PRE-SERVICE AND IN-SERVICE MATHEMATICS TEACHERS' CONCEPT IMAGES OF RADIAN

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This study investigates pre-service and in-service mathematics teachers' subject knowledge of radian. Subject knowledge is investigated under the theoretical frameworks of concept images and cognitive units. Qualitative and quantitative research methods were designed for this study. Thirty seven pre-service and fourteen in-service mathematics teachers' completed a questionnaire which aims to assess their understanding of radian. Three pre-service and one in-service teachers were selected for individual interviews on the basis of theoretical sampling. The data indicated that participants' concept images of radian were dominated by concept images of degree.

INTRODUCTION

Trigonometry is one of the topics in mathematics education research which did not receive enough attention. As stated by Fi (2003), much of the literature on trigonometry has focused on trigonometric functions (Even, 1989; Even, 1990; Bolte, 1993; Howald, 1998). Other studies focus on the learning and teaching of trigonometry and trigonometric functions with computers and calculators (Blacket & Tall, 1991; Wenzelbuer, 1992; Silva, 1994; Lobo da Costa & Magina, 1998). A few researchers studied more specific issues in trigonometry such as simplification of trigonometric expressions (Delice, 2002) and relationship between trigonometry and forces in physics (Doerr & Confrey, 1994).

There is little research on teachers' understanding of trigonometry (Doerr, 1996; Fi, 2003). This study focuses on pre-service and in-service mathematics teachers' understanding of a specific concept in trigonometry, namely the radian. Fi (2003) found that although pre-service mathematics teachers were successful with conversation between radians and degrees, none of them was able to accurately define the radian measure as a ratio of two lengths: the length of the arc of a central angle of a circle and the radius of the circle.

THEORETICAL FRAMEWORK

The subject knowledge of pre-service and in-service mathematics teachers' concept images of radian was investigated under the theoretical frameworks of concept images and cognitive units. Tall & Vinner (1981) introduce the notions of concept definition and concept image and makes a distinction between the two. They define concept definition as the 'form of words used to specify that concept' (p. 152). A formal concept definition is one accepted by the mathematical community at large. As Tall & Vinner (1981) assert, we can use mathematical concepts without knowing the formal definitions. To explain how this occurs, they define concept image as 'the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes' (p. 152). They assert that it is built up over the years by experience, and that different stimuli at different times can activate different parts of the concept image developing them in a way which need not be a coherent whole. Vinner (1992) asserts that specific individuals create idiosyncratic images and also the same individual might react differently to a concept encountered within different situations.

A chunk of the concept image in which an individual can consciously focus attention at a time is called a *cognitive unit* by Barnard & Tall (1997). A cognitive unit has also links (many of which are unconscious) to the other parts of our cognitive structure. Cognitive units can be symbols, representations, theorems, properties. The authors claim that powerful mathematical thinking requires compressing the information to fit into cognitive units and making connections between them. The authors also claim that 'what is a cognitive unit for one individual may not be a cognitive unit for another' (p. 41).

Under these theoretical frameworks, this study tries to answer the following research questions:

- What kind of concept images of radian do participants have?
- What are the sources of such concept images?

METHODOLOGY

To be able to answer the research questions above, qualitative and quantitative research methods were designed for this study. The data was collected using questionnaires and semi-structured interviews. Thirty seven pre-service and fourteen in-service mathematics teachers' completed a questionnaire which aims to assess their understanding of radian. Three pre-service and one in-service teacher were selected for individual interviews on the basis of theoretical sampling. As Mason (1996) asserts theoretical sampling means selecting a sample on the basis of their relevance to the research questions and theoretical positions to be able to build in certain characteristics or criteria which help to develop and test the theory and explanation. Therefore, two participants (one pre-service and one-in-service teacher who are represented by R1 and R2) were selected to have stronger concept images of radian and two participants (who are both pre-service teachers and are represented by the D1 and D2) were selected to have strong concept images of degree.

Research instruments

The aim of the questionnaire was to reveal concept images of radian. Participants were asked to evaluate the values of outputs of trigonometric functions when the inputs are given and visa versa.

The aim of the interview was to investigate the concept images in detail and the sources of these concept images. We are particularly interested in the role of right triangle and unit circle as cognitive units in the concept images. The interviews were video-taped and were transcribed. Interviews were semi-structured and included three questions. First, participants were asked to prepare a concept map of trigonometry using little post-it papers. They were asked to draw the concept maps after they finished organising the concept maps. They were also asked follow-up questions about their concept maps. Second, they were asked to draw the graph of a trigonometric function. In the third question they were asked the definition of radian.

