THE FUNCTION CONCEPT: COMPREHENSION AND COMPLICATION

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In this paper, we contrast the mathematical simplicity of the function concept that is appreciated by some students and the spectrum of cognitive complications that most students have in coping with the function definition in its many representations. Our data is based on interviews with nine (17-year old) students selected as a cross-section from 114 responses to a questionnaire. We distinguish four categories in a spectrum from those who have a simple grasp of the core function concept applicable to the full range of representations to those who see only complicated details in different contexts without any overall grasp of the conceptual structure.

INTRODUCTION

The concept of function is one of the most fundamental concepts in mathematics, which appears from primary school through to university. Extensive research on functions investigates the topic using different theoretical frameworks. Tall & Vinner (1981) make a distinction between concept definition (the 'form of words used to specify that concept' (p. 152)) and the concept image ('the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes' (p. 152)). Vinner (1983) categorized students' concept images which conflict with the formal definition e.g. relationships with 'reasonable graphs', including those that are physically 'continuous' or 'smooth', given by one rule, whilst rejecting those given by more than one rule or having unfamiliar graphs. Sfard (1991) makes a distinction between operational and structural conception of a function in which finding the value of a function for each input of x corresponds to the operational conception while considering a graph as an integrated whole, as an object, corresponds to the structural conception of the function concept. She suggests that a structural conception of functions is difficult to achieve. Similarly, Breidenbach *et al*. (1992) distinguish between action and process conception of function. Action requires an explicit recipe or formula in a step-by-step manner while process requires a grasp of the whole without the details of each step. Other research (e.g. Kaput, 1992; Keller & Hirsch, 1998; Yerushalmy, 1991) focuses on the multiple representations of functions especially with the availability of computers and graphical calculators. In this research, students' flexibility among different representations is seen as an indication of a better understanding of the function concept.

THEORETICAL FRAMEWORK

Our theoretical framework starts with the notion of core concept of function as defined by Thompson (1994) and its relationship to other aspects of the concept image:

> …the core concept of "function" is not represented by any of what are commonly called the multiple representations of function, but instead our making connections among representational activities produces a subjective sense of invariance … it may be wrongheaded to focus on graphs, expressions, or tables as representations of function. We should instead focus on them as representations of something that, from students' perspective, is representable, such as aspects of a specific situation. (Thompson, 1994, p. 39)

Thompson (1994) claims that if students do not see something remain the same as they move among different representations then they see each representation as a topic to be learnt in isolation.

Our theoretical framework contrasts the *mathematical simplicity* of the core concept and the *cognitive complication* that arises for students who fail to make sense of the core concept. The core concept simply specifies two sets, in which each element in the first is assigned to a unique element in the second. However, though the idea is mathematically simple, it has a generative power which gives rise to highly complex ideas which prove to be cognitively complicated for students who focus on the variety of complications arising from different aspects of different representations.

BACKGROUND OF THE STUDY

This study was conducted in a Turkish context where the function topic is taught in a more formal way. In the textbooks, the definition is given as follows:

> Let *A* and *B* be two non-empty sets. A relation *f* from *A* to *B* is called a function if it assigns every element in *A* to a unique element in *B* (Demiralp *et al*., 2000, our translation).

This definition is followed by an explanation as follows:

> A function *f* defined from *A* to *B* assigns:
>
> 1. All elements in *A* to elements in *B*.
>
> 2. Every element in *A* to a unique element in *B*.

This will be called the colloquial definition. In the textbooks, it is explained in a prototypical way using a set-correspondence diagram as follows:

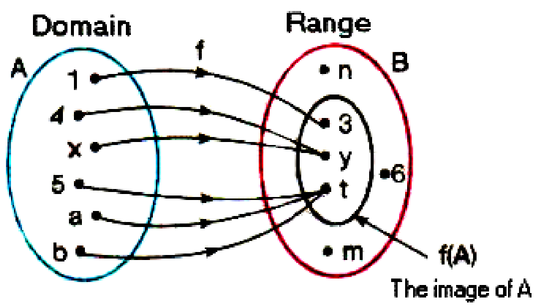
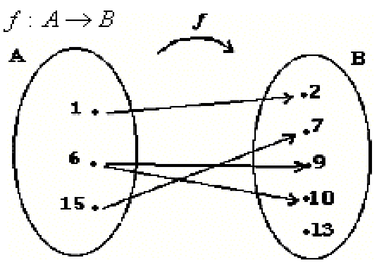
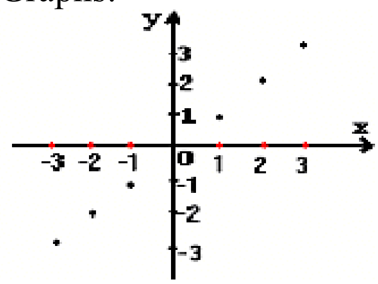
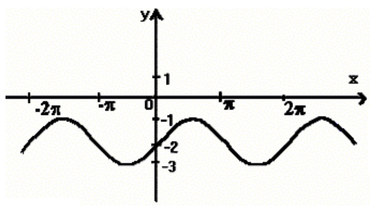


Figure 1. A visual explanation of the function definition

Introducing the function topic with a colloquial definition, the teaching focuses successively on different aspects of functions such as sets of ordered pairs, graphs and expressions. Some aspects (such as the visual set theoretic representation) are presented as prototypes, and some (such as graphs and expressions) are based on exemplars. Rosch (1975) explains a prototype in terms of the ‘clearest cases’ or ‘best examples’ or people’s judgements of goodness of membership in the category (Rosch, 1978). On the other hand, exemplars are more specific cases and are often seen as clusters (Ross & Makin, 1999). For instance, trigonometric functions and logarithmic functions are two different clusters. As will be discussed later on, this distinction has important implications for the categorization of student conceptions.

METHODOLOGY

The data in this study is obtained from semi-structured interviews. The sample is nine students in grade 3 (17 year-old) in two high schools in Turkey. The students have been studying functions for over two years since their introduction to the colloquial definition in grade 1 of high school. The students were selected by a questionnaire among a hundred and fourteen students from different subject groups (maths & science, Turkish & maths, social subjects). All students were presented four different aspects of functions as follows:

Set-correspondence diagram:	Expression:	Graphs:
	$f: R \to R$ $f(x)=\begin{cases} 1, & \text{if } x^2-2x+1>0 \\ 0, & \text{if } x^2-2x+1=0 \\ -1, & \text{if } x^2-2x+1<0 \end{cases}$	
Set of ordered pairs: $A=\{1,2,3,4\}$ $f: A \to R$, $f=\{(1,1),(1,2),(2,2),(3,3),(4,3)\}$	$f: R \to R$ $f(x)=\sin x-2$	

Students were then asked the following question:

Is this a function? ... Can you explain why?

Follow-up questions were asked according to the explanations given by the students.

ANALYSIS OF THE DATA

Students spoken responses to the above questions were transcribed and analyzed with their written responses. The aim of the analysis is to categorize their responses to distinguish students who could handle the simplicity and complexity of the core concept of function from those for whom function concept is cognitively complicated. To do the analysis, the following categories of responses were distinguished:

- Colloquial definition (CD): The use of the colloquial definition.
- Colloquial definition wrongly used (CDW): Either recalling the colloquial definition wrongly (e.g. saying that two elements in the domain can be assigned to the same element in the range) or using it in a wrong way (missing out that one element in the domain is not assigned to any element in the range).
- Exemplar-based focus (EBF): Recalling specific examples e.g. recognizing the expression $f(x) = \sin x - 2$ as a trigonometric function without any reference to the definitional properties. Responses that focus on the visual hints from the graphs, the notational hints such as existence of ‘ $f(x)$ ’ and the general appearances of the graphs are also considered in this category.
- Vertical line test (VLT): Drawing vertical lines through the graph.
- Set diagram (SD): Drawing a set diagram to decide whether or not the given item is a function.
- Graph (GR): Drawing the graph of the given item.
- Wrong graph (WGR): Drawing the wrong graph for the given item.
- Domain-range confusion (DRC): Considering the domain as the range of the function or vice versa.
- Other (OTH)

