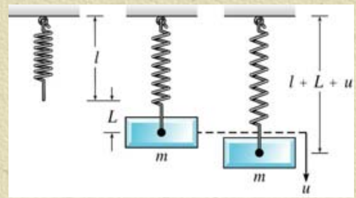
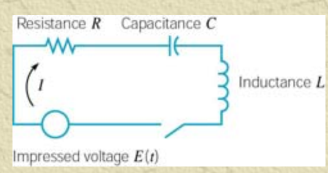


DE-2013

Dr. M. Sakalli



Ch 3.7: Variation of Parameters, with an example

- $g(t)$ is a quotient of sin$t$ or cos$t$, not a sum of or product of such. So cannot use method of undetermined coefficients, here the solution to the homogeneous equation is oscillatory. y'(t)=0. General Mtd.

$$y'' + 4y = 3\csc t = 3/\sin(t)$$

$$y'' + w_n{}^2 y = 0$$

$$y_C(t) = c_1 \cos 2t + c_2 \sin 2t$$

- Introducing u(t) into the solution, y(t)=u1(t) cos(t)+u2(t) sin(t), substitute supposed sol and its derivates. Using hmg solutions..

$$y'(t) = u_1'(t)\cos 2t - 2u_1(t)\sin 2t + u_2'(t)\sin 2t + 2u_2(t)\cos 2t$$

$$y'(t) = -2u_1(t)\sin 2t + 2u_2(t)\cos 2t \qquad u_1'(t)\cos 2t + u_2'(t)\sin 2t = 0$$

$$y''(t) = -2u_1'(t)\sin 2t - 4u_1(t)\cos 2t + 2u_2'(t)\cos 2t - 4u_2(t)\sin 2t$$

$$-2u_1'(t)\sin 2t + 2u_2'(t)\cos 2t = 3\csc t$$

$$u_1'(t)\cos 2t + u_2'(t)\sin 2t = 0$$

$$-2u_1'(t)\sin 2t + 2u_2'(t)\cos 2t = 3\csc t$$
$$u_1'(t)\cos 2t + u_2'(t)\sin 2t = 0$$

$$u_2'(t) = -u_1'(t)\frac{\cos 2t}{\sin 2t},$$

$$-2u_1'(t)\left[\sin^2(2t)+\cos^2(2t)\right] = 3\left[\frac{2\sin t\cos t}{\sin t}\right], \quad u_1'(t) = -3\cos t$$

$$u_2'(t) = 3\cos t\left[\frac{\cos 2t}{\sin 2t}\right] = 3\cos t\left[\frac{1-2\sin^2 t}{2\sin t\cos t}\right] = 3\left[\frac{1-2\sin^2 t}{2\sin t}\right]$$
$$= 3\left[\frac{1}{2\sin t} - \frac{2\sin^2 t}{2\sin t}\right] = \frac{3}{2}\csc t - 3\sin t$$

$$u_1(t) = \int u_1'(t)dt = \int -3\cos t dt = -3\sin t + c_1$$
$$u_2(t) = \int u_2'(t)dt = \int\left(\frac{3}{2}\csc t - 3\sin t\right)dt = \frac{3}{2}\ln\left|\csc t - \cot t\right| + 3\cos t + c_2$$

Summary

$$y'' + p(t)y' + q(t)y = g(t)$$
$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

- Suppose y_1, y_2 are fundamental solutions to the homogeneous and the coefficient on y'' is 1. Using the Wronskian to find u_1 and u_2, and integrate the results.

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0$$
$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t)$$

$$u_1'(t) = -\frac{y_2(t)g(t)}{W(y_1, y_2)(t)}, \quad u_2'(t) = \frac{y_1(t)g(t)}{W(y_1, y_2)(t)}$$

- Theorem 3.7.1 DE of (1), and its hmg form (2), if the functions p, q and g are continuous on an open [I], and if y_1 and y_2 are fundamental solutions to Eq. (2), then a particular solution of Eq. (1) is (3) and the general solution is (4).

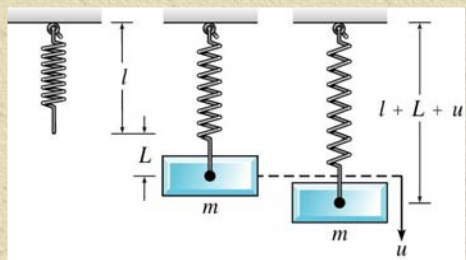
$$y'' + p(t)y' + q(t)y = g(t) \qquad (1)$$
$$y'' + p(t)y' + q(t)y = 0 \qquad (2)$$
$$Y(t) = -y_1(t)\int\frac{y_2(t)g(t)}{W(y_1, y_2)(t)}dt + y_2(t)\int\frac{y_1(t)g(t)}{W(y_1, y_2)(t)}dt \qquad (3)$$
$$y(t) = c_1y_1(t) + c_2y_2(t) + Y(t) \qquad (4)$$

$$u_1'(t)y_1(t)+u_2'(t)y_2(t)=0$$
$$u_1'(t)y_1'(t)+u_2'(t)y_2'(t)=g(t)$$

- y"-4y'+4y=e^{2x}/x
- y"+2y'+4y=e^{-x} ln(x), x>0
- py"+qy'+ry=g(x)
- c1y1+c2y2=y_c, complementary
- v1(t)y1+v(2)y2=y_p, particular.
- (v'y+vy')_1+(v'y+vy')_2=y'_p
- If set (v'y)_1+(v'y)_2=0 (1 assumption)
- (vy')_1+(vy')_2=y'_p
- (v'y'+vy")_1+(v'y'+vy")_2=y"_p
- p{(v'y