15.4 Triple Integrals in Rectangular Coordinates

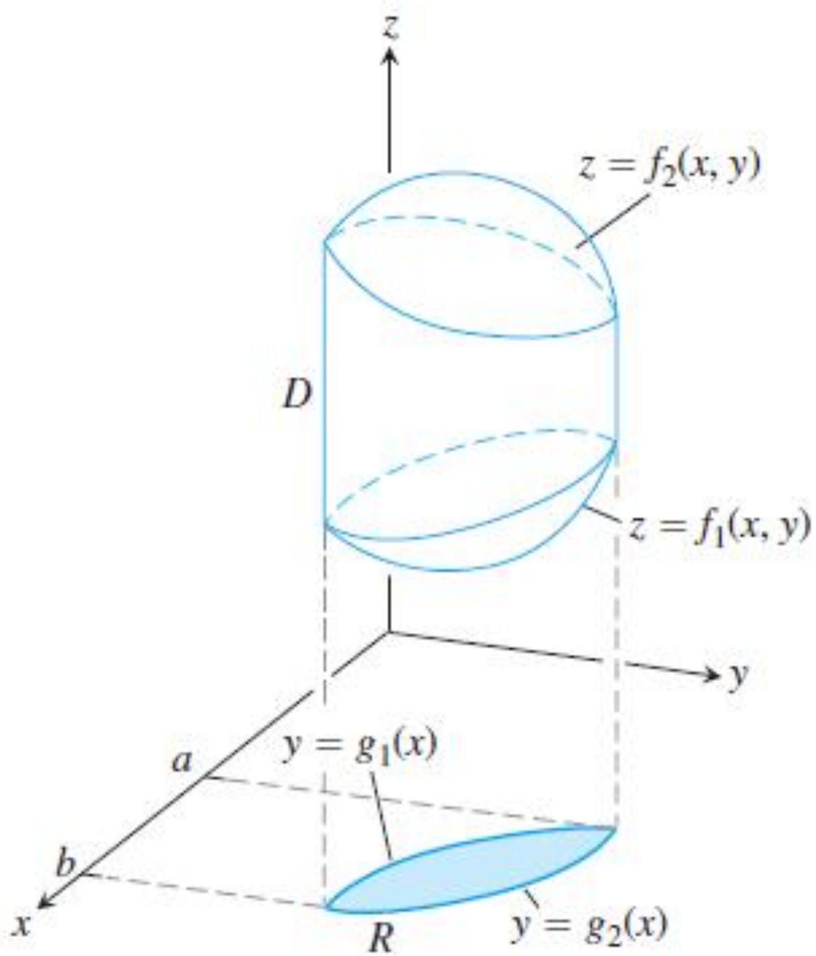
We use triple integrals to calculate:

- the volumes of three-dimensional shapes
- the masses and moments of solids of varying density
- the average value of a function over a three dimensional region.

