14.1 Functions of Several Variables
• A real-world phenomenon usually depends on two or more independent variables.

• We need to extend the basic ideas of functions of a single variable to functions of several variables.

**Evaluating a Function**

The value of \( f(x, y, z) = \sqrt{x^2 + y^2 + z^2} \) at the point \((3, 0, 4)\) is

\[
f(3, 0, 4) = \sqrt{(3)^2 + (0)^2 + (4)^2} = \sqrt{25} = 5.
\]
Domains and Ranges

- Avoid complex numbers or division by zero

<table>
<thead>
<tr>
<th>Function</th>
<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w = \sqrt{y - x^2} )</td>
<td>( y \geq x^2 )</td>
<td>([0, \infty))</td>
</tr>
<tr>
<td>( w = \frac{1}{xy} )</td>
<td>( xy \neq 0 )</td>
<td>((-\infty, 0) \cup (0, \infty))</td>
</tr>
<tr>
<td>( w = \sin xy )</td>
<td>Entire plane</td>
<td>([-1, 1])</td>
</tr>
</tbody>
</table>

(b) Functions of Three Variables

<table>
<thead>
<tr>
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<th>Domain</th>
<th>Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w = \sqrt{x^2 + y^2 + z^2} )</td>
<td>Entire space</td>
<td>([0, \infty))</td>
</tr>
<tr>
<td>( w = \frac{1}{x^2 + y^2 + z^2} )</td>
<td>((x, y, z) \neq (0, 0, 0))</td>
<td>((0, \infty))</td>
</tr>
<tr>
<td>( w = xy \ln z )</td>
<td>Half-space ( z &gt; 0 )</td>
<td>((-\infty, \infty))</td>
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Graphs, Level Curves, and Contours of Functions of Two Variables

DEFINITIONS  Level Curve, Graph, Surface

The set of points in the plane where a function \( f(x, y) \) has a constant value \( f(x, y) = c \) is called a level curve of \( f \). The set of all points \( (x, y, f(x, y)) \) in space, for \( (x, y) \) in the domain of \( f \), is called the graph of \( f \). The graph of \( f \) is also called the surface \( z = f(x, y) \).
Graphing a Function of Two Variables

Graph \( f(x, y) = 100 - x^2 - y^2 \) and plot the level curves \( f(x, y) = 0 \), \( f(x, y) = 51 \), and \( f(x, y) = 75 \) in the domain of \( f \) in the plane.

The surface \( z = f(x, y) = 100 - x^2 - y^2 \) is the graph of \( f \).

FIGURE 14.4 The graph and selected level curves of the function
\( f(x, y) = 100 - x^2 - y^2 \)
Functions of Three Variables

**DEFINITION**  
**Level Surface**  
The set of points \((x, y, z)\) in space where a function of three independent variables has a constant value \(f(x, y, z) = c\) is called a level surface of \(f\).

Describe the level surfaces of the function

\[
f(x, y, z) = \sqrt{x^2 + y^2 + z^2}
\]

- \(\sqrt{x^2 + y^2 + z^2} = 1\)
- \(\sqrt{x^2 + y^2 + z^2} = 2\)
- \(\sqrt{x^2 + y^2 + z^2} = 3\)
Modeling Temperature Beneath Earth’s Surface

\[ w = \cos \left( 1.7 \times 10^{-2} t - 0.2x \right) e^{-0.2x} . \]
Computer-generated graphs and level surfaces of typical functions of two variables.

(a) $z = e^{-\frac{(x^2 + y^2)}{8}} \sin x^2 + \cos y^2$

(b) $z = \sin x + 2\sin y$
(c) \( z = (4x^2 + y^2)e^{-x^2-y^2} \)

(d) \( z = xye^{-y^2} \)
\[ z = (\cos x)(\cos y) e^{-\sqrt{x^2 + y^2}/4} \]

\[ z = -\frac{xy^2}{x^2 + y^2} \]
c. 

\[ z = \frac{1}{4x^2 + y^2} \]
14.2  Limits and Continuity in Higher Dimensions

**THEOREM 1**  Properties of Limits of Functions of Two Variables

The following rules hold if $L$, $M$, and $k$ are real numbers and

$$
\lim_{(x,y) \to (x_0, y_0)} f(x, y) = L \quad \text{and} \quad \lim_{(x,y) \to (x_0, y_0)} g(x, y) = M.
$$

1. **Sum Rule:**
   $$\lim_{(x,y) \to (x_0, y_0)} (f(x, y) + g(x, y)) = L + M$$

2. **Difference Rule:**
   $$\lim_{(x,y) \to (x_0, y_0)} (f(x, y) - g(x, y)) = L - M$$

3. **Product Rule:**
   $$\lim_{(x,y) \to (x_0, y_0)} (f(x, y) \cdot g(x, y)) = L \cdot M$$

4. **Constant Multiple Rule:**
   $$\lim_{(x,y) \to (x_0, y_0)} (kf(x, y)) = kL \quad \text{any number } k$$

5. **Quotient Rule:**
   $$\lim_{(x,y) \to (x_0, y_0)} \frac{f(x, y)}{g(x, y)} = \frac{L}{M} \quad M \neq 0$$

6. **Power Rule:** If $r$ and $s$ are integers with no common factors, and $s \neq 0$, then
   $$\lim_{(x,y) \to (x_0, y_0)} (f(x, y))^{r/s} = L^{r/s}$$

provided $L^{r/s}$ is a real number. (If $s$ is even, we assume that $L > 0$.)
Continuity

As with functions of a single variable, continuity is defined in terms of limits.

