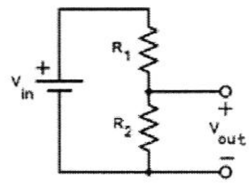
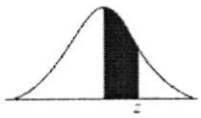
Student Name and Number:_______________________________________________

## *STAT2056 Exam #2*

May 9, 2018

(25 pts.) **1)** The following is a voltage divider circuit where the input voltage $V_{in} = 1$ volt, and, $R_1 = 1$ ohm. $R_2$ is taken from a box of random resistors which are normally distributed with the mean of 1 ohm and the standard deviation of 0.1 ohm. Find the probabilities $P[V_{out} > 8/17]$ and $P[V_{out} > 9/17]$.




$$\left( Z = \frac{X - \mu}{\sigma} \right)$$

**STANDARD NORMAL TABLE (Z)**

| z | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| **0.0** | 0.0000 | 0.0040 | 0.0080 | 0.0120 | 0.0160 | 0.0190 | 0.0239 | 0.0279 | 0.0319 | 0.0359 |
| **0.1** | 0.0398 | 0.0438 | 0.0478 | 0.0517 | 0.0557 | 0.0596 | 0.0636 | 0.0675 | 0.0714 | 0.0753 |
| **0.2** | 0.0793 | 0.0832 | 0.0871 | 0.0910 | 0.0948 | 0.0987 | 0.1026 | 0.1064 | 0.1103 | 0.1141 |
| **0.3** | 0.1179 | 0.1217 | 0.1255 | 0.1293 | 0.1331 | 0.1368 | 0.1406 | 0.1443 | 0.1480 | 0.1517 |
| **0.4** | 0.1554 | 0.1591 | 0.1628 | 0.1664 | 0.1700 | 0.1736 | 0.1772 | 0.1808 | 0.1844 | 0.1879 |
| **0.5** | 0.1915 | 0.1950 | 0.1985 | 0.2019 | 0.2054 | 0.2088 | 0.2123 | 0.2157 | 0.2190 | 0.2224 |
| **0.6** | 0.2257 | 0.2291 | 0.2324 | 0.2357 | 0.2389 | 0.2422 | 0.2454 | 0.2486 | 0.2517 | 0.2549 |
| **0.7** | 0.2580 | 0.2611 | 0.2642 | 0.2673 | 0.2704 | 0.2734 | 0.2764 | 0.2794 | 0.2823 | 0.2852 |
| **0.8** | 0.2881 | 0.2910 | 0.2939 | 0.2969 | 0.2995 | 0.3023 | 0.3051 | 0.3078 | 0.3106 | 0.3133 |
| **0.9** | 0.3159 | 0.3186 | 0.3212 | 0.3238 | 0.3264 | 0.3289 | 0.3315 | 0.3340 | 0.3365 | 0.3389 |
| **1.0** | 0.3413 | 0.3438 | 0.3461 | 0.3485 | 0.3508 | 0.3513 | 0.3554 | 0.3577 | 0.3529 | 0.3621 |
| **1.1** | 0.3643 | 0.3665 | 0.3686 | 0.3708 | 0.3729 | 0.3749 | 0.3770 | 0.3790 | 0.3810 | 0.3830 |
| **1.2** | 0.3849 | 0.3869 | 0.3888 | 0.3907 | 0.3925 | 0.3944 | 0.3962 | 0.3980 | 0.3997 | 0.4015 |
| **1.3** | 0.4032 | 0.4049 | 0.4066 | 0.4082 | 0.4099 | 0.4115 | 0.4131 | 0.4147 | 0.4162 | 0.4177 |
| **1.4** | 0.4192 | 0.4207 | 0.4222 | 0.4236 | 0.4251 | 0.4265 | 0.4279 | 0.4292 | 0.4306 | 0.4319 |
| **1.5** | 0.4332 | 0.4345 | 0.4357 | 0.4370 | 0.4382 | 0.4394 | 0.4406 | 0.4418 | 0.4429 | 0.4441 |

*Hint:* Note that $V_{out} = \frac{R_2}{1+R_2}$. First find the corresponding $R_2$ range; then, use the given table.

$$\underline{V_{out} > \frac{8}{17}} \Rightarrow \frac{R_2}{1+R_2} > \frac{8}{17} \Rightarrow 17R_2 > 8\cdot(1+R_2) \Rightarrow R_2 > \frac{8}{9}$$

$$\Rightarrow z = \frac{8/9 - 1}{0.1} = -\frac{10}{9} = -1.11$$


$$\Rightarrow P\left[V_{out} > \frac{8}{17}\right] = 0.3665 + 0.5 = \underline{\underline{0.8665}}$$

$$\underline{V_{out} > \frac{9}{17}} \Rightarrow \frac{R_2}{1+R_2} > \frac{9}{17} \Rightarrow 17R_2 > 9(1+R_2) \Rightarrow R_2 > \frac{9}{8}$$

$$\Rightarrow z = \frac{9/8 - 1}{0.1} = \frac{10}{8} = 1.25$$


$$\Rightarrow P\left[V_{out} > \frac{9}{17}\right] = 0.5 - 0.3944 = \underline{\underline{0.1056}}$$

*Note:* Total time allowed is 90 min. Please show all your work and write legibly.

(25 pts.) **2)** Consider Problem 1 again. Find the PDF of the output voltage. Try to approximately draw it.

*Hint:* Consider $X = R_2$ and $Y = V_{out}$. Note that $Y = g(X) = \frac{X}{1+X}$. Then, $p_Y(y) = p_X(g^{-1}(y))\left|\frac{dg^{-1}(y)}{dy}\right|$.
For $X \sim N(\mu,\sigma^2)$, $p_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$.

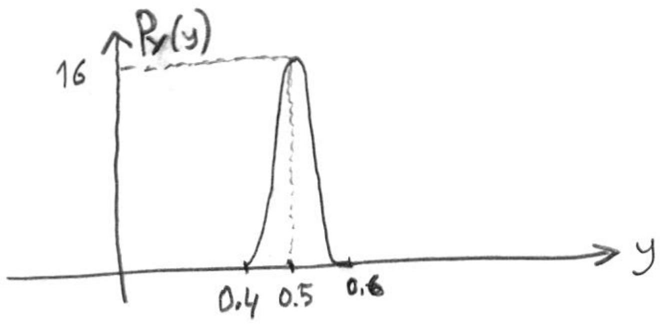
$$y = g(x) = \frac{x}{1+x} \Rightarrow \frac{1}{y} = \frac{1+x}{x} = 1 + \frac{1}{x} \Rightarrow \frac{1}{x} = \frac{1}{y} - 1 = \frac{1-y}{y}$$

$$\Rightarrow x = g^{-1}(y) = \frac{y}{1-y} = \frac{1}{1-y} - 1$$

$$\Rightarrow \frac{d g^{-1}(y)}{dy} = \frac{1}{(1-y)^2}$$

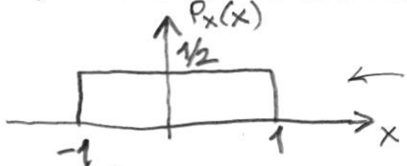
$$P_X(x) = \frac{10}{\sqrt{2\pi}} \exp\left(-50(x-1)^2\right)$$

$$\Rightarrow P_Y(y) = \frac{10}{\sqrt{2\pi}} \exp\left(-50\left(\frac{1}{1-y} - 2\right)^2\right) \cdot \frac{1}{(1-y)^2}$$



(25 pts.) **3)** Assume that $X$ and $N$ are *uniform* independent continuous random variables taking values between -1, and, 1. The random variable $Y$ is defined as $Y = 2X+1+N$. Find the following values:

a) $E(X)$




← total area $= 1$

$$E(X) = \int_{-1}^{1} \frac{1}{2} x \, dx = \underline{\underline{0}} \quad \text{(symmetric)}$$

b) $E(Y) = E[2X+1+N] = 2 \cdot E[X] + 1 + E[N] = 2 \cdot 0 + 1 + 0 = \underline{\underline{1}}$

c) $var(X) = E[X^2] - E[X]^2 = E[X^2] = \int_{-1}^{1} \frac{1}{2} x^2 dx = \left. \frac{x^3}{6} \right|_{-1}^{1} = \underline{\underline{\frac{1}{3}}}$

d) $cov(X,Y) = E[XY] - \underbrace{E[X]}_{0}\underbrace{E[Y]}_{1} = E[XY] = E[2X^2 + X + XN]$

$$= 2 \cdot E[X^2] + E[X] + \overbrace{E[X] \cdot E[N]}^{\text{independent}}$$

$$= 2 \cdot \frac{1}{3} + 0 + 0 \cdot 0 = \underline{\underline{\frac{2}{3}}}$$

e) $var(Y) = E[Y^2] - E[Y]^2 = E[4X^2 + 1 + N^2 + 4X + 4XN + 2N] - 1^2$

$$= \frac{4}{3} + 1 + \frac{1}{3} + 0 + 0 + 0 - 1$$

$$= \underline{\underline{\frac{5}{3}}}$$

f) The correlation coefficient $\rho_{X,Y} = \frac{cov(X,Y)}{\sqrt{var(X)var(Y)}} = \frac{2/3}{\sqrt{5/9}} = \frac{2}{\sqrt{5}} = \underline{\underline{0.894}}$

(25 pts.) **4)** Assume that $X$ and $Y$ are independent exponential random variables with the PDF of $p(x)=e^{-x}$ for $x\geq 0$, and, $p(x)=0$ for $x<0$. Find the formula of the PDF of $Z = X - Y$ and draw it.

*Hint*: If $X$ and $Y$ are independent continuous random variables, an expression for the PDF of $Z = X - Y$ can be given as $p_Z(z) = \int_{-\infty}^{\infty} p_X(x)p_Y(x-z)dx$.

$z \geq 0$:

$$P_z(z) = \int_z^{\infty} e^{-x} \cdot e^{-(x-z)} dx = e^{z} \int_z^{\infty} e^{-2x} dx = e^{z}\left(-\frac{1}{2} e^{-2x}\Big|_z^{\infty}\right)$$

$$= e^{z}\left(\frac{1}{2} e^{-2z}\right) = \frac{1}{2} e^{-z}$$

$z < 0$:

$$P_z(z) = \int_0^{\infty} e^{-x} \cdot e^{-(x-z)} dx = e^{z}\left(-\frac{1}{2} e^{-2x}\Big|_0^{\infty}\right) = \frac{1}{2} e^{z}$$

$$\Rightarrow P_z(z) = \frac{1}{2} e^{-|z|}$$