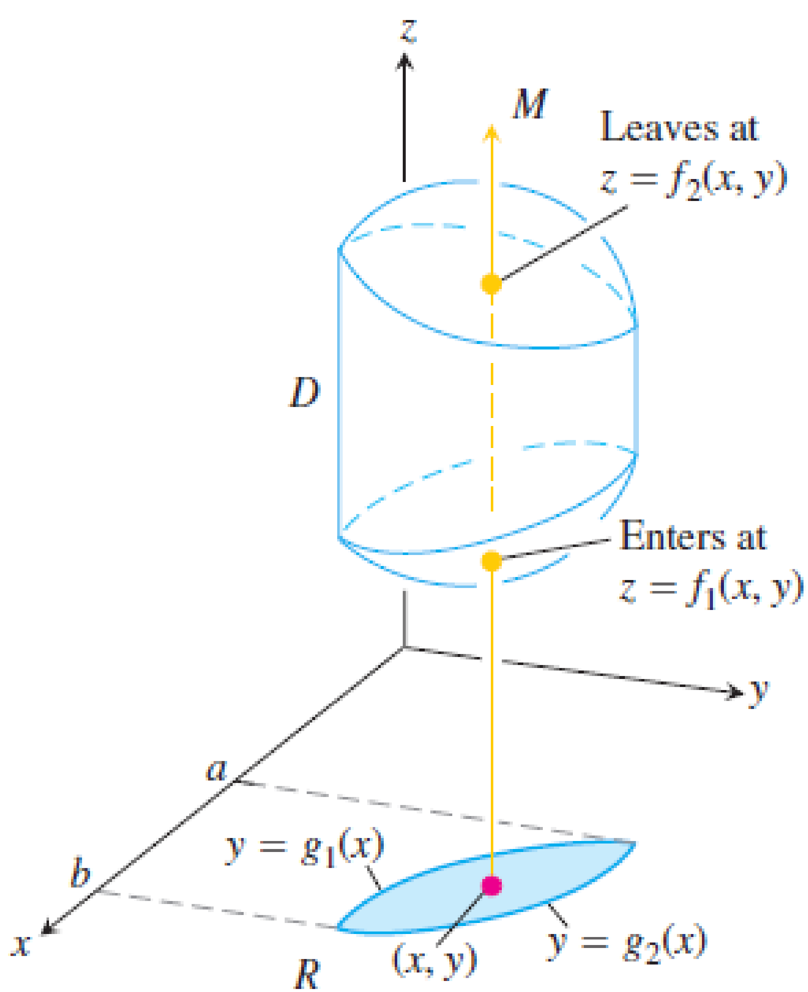
Finding Limits of Integration

We evaluate a triple integral by applying a three-dimensional version of Fubini's Theorem

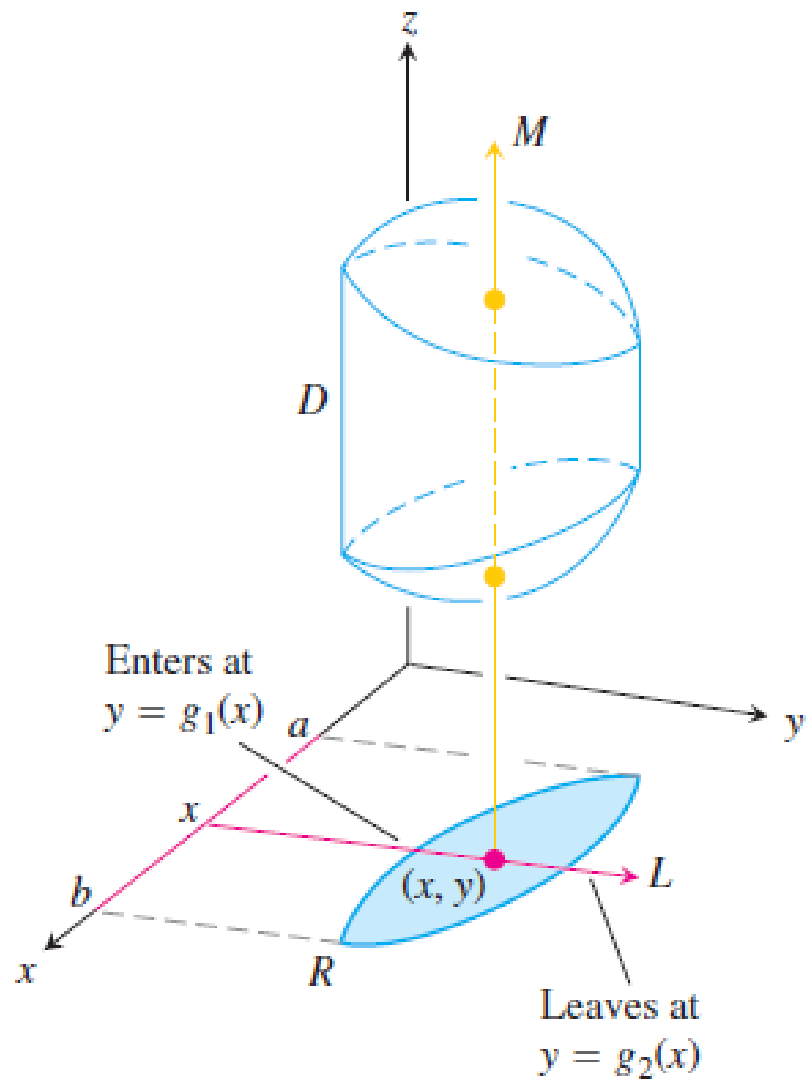
1. *Sketch:* Sketch the region D along with its "shadow" R (vertical projection) in the xy-plane. Label the upper and lower bounding surfaces of D and the upper and lower bounding curves of R.



2. *Find the z-limits of integration:* Draw a line M passing through a typical point (x, y) in R parallel to the z-axis. As z increases, M enters D at $z = f_1(x, y)$ and leaves at $z = f_2(x, y)$. These are the z-limits of integration.