To be able to compare students’ responses and to categorize them, these responses are put into a grid. The labels “CD”, “CD with another response (e.g. SD)” and “CDW” are marked in different scales of the grey colour so that as it gets bolder it indicates a stronger focus on the definitional properties:

The grid is presented below:

		Ali	**A**hmet	**A**ysel	**A**rif	**B**elma	**B**elgin	**C**em	**D**eniz	**D**emet
SET DIAGRAM		**CD**	**CD**	**CD**	**CD**	**CD**	**CD**	**CDW**	**EBF**	**EBF**
ORDERED PAIRS		**CDW CD**	**CD SD**	**CD**	**CD SD**	**CD**	**———**	**CDW**	**EBF**	**OTH**
GRAPHS	Points on a line	**CD**	**CD**	**CD VLT**	**CD**	**EBF**	**CDW**	**OTH**	**EBF**	**EBF**
	The graph of f(x)=sinx–2	**CD**	**CD VLT SD**	**CD**	**OTH**	**EBF**	**EBF**	**EBF**	**EBF**	**EBF**
EXPRES-SIONS	Signum function	**EBF GR SD**	**EBF GR VLT**	**EBF WGR**	**EBF**	**EBF**	**DRC**	**EBF**	**OTH**	**OTH**
	f(x)=sinx-2	**CD**	**EBF-CD**	**CD**	**OTH**	**———**	**EBF**	**———**	**EBF**	**WGR**

Table 1. CD: Colloquial Definition; CDW: Colloquial definition wrongly used; EBF: Exemplar-Based Focus; SD: Set Diagram; VLT: Vertical Line Test; GR: Graph; WGR: Wrong Graph; DR: Domain-Range Confusion; OTH: Other; ---: No Response

CATEGORIZATION OF STUDENTS' RESPONSES

The analysis of the responses from the nine students in the interview revealed a spectrum of performances as summarized in the grid in table 1 above. Four categories were distinguished. The first contains four students (Ali, Ahmet, Ayseel, Arif) who could focus on the simplicity of the core concept of function using the definitional properties for all aspects of functions in a coherent way. The second contains two students (Belma and Belgin) who could focus on the definitional properties only for the set-correspondence diagram and the set of ordered pairs. They gave complicated responses for the graphs and expressions with Belma giving mainly exemplar-based explanations. For example, she rejected the graph of $f(x) = \sin x - 2$ as a function since the graph passes only through the y-axis but not the x-axis. Belgin considered $f(x) = \sin x - 2$ as a function because the general appearance of the graph increases and decreases. The third category consists of one student (Cem) who used the colloquial definition wrongly for the set-correspondence diagram and the set of ordered pairs and focused on irrelevant properties of graphs and expressions. He could not remember the colloquial definition correctly. He considered the set-correspondence diagram as a function. He said that 6 in the domain could be assigned to two elements but not three elements. In the fourth category, there are two students (Deniz and Demet) who gave very complicated responses to all different aspects of functions, most of which were exemplar-based. Unlike students in the other categories they did not refer to the colloquial definition even for the set-correspondence diagram and set of ordered pairs. For instance, they rejected the set-correspondence diagram because, unlike the picture in figure 1, the arrows cross each other. This shows the subtle coercion of the concept image with unintended incidental properties that are noticed implicitly by the students.

CONCLUSION

Students in the interviews treated various aspects of functions in cognitively different ways. They dealt with set-correspondence diagrams and set of ordered pairs as prototypes and graphs and expressions as exemplars. The students in the second category used the colloquial definition for set-correspondence diagrams and set of ordered pairs. These two aspects cause less complication which we interpret as being a consequence of the fact that the set-correspondence diagram and set of ordered pairs are themselves prototypes (although with certain limitations, for instance the finite nature of the sets pictured). The exemplars, graphs and expressions, caused more complication. Only successful students coped with the complexity of the function concept in all different contexts and could handle the possible cognitive complications by applying the colloquial definition to each context.

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