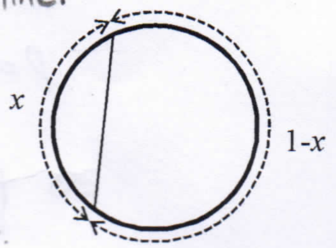
Student Name:______________________

STAT2056 Midterm Exam #1 Mar. 27, 2018

(25 pts.) **1)** In this question, we will find the probability of *crossing* of two random lines inside a circle. Assume the endpoints of the lines are chosen randomly on the perimeter. Let us follow the steps below.

a) First consider that we know the first line in the circle. Assume that the circumference is 1 (i.e. the radius is $1/2\pi$), and the arc length corresponding to the line is x as shown in the figure. In terms of the variable x, find the probability of crossing when a second line is randomly drawn in the circle.



Two endpoints need to be selected for the second line.
There are 4 possible situations:

Endpoint 1	Endpoint 2
x	x
x	$(1-x)$
$(1-x)$	x
$(1-x)$	$(1-x)$

$\}$ These two cases give us crossing.

Therefore, the probability is $\underline{2x(1-x)}$

b) Once you obtain the result in part a, considering the range of x, find the average probability using integral operation. This should give us the desired probability.

$\underline{0 \leq x \leq 1} \Rightarrow \int_0^1 2x(1-x)\,dx = x^2 - \frac{2}{3}x^3 \Big|_0^1 = \frac{1}{3}$

Note that, if you consider the range $0 \leq x < \frac{1}{2}$, we have

$\frac{1}{1/2}\int_0^{1/2} 2x(1-x)\,dx = 2x^2 - \frac{4}{3}x^3 \Big|_0^{1/2} = \frac{1}{3}$ same result.

c) Now, think in a different way. Consider four random points on the circle that represents the endpoints of the lines. How many different line couple arrangements are possible? How many of them are crossing each other? From this simple ratio you can calculate the desired probability. Is it same as you found on part b?

Three possible arrangements are:

Only one of them gives us crossing.
Therefore the probability is $\frac{1}{3}$.

(25 pts.) **2)** A factory manufactures an item, and, has 3 machines available. Following table indicates the daily total produce together with corresponding average defective item amounts for each machine:

Machine	Daily Total Produce	Daily Average of Defective Items
A	300	3
B	400	2
C	500	3

a) Find the probability that Machine A produces no defective items in a certain day:

a1) Using Binomial Distribution $p_X[k] = \binom{M}{k} p^k (1-p)^{M-k}$

$p = \frac{3}{300} = 0.01$

$\Rightarrow \binom{300}{0} 0.01^0 (1-0.01)^{300} = 0.99^{300} = \underline{\underline{0.0490}}$

a2) Approximately Using Poisson Distribution $p_X[k] = e^{-\lambda} \frac{\lambda^k}{k!}$

$\lambda = 3$

$\Rightarrow e^{-3} \frac{3^0}{0!} = e^{-3} = \underline{\underline{0.0498}}$

b) If a randomly selected item is defective, what is the probability that the item is produced in Machine A?

Defective

$P(A|D) = \frac{P(A \cap D)}{P(D)} = \frac{3/1200}{8/1200} = \underline{\underline{\frac{3}{8}}}$ ← total produce

$\left(P(A \cap D) = P(D|A) \cdot P(A) = \frac{3}{300} \cdot \frac{300}{1200} = \frac{3}{1200} \right)$

or directly: $\frac{3}{8}$ ← defectives of A / ← total defectives

c) If we know that the defective item is not produced in Machine A, what is the probability that the item is produced in Machine B?