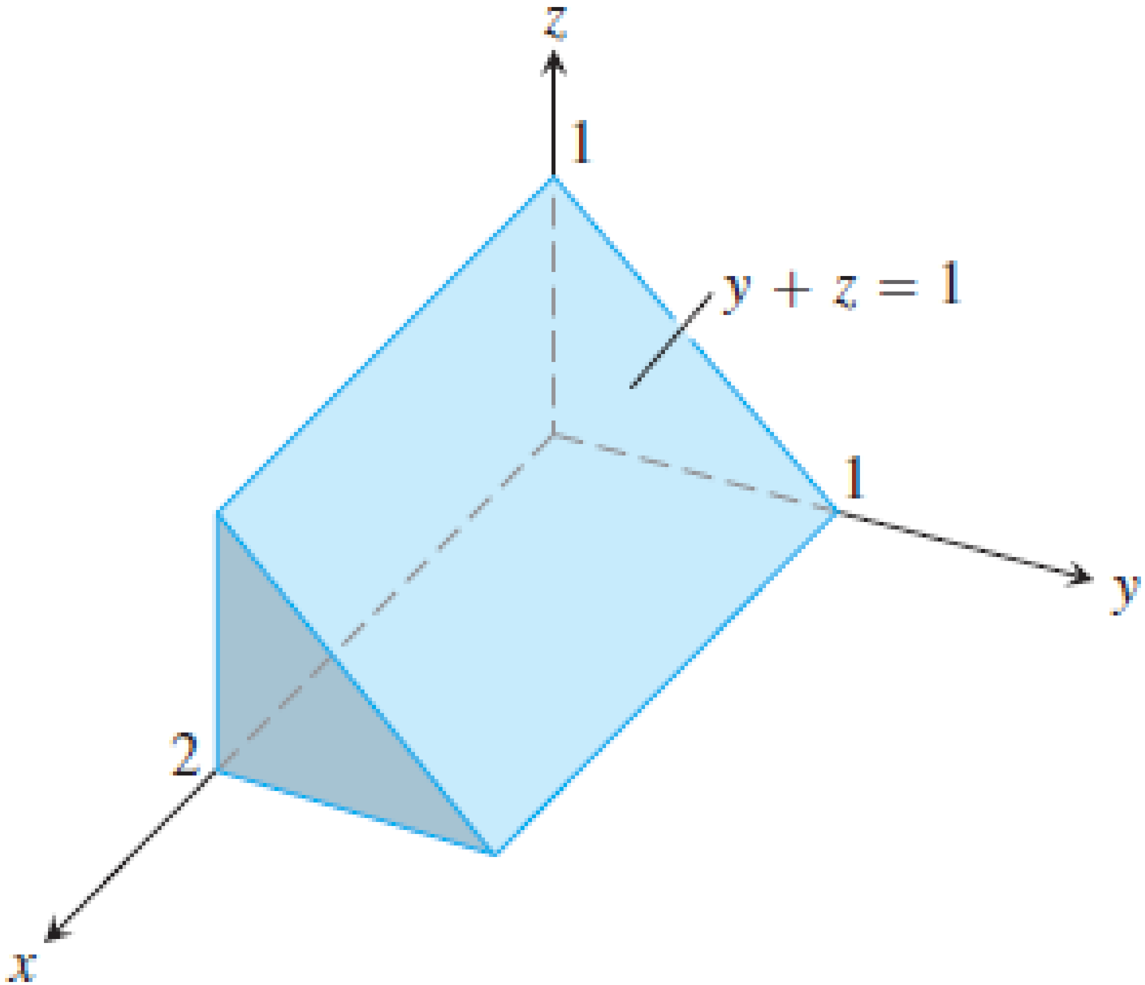
3. *Find the y-limits of integration:* Draw a line L through (x, y) parallel to the y-axis. As y increases, L enters R at $y = g_1(x)$ and leaves at $y = g_2(x)$. These are the y-limits of integration.