**DEFINITION  Continuous Function of Two Variables**

A function $f(x, y)$ is continuous at the point $(x_0, y_0)$ if

1. $f$ is defined at $(x_0, y_0)$,
2. $\lim_{(x, y) \to (x_0, y_0)} f(x, y)$ exists,
3. $\lim_{(x, y) \to (x_0, y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.
• The calculus of several variables is basically single-variable calculus applied one at a time.

• Hold all but one of the independent variables constant and differentiate with respect to that one variable, we get a “partial” derivative.
Partial Derivatives of a Function of Two Variables

If \((x_0, y_0)\) is a point in the domain of a function \(f(x, y)\), the vertical plane \(y = y_0\) will cut the surface \(z = f(x, y)\) in the curve \(z = f(x, y_0)\) (Figure 14.13). This curve is the graph

\[
\begin{align*}
\frac{\partial f}{\partial x} \bigg|_{(x_0, y_0)} &= \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},
\end{align*}
\]

provided the limit exists.
The curve $z = f(x, y_0)$ in the plane $y = y_0$.

Tangent line

$P(x_0, y_0, f(x_0, y_0))$

Vertical axis in the plane $y = y_0$

$z = f(x, y)$

Horizontal axis in the plane $y = y_0$. 

$(x_0, y_0)$

$(x_0 + h, y_0)$
The notation for a partial derivative depends on what we want to emphasize:

\[ \frac{\partial f}{\partial x}(x_0, y_0) \text{ or } f_x(x_0, y_0) \]  
“Partial derivative of \( f \) with respect to \( x \) at \((x_0, y_0)\)” or “\( f \) sub \( x \) at \((x_0, y_0)\).” Convenient for stressing the point \((x_0, y_0)\).

\[ \frac{\partial z}{\partial x} \bigg|_{(x_0, y_0)} \]  
“Partial derivative of \( z \) with respect to \( x \) at \((x_0, y_0)\).” Common in science and engineering when you are dealing with variables and do not mention the function explicitly.

\[ f_x, \frac{\partial f}{\partial x}, z_x, \text{ or } \frac{\partial z}{\partial x} \]  
“Partial derivative of \( f \) (or \( z \)) with respect to \( x \).” Convenient when you regard the partial derivative as a function in its own right.
The definition of the partial derivative of \( f(x, y) \) with respect to \( y \) at a point \((x_0, y_0)\) is similar to the definition of the partial derivative of \( f \) with respect to \( x \). We hold \( x \) fixed at the value \( x_0 \) and take the ordinary derivative of \( f(x_0, y) \) with respect to \( y \) at \( y_0 \).

**DEFINITION**  \textbf{Partial Derivative with Respect to \( y \)}

The partial derivative of \( f(x, y) \) with respect to \( y \) at the point \((x_0, y_0)\) is

\[
\frac{\partial f}{\partial y} \bigg|_{(x_0, y_0)} = \frac{d}{dy} f(x_0, y) \bigg|_{y=y_0} = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},
\]

provided the limit exists.

The slope of the curve \( z = f(x_0, y) \) at the point \( P(x_0, y_0, f(x_0, y_0)) \) in the vertical plane \( x = x_0 \) (Figure 14.14) is the partial derivative of \( f \) with respect to \( y \) at \((x_0, y_0)\). The tangent line to the curve at \( P \) is the line in the plane \( x = x_0 \) that passes through \( P \) with this slope. The partial derivative gives the rate of change of \( f \) with respect to \( y \) at \((x_0, y_0)\) when \( x \) is held fixed at the value \( x_0 \). This is the rate of change of \( f \) in the direction of \( j \) at \((x_0, y_0)\).

The partial derivative with respect to \( y \) is denoted the same way as the partial derivative with respect to \( x \):

\[
\frac{\partial f}{\partial y} (x_0, y_0), \quad f_y(x_0, y_0), \quad \frac{\partial f}{\partial y}, \quad f_y.
\]

Notice that we now have two tangent lines associated with the surface \( z = f(x, y) \) at
FIGURE 14.15  Figures 14.13 and 14.14 combined. The tangent lines at the point \((x_0, y_0, f(x_0, y_0))\) determine a plane that, in this picture at least, appears to be tangent to the surface.
EXAMPLE 1  Finding Partial Derivatives at a Point

Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(4, -5)$ if

$$f(x, y) = x^2 + 3xy + y - 1.$$  

Solution  To find $\frac{\partial f}{\partial x}$, we treat $y$ as a constant and differentiate with respect to $x$:

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (x^2 + 3xy + y - 1) = 2x + 3 \cdot 1 \cdot y + 0 - 0 = 2x + 3y.$$  

The value of $\frac{\partial f}{\partial x}$ at $(4, -5)$ is $2(4) + 3(-5) = -7$.