RESULTS FROM THE QUESTIONNAIRE

Participants responded to the following questions in the questionnaire:

1) $f: R \to R$ and $f(x) = x \sin x$ is given. Plot the following points on the Cartesian plane.
a) $(30, f(30)) = ?$ b) $(\pi/2, f(\pi/2)) = ?$ c) $(\pi/6, f(60)) = ?$ d) $(2, f(\pi/3)) = ?$

2) $f: R \to R$ and $f(x) = \cos x$ is given. If $f(x) = -\sqrt{3}/2$ $x = ?$

3) $f: R \to R$ and $f(x) = \sin x$ is given. If $\sin x = a \Leftrightarrow \arcsin(a) = x$ then find the following: a) $\arctan(1) = ?$ b) $\arctan(-\sqrt{3}) = ?$

The analysis of the responses to the questions indicated the following categories:

- *Correct*: Seeing 30 as a reel number and considering it as an angle in radians
- *Degree image*: Considering 30 as 30^o, but not seeing it as a reel number
- *No plotting*: Finding the values correctly without plotting the points
- *Other*: Other responses which cannot be categorised further
- *No response*: Giving no response
- *NA*: The category is not applicable for that question

The results from the questionnaire were summarised in the table below:

N=51	Correct	Degree image	No plotting	Other	No Response
1-a	3.9	90.2	0	5.9	0
1-b	62.7	23.5	3.9	9.8	0
1-c	5.9	92.2	0	0	2
1-d	62.7	25.5	2	3.9	3
2	39.2	56.9	NA	3.9	0
3	56.9	39.2	NA	0	2

Table 1. Frequencies for the questions in the questionnaire

As seen in the table above, participants' concept images of radian were dominated by the concept of degree. For instance, although it was mentioned that the function was defined from real numbers to real numbers, 90.2% of them considered "sin 30" in question 1a in degrees. In other words, they found the value for $\sin 30^0$ but did not consider 30 as a real number. Only 3.9% of them considered sin 30 in radians. Similarly, in question 2, only 39.2% of the participants found x in radians while 56.9% of them found x in degrees.

RESULTS FROM THE INTERVIEWS

The aim of the interview was to investigate the concept images in detail and the sources of these concept images. Interviews were semi-structured and included three questions. Below an account of participants' responses were presented.

Concept maps

The aim of the concept maps was to investigate the sources of participants' concept images. We were particularly interested in the place of the unit circle and the right triangle in the concept maps. Participants were also asked to explain how did they prepare their maps. The original concept maps were drawn electronically since the words were translated into English. Therefore, they should be considered as representations of participants' original concept maps.

R1 prepared the following concept map in figure 1:

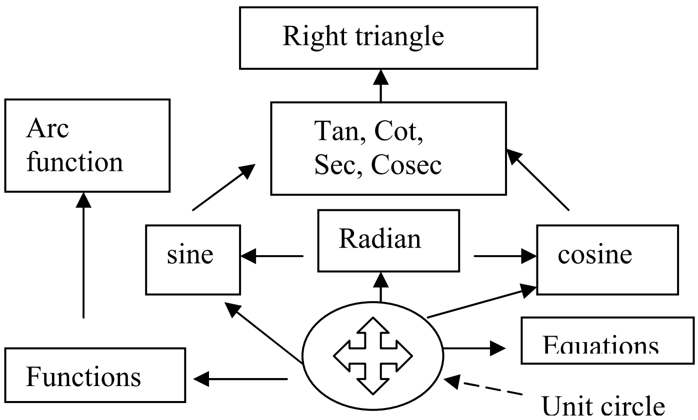

Figure 1. Concept map of R1

He mentioned that the unit circle is almost linked to every concept in trigonometry. He also stated that when he teaches trigonometry he always introduce it with the unit circle.