4. *Find the x-limits of integration:* Choose x-limits that include all lines through R parallel to the y-axis ($x = a$ and $x = b$ in the preceding figure). These are the x-limits of integration. The integral is

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x, y, z)\, dz\, dy\, dx.$$

Follow similar procedures if you change the order of integration. The "shadow" of region D lies in the plane of the last two variables with respect to which the iterated integration takes place.



$$\int_0^1 \int_0^{1-z} \int_0^2 dx\, dy\, dz$$

$$\int_0^1 \int_0^2 \int_0^{1-z} dy\, dx\, dz$$

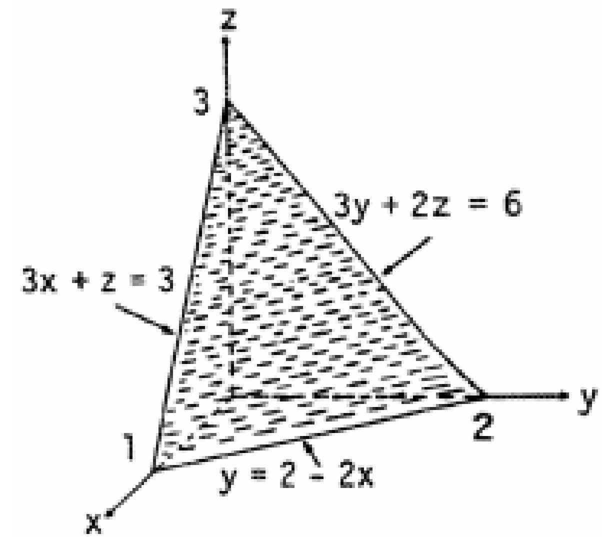
$$\int_0^1 \int_0^2 \int_0^{1-y} dz\, dx\, dy$$

$$\int_0^1 \int_0^{1-y} \int_0^2 dx\, dz\, dy$$

$$\int_0^2 \int_0^1 \int_0^{1-z} dy\, dz\, dx$$

$$\int_0^2 \int_0^1 \int_0^{1-y} dz\, dy\, dx$$

Volume of tetrahedron Write six different iterated triple integrals for the volume of the tetrahedron cut from the first octant by the plane $6x + 3y + 2z = 6$. Evaluate one of the integrals.



$$\int_0^1 \int_0^{2-2x} \int_0^{3-3x-3y/2} dz\, dy\, dx$$

$$= \int_0^1 \int_0^{2-2x} \left(3 - 3x - \tfrac{3}{2}y\right) dy\, dx$$

$$= \int_0^1 \left[3(1-x)\cdot 2(1-x) - \tfrac{3}{4}\cdot 4(1-x)^2\right] dx$$

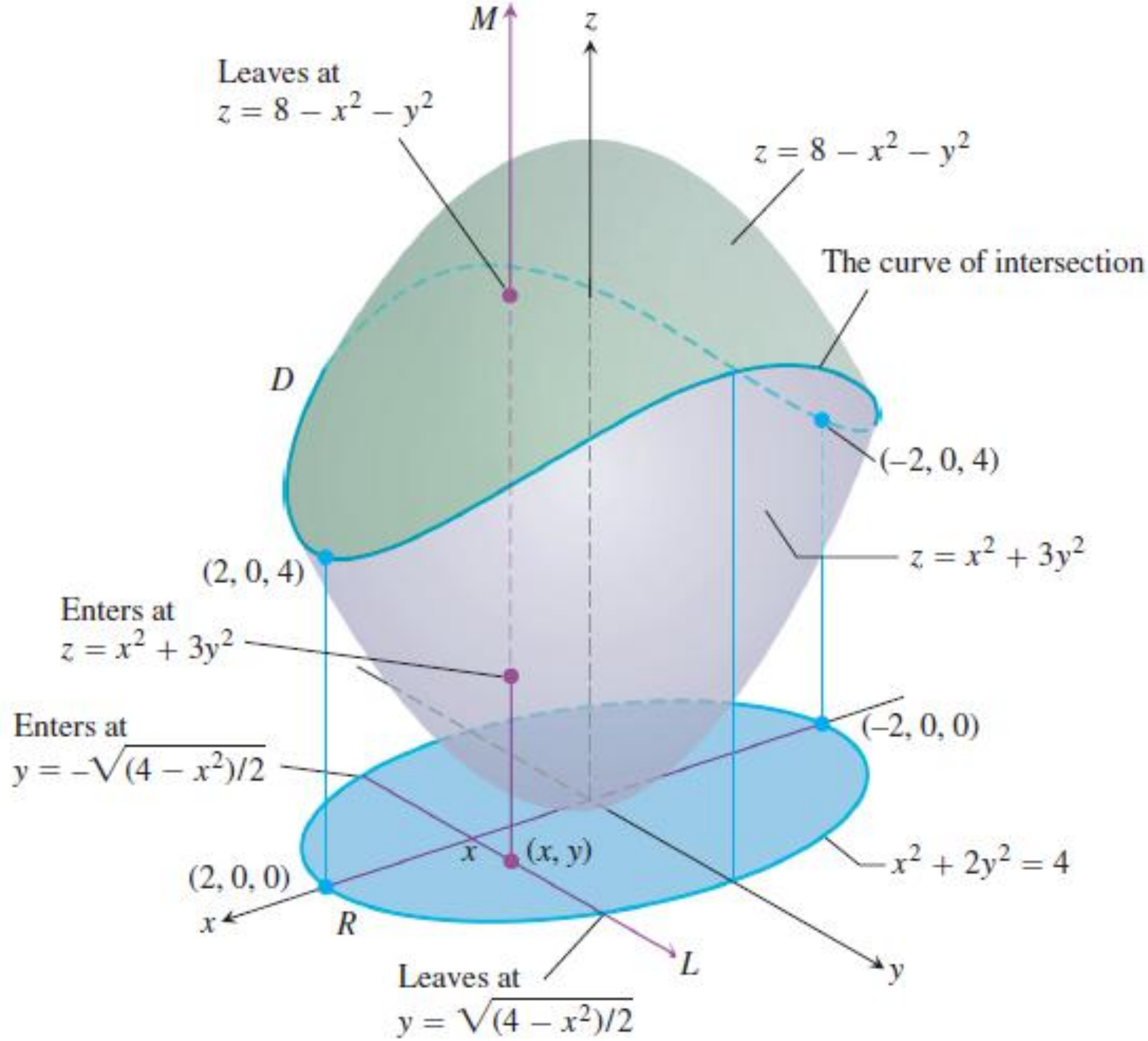
$$= 3\int_0^1 (1-x)^2\, dx = \left[-(1-x)^3\right]_0^1 = 1,$$