To find $\frac{\partial f}{\partial y}$, we treat $x$ as a constant and differentiate with respect to $y$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (x^2 + 3xy + y - 1) = 0 + 3 \cdot x \cdot 1 + 1 - 0 = 3x + 1.$$  

The value of $\frac{\partial f}{\partial y}$ at $(4, -5)$ is $3(4) + 1 = 13$.

EXAMPLE 2  Finding a Partial Derivative as a Function

Find $\frac{\partial f}{\partial y}$ if $f(x, y) = y \sin xy$.

Solution  We treat $x$ as a constant and $f$ as a product of $y$ and $\sin xy$:

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (y \sin xy) = y \frac{\partial}{\partial y} \sin xy + (\sin xy) \frac{\partial}{\partial y} (y)$$

$$= (y \cos xy) \frac{\partial}{\partial y} (xy) + \sin xy = xy \cos xy + \sin xy.$$
EXAMPLE 4  Implicit Partial Differentiation

Find $\frac{\partial z}{\partial x}$ if the equation

$$yz - \ln z = x + y$$

defines $z$ as a function of the two independent variables $x$ and $y$ and the partial derivative exists.

Solution  We differentiate both sides of the equation with respect to $x$, holding $y$ constant and treating $z$ as a differentiable function of $x$:

$$\frac{\partial}{\partial x} (yz) - \frac{\partial}{\partial x} \ln z = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial x}$$

$$y \frac{\partial z}{\partial x} - \frac{1}{z} \frac{\partial z}{\partial x} = 1 + 0$$

$$+ y^2$$

$$\left( y - \frac{1}{z} \right) \frac{\partial z}{\partial x} = 1$$

With $y$ constant,

$$\frac{\partial}{\partial x} (yz) = y \frac{\partial z}{\partial x}.$$
EXAMPLE 5  Finding the Slope of a Surface in the $y$-Direction

The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$ in a parabola. Find the slope of the tangent to the parabola at $(1, 2, 5)$ (Figure 14.16).

Solution  The slope is the value of the partial derivative $\frac{\partial z}{\partial y}$ at $(1, 2)$:

$$\left. \frac{\partial z}{\partial y} \right|_{(1,2)} = \left. \frac{\partial}{\partial y} (x^2 + y^2) \right|_{(1,2)} = 2y \bigg|_{(1,2)} = 2(2) = 4.$$

$$\left. \frac{dz}{dy} \right|_{y=2} = \left. \frac{d}{dy} (1 + y^2) \right|_{y=2} = 2y \bigg|_{y=2} = 4.$$
Finding Second-Order Partial Derivatives

If \( f(x, y) = x \cos y + ye^x \), find

\[
\frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial^2 f}{\partial y \partial x}, \quad \frac{\partial^2 f}{\partial y^2}, \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}.
\]

Solution

\[
\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (x \cos y + ye^x)
\]

\[= \cos y + ye^x\]

So

\[
\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = -\sin y + e^x
\]

\[
\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = ye^x.
\]

\[
\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (x \cos y + ye^x)
\]

\[= -x \sin y + e^x\]

So

\[
\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = -\sin y + e^x
\]

\[
\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = -x \cos y.
\]

23
The Mixed Derivative Theorem

You may have noticed that the “mixed” second-order partial derivatives

$$\frac{\partial^2 f}{\partial y \partial x} \quad \text{and} \quad \frac{\partial^2 f}{\partial x \partial y}$$

in Example 9 were equal. This was not a coincidence. They must be equal whenever $f$, $f_x$, $f_y$, $f_{xy}$, and $f_{yx}$ are continuous, as stated in the following theorem.

**THEOREM 2**  The Mixed Derivative Theorem

If $f(x, y)$ and its partial derivatives $f_x$, $f_y$, $f_{xy}$, and $f_{yx}$ are defined throughout an open region containing a point $(a, b)$ and are all continuous at $(a, b)$, then

$$f_{xy}(a, b) = f_{yx}(a, b).$$
EXAMPLE 10  Choosing the Order of Differentiation

Find $\frac{\partial^2 w}{\partial x \partial y}$ if

$$w = xy + \frac{e^y}{y^2 + 1}.$$  

Solution  The symbol $\frac{\partial^2 w}{\partial x \partial y}$ tells us to differentiate first with respect to $y$ and then with respect to $x$. If we postpone the differentiation with respect to $y$ and differentiate first with respect to $x$, however, we get the answer more quickly. In two steps,

$$\frac{\partial w}{\partial x} = y \quad \text{and} \quad \frac{\partial^2 w}{\partial y \partial x} = 1.$$  

If we differentiate first with respect to $y$, we obtain $\frac{\partial^2 w}{\partial x \partial y} = 1$ as well.