + x^2 - \frac{3}{2} \right) dx = \frac{32}{15}$; $M_x = \int_{-1}^{1} \int_{x^2}^{1} y(y + 1)\, dy\, dx = \int_{-1}^{1} \left(\frac{5}{6} - \frac{x^6}{3} - \frac{x^4}{2} \right) dx$

$= \frac{48}{35}$; $M_y = \int_{-1}^{1} \int_{x^2}^{1} x(y + 1)\, dy\, dx = \int_{-1}^{1} \left(\frac{3x}{2} - \frac{x^5}{2} - x^3 \right) dx = 0 \Rightarrow \bar{x} = 0$ and $\bar{y} = \frac{9}{14}$; $I_y = \int_{-1}^{1} \int_{x^2}^{1} x^2(y + 1)\, dy\, dx$

$= \int_{-1}^{1} \left(\frac{3x^2}{2} - \frac{x^6}{2} - x^4 \right) dx = \frac{16}{35}$; $R_y = \sqrt{\frac{I_y}{M}} = \sqrt{\frac{3}{14}}$