$$\int_0^2 \int_0^{1-y/2} \int_0^{3-3x-3y/2} dz\, dx\, dy, \quad \int_0^1 \int_0^{3-3x} \int_0^{2-2x-2z/3}$$

$$\int_0^3 \int_0^{1-z/3} \int_0^{2-2x-2z/3} dy\, dx\, dz, \quad \int_0^2 \int_0^{3-3y/2} \int_0^{1-y/2-z/3}$$

$$\int_0^3 \int_0^{2-2z/3} \int_0^{1-y/2-z/3} dx\, dy\, dz$$

Find the volume of the region D enclosed by the surfaces $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.



$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\, dy\, dx$$

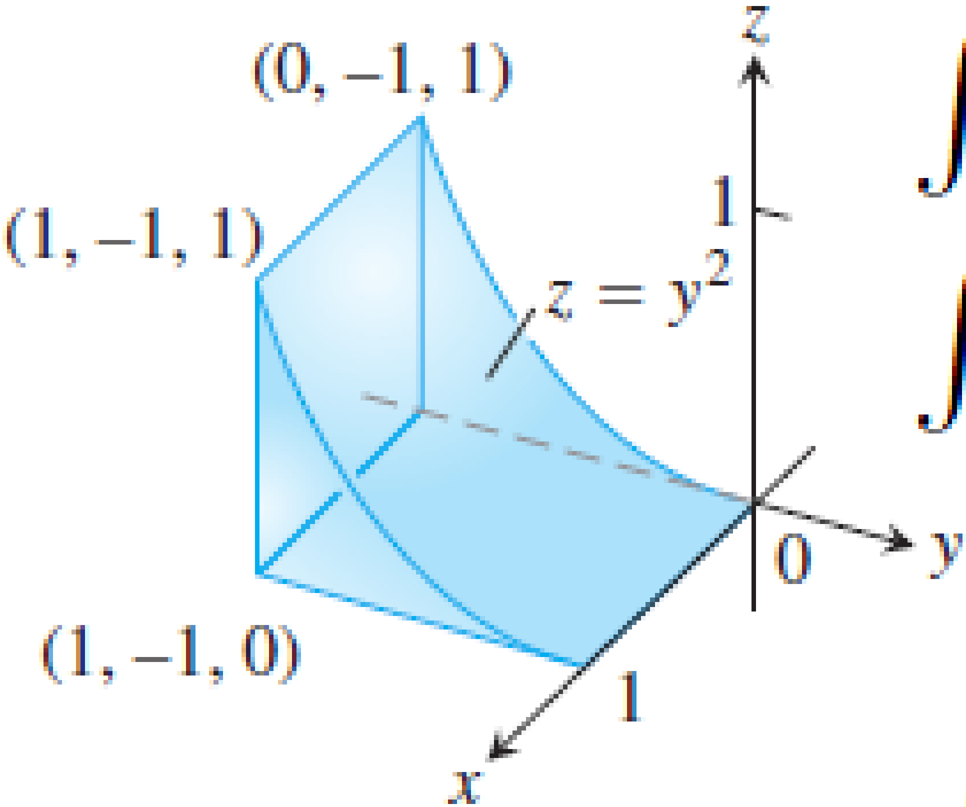
$$= \int_{-2}^{2} \int_{-\sqrt{(4-x^2)/2}}^{\sqrt{(4-x^2)/2}} (8 - 2x^2 - 4y^2)\, dy\, dx$$

$$= \int_{-2}^{2} \left[(8 - 2x^2)y - \frac{4}{3} y^3 \right]_{y=-\sqrt{(4-x^2)/2}}^{y=\sqrt{(4-x^2)/2}} dx$$

$$= \int_{-2}^{2} \left(2(8 - 2x^2)\sqrt{\frac{4 - x^2}{2}} - \frac{8}{3} \left(\frac{4 - x^2}{2} \right)^{3/2} \right) dx$$

$$= \int_{-2}^{2} \left[8\left(\frac{4 - x^2}{2} \right)^{3/2} - \frac{8}{3} \left(\frac{4 - x^2}{2} \right)^{3/2} \right] dx = \frac{4\sqrt{2}}{3} \int_{-2}^{2} (4 - x^2)^{3/2}\, dx$$

$= 8\pi\sqrt{2}.$ After integration with the substitution $x = 2 \sin u$.



$$\int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy\, dz\, dx$$

$$\int_{-1}^0 \int_0^{y^2} \int_0^1 dx\, dz\, dy$$

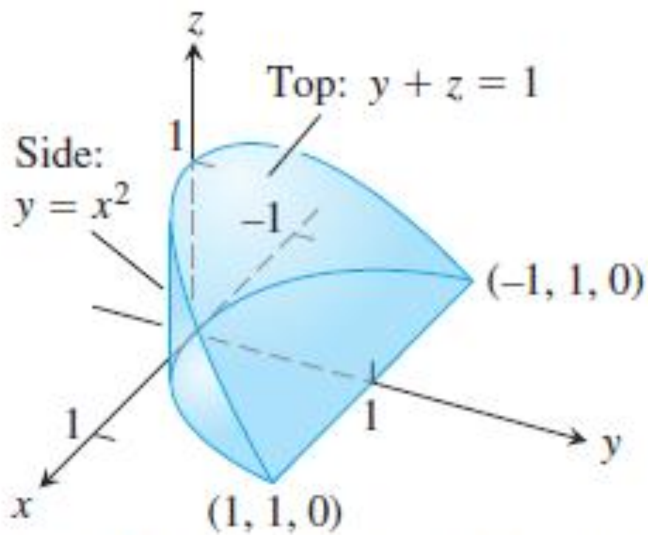
$$\int_0^1 \int_0^1 \int_{-1}^{-\sqrt{z}} dy\, dx\, dz$$

$$\int_{-1}^0 \int_0^1 \int_0^{y^2} dz\, dx\, dy$$

$$\int_0^1 \int_{-1}^0 \int_0^{y^2} dz\, dy\, dx.$$

Here is the region of integration of the integral

$$\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} dz\, dy\, dx\,.$$



Rewrite the integral as an equivalent iterated integral in the order

a. *dy dz dx* **b.** *dy dx dz*

c. *dx dy dz* **d.** *dx dz dy*

e. *dz dx dy*.

(a) $\int_{-1}^{1}\int_{0}^{1-x^2}\int_{x^2}^{1-z} dy\,dz\,dx$

(b) $\int_{0}^{1}\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_{x^2}^{1-z} dy\,dx\,dz$

(c) $\int_{0}^{1}\int_{0}^{1-z}\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy\,dz$

(d) $\int_{0}^{1}\int_{0}^{1-y}\int_{-\sqrt{y}}^{\sqrt{y}} dx\,dz\,dy$

(e) $\int_{0}^{1}\int_{-\sqrt{y}}^{\sqrt{y}}\int_{0}^{1-y} dz\,dx\,dy$