R2 prepared the following concept map in figure 2 below:

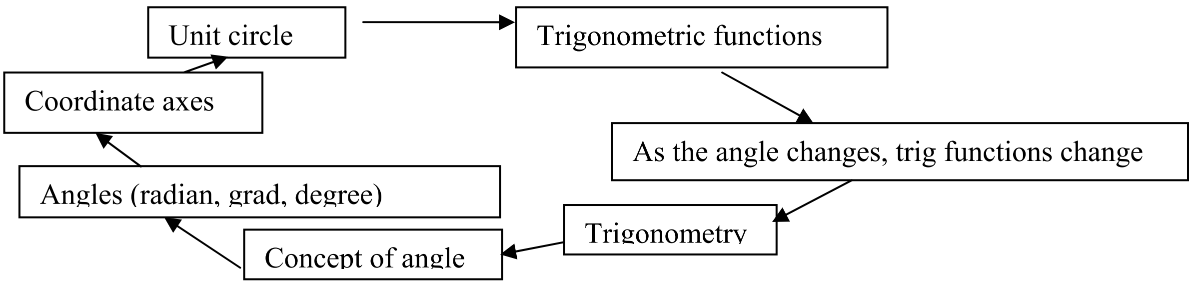

Figure 2. Concept map of R2.

He started his concept map from “trigonometry” cell in the bottom right and continued clockwise. He stated that he defines trigonometric functions using the unit circle. When he was asked where he would put the “right angle” in his concept map, he said that “it is like your name which is obvious to you. It is no where but actually everywhere”.

D1 prepared the following concept map:

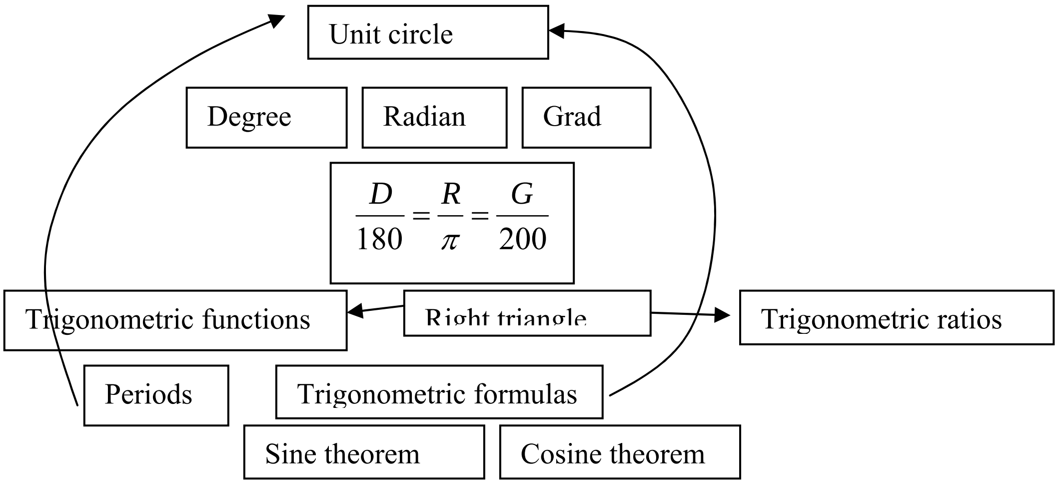

Figure 3. Concept map of D1.

She started her concept map with the unit circle on the top. When she was explaining trigonometric functions, she defined the sine, cosine, tangent and cotangent using the right triangle. However, when she was explaining the periods of these functions and where the sine and cosine are positive and negative, she used the unit circle.

D2 prepared a hierarchical concept map as follows:

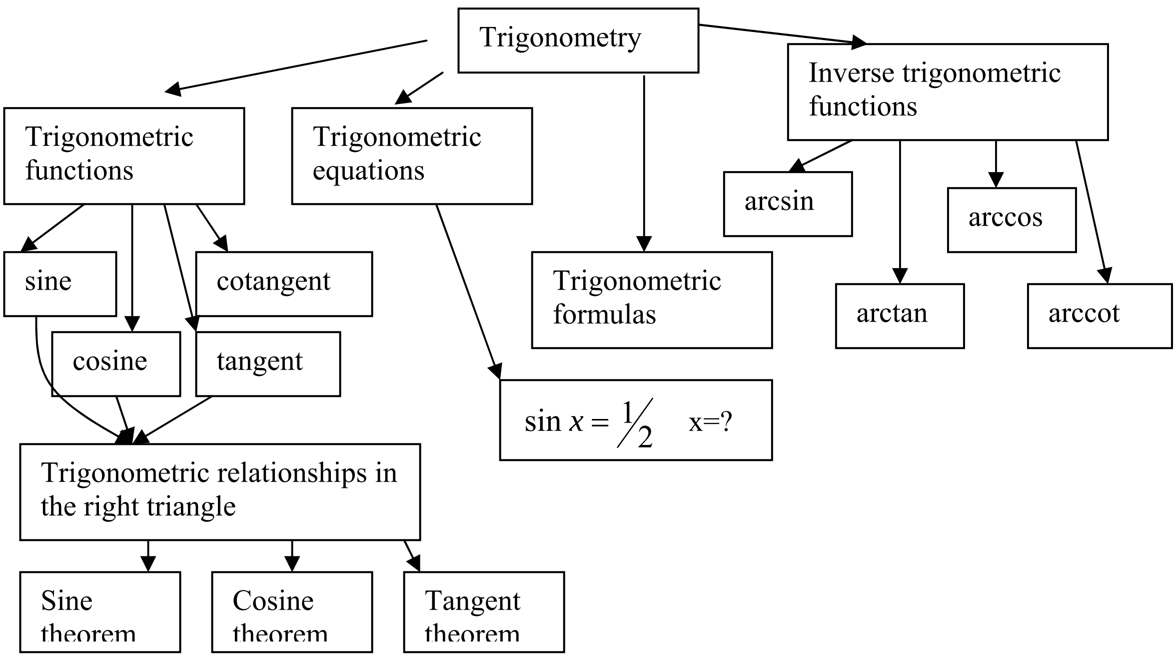

Figure 4. Concept map of D2.

Unit circle did not appear in the concept map of D2. She related the trigonometric functions (sine, cosine, tangent, cotangent) to the right triangle.

As can be interpreted from the concept maps of the participants, unit circle had richer connections in the concept maps of R1 and R2 (who have stronger concept image of radian) while right triangle exists in the concept maps of D1 and D2.

The graph of sine function

In the interview, participants were secondly asked the following question after they drew the graph of sine function:

> *Suppose you are teaching your students how to draw the graph of sine function and suppose you plotted the points* $(0,0)$ *and* $(\pi/2, 1)$ *then drew the graph in that interval. How would you explain if one of your students asks you why you did not draw the graph straight?*

R1 drew the graph correctly but did not explain why it not straight using the unit circle. He just stated that the sine function is not a linear function.

R2 explained it using the unit circle:

R2: I would use the unit circle. First I look at the value of sine at zero, then bigger angles. As I go towards ninety degrees my vector (along the y-axis) gets bigger. However, it gets bigger faster first, until forty-five degrees. After that it gets bigger slower. That's why it is not straight.

D1 also made use of the unit circle. She referred to a computer software as shown below and explained that the values do not grow proportionally:

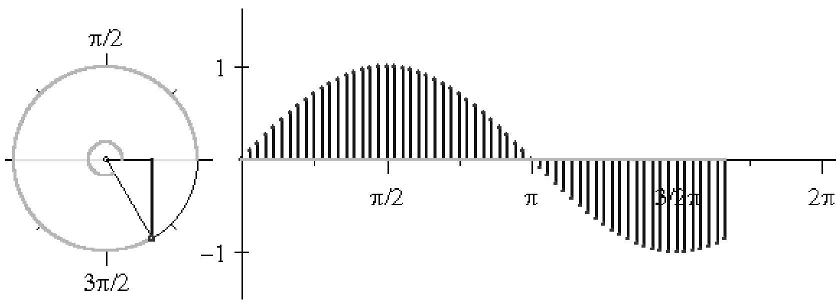

Figure 5. Picture from the computer screen explaining the graph of sine function

D2 could not explain why the graph was not straight. She first said that the function was increasing; therefore it could not be straight. When she was asked the reason why, she mentioned that straight lines are not increasing but functions such as sine and cosine are increasing and decreasing. She said that

D2: This is an increasing function. When it is increasing or decreasing it cannot be a line. Because an equation for a line is like $y = x + 2$. When you substitute values for x then y is constant. But functions like sine and cosine are decreasing and increasing. That is why the graph cannot be straight. If the graph is towards up, it is increasing. If it is going down, it is decreasing.

As seen from her response she could not relate the drawing of the graph to the unit circle.

Personal concept definition of radian

When the participants were asked the definition of radian, the equation $\frac{D}{180} = \frac{R}{\pi}$ acted as a cognitive unit. Their personal concept definition is dominated by that equation. Excerpts from their responses are given below:

R1: Divide the degree by a hundred and eighty and multiply it by π. One radian represents the ratio of 360^0 which is in the centre of the unit circle to the circumference of the circle.

R2: If we divide a circle into three hundred and sixty pieces, we call the one piece as degree. If we divide the circle into two hundred pieces, we get the grad. If we divide it into 2π we get a radian.

D1: Radian is the whole of 360^0, it is 2π radian. Consider the unit circle…these are all choices, I can define my own unit. Let's divide it into six hundred, call it something else, like the grad.

D2: First of all, degree divided by a hundred and eighty equals to radian divided by π. Radian means the angle, it is the multiples of π when we think of π as a hundred and eighty degrees…it is the value in terms of π. 30^0 is $\pi/6$. Radian of angle means its value in terms of π.

As can be interpreted from the responses above, none of the participants successfully defined the radian as a ratio of two lengths: the length of the arc of a central angle of a circle and the radius of the circle. R1's personal concept definition is the closest to the accurate definition.

DISCUSSION AND CONCLUSION

This study investigated pre-service and in-service mathematics teachers' concept images of radian and the sources of such concept images. The data from the questionnaire revealed that most of the participants' concept images of radian was not rich enough and were dominated by their concept images of degree. Participants did not consider radian as a real number although the trigonometric functions that were given to them were explicitly defined on the set of real numbers. The interview data suggested possible sources of such concept images. First of all, the equation $\frac{D}{180} = \frac{R}{\pi}$ acted as a cognitive unit for the participants. None of the participants defined the radian as a ratio of two lengths. Secondly, participants who have stronger concept images of radian established richer connections between unit circle and other concepts in trigonometry as revealed from the concept maps and the graph drawing task. On the other hand, participants who have stronger degree images have right triangle as one of their cognitive units. Another possible source might be that the concept image of π in the context of trigonometry is different from the concept image of π as a real number. Further research is needed to investigate these findings in detail especially the role of the unit circle and right triangle as cognitive units in trigonometry.

References

Barnard, T. & Tall, D.O. (1997). Cognitive Units, Connections and Mathematical Proof. *Proc. 21st Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 41-48). Lahti, Finland: PME.

Blacket, N. & Tall, D.O. (1991). Gender and the versatile learning of trigonometry using computer software. *Proc. 15th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 1, pp. 144-151). Lisbon, Portugal: PME.

Delice, A. (2002). Recognising, recalling and doing in the 'simplification' of trigonometric expressions. *Proc. 26th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 1, p. 274). Norwich, UK: PME.

Doerr, H. (1996). Integrating the study of trigonometry, vectors, and force through modelling. *School Science and Mathematics*, 96 (8), 407 – 418.

Doerr, H. M. & Confrey, J. (1994). A modelling approach to understanding the trigonometry of forces: a classroom study. *Proc. 18th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 264 - 271). Lisbon, Portugal: PME.

Fi, C. (2003). Preservice Secondary School Mathematics Teachers' Knowledge of Trigonometry: Subject Matter Content Knowledge, Pedagogical Content Knowledge and Envisioned Pedagogy. *Unpublished PhD Thesis*, University of Iowa: USA.

Mason, J. (1996). *Qualitative Researching*. London: Sage.

Nielce Lobo da Costa & Sandra Magina (1998). Making sens of sine and cosine functions through alternative aproaches: Computer and "experimental world" contexts. *Proc. 22th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 224-231). Stellenbosch, South Africa: PME.

Silva, Z. L. (1994). A geometric – algebraic approximate to the trigonometric functions with the use of supercalculators. *Proc. 18th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 1, p. 71). Lisbon, Portugal: PME.

Tall, D.O. & Vinner, S. (1981). Concept Image and Concept Definition in Mathematics with Particular Reference to Limit and Continuity. *Educational Studies in Mathematics*, Vol. 12, pp. 151-169.

Vinner, S. (1992). The Function Concept as a Prototype for Problems in Mathematics Learning. In G. Harel, & E. Dubinsky, (Eds) The Concept of Function: Aspects of Epistemology and Pedagogy, MAA, pp. 195-213.

Wenzelbuer, E. (1992). The learning of trigonometric functions in a graphical computer environment. *Proc. 16th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 3, pp. 106-114). Durham, USA: PME.