$\underline{\underline{\frac{2}{5}}}$ ← defectives of B / ← defectives of B & C

(25 pts.) **3)** Assume that X and N are *uniform* independent discrete random variables taking values of 0, 1, and, 2 only. (i.e. their PMF is $p[i] = \frac{1}{3}$ for i=0,1,2). And the random variable Y is defined as $Y = 2X+1+N$. Find the following values:

a) $E[X] = 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} = 1$

b) $E[Y] = 2E[X] + 1 + E[N] = 2 \cdot 1 + 1 + 1 = 4$

c) $E[X^2] = 0^2 \cdot \frac{1}{3} + 1^2 \cdot \frac{1}{3} + 2^2 \cdot \frac{1}{3} = \frac{5}{3}$

d) $var(X) = E[X^2] - E[X]^2 = \frac{5}{3} - 1^2 = \frac{2}{3}$

e) $E[XN] = E[X] \cdot E[N] = 1 \cdot 1 = 1$
(independent)

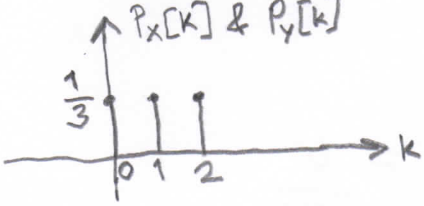
f) $cov(X, Y) = E[XY] - E[X]E[Y] = E[2X^2 + X + XN] - E[X]E[Y]$
$= 2 \cdot \frac{5}{3} + 1 + 1 - 1 \cdot 4 = \frac{4}{3}$

g) $var(Y) = E[Y^2] - E[Y]^2 = E[4X^2 + 1 + N^2 + 4X + 4XN + 2N] - E[Y]^2$
$= 4 \cdot \frac{5}{3} + 1 + \frac{5}{3} + 4 \cdot 1 + 4 \cdot 1 + 2 \cdot 1 - 4^2 = \frac{10}{3}$

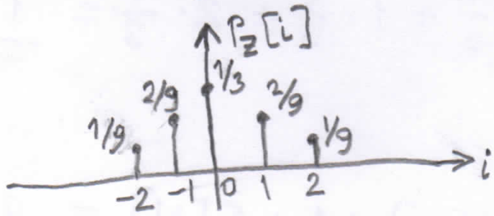
h) The correlation coefficient $\rho_{X,Y} = \frac{cov(X,Y)}{\sqrt{var(X)var(Y)}} = \frac{4/3}{\sqrt{2/3 \cdot 10/3}} = \frac{2}{\sqrt{5}} = 0.8944$

(25 pts.) **4)** Let *X* and *Y* are *independent* discrete random variables taking on *integer* values from -∞ to +∞. An expression for the PMF of $Z = X - Y$ can be given as $p_Z[i] = \sum_{k=-\infty}^{\infty} p_X[k]p_Y[k-i]$, using the PMFs of *X* and *Y*.

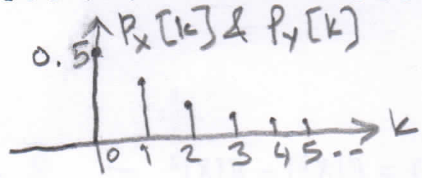
a) Assume that *X* and *Y* are *uniform* discrete random variables taking values of 0, 1, and, 2 only. Draw the PMF of $Z = X - Y$.



$\Rightarrow$



b) Now assume that *X* and *Y* are random variables with the PMF of $p[k]=(0.5)^{k+1}$ for $k \geq 0$, and, $p[k]=0$ for $k<0$. Find the formula for the PMF of $Z = X - Y$.



For $i \geq 0$:

$$P_Z[i] = \sum_{k=-\infty}^{\infty} P_X[k] \cdot P_Y[k-i]$$

$$= \sum_{k=i}^{\infty} 0.5^{k+1} \cdot 0.5^{k-i+1}$$

$$= 0.5^i \sum_{k=i}^{\infty} (0.25)^{k-i+1}$$

$$= 0.5^i \cdot \left(\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots\right)$$

$$= 0.5^i \frac{1/4}{1-\frac{1}{4}}$$

$$= \frac{1}{3} \cdot 0.5^i$$

For $i < 0$: $P_Z[i] = P_Z[-i]$

There fore we obtain : $P_Z[i] = \frac{1}{3} \cdot (0.5)^{|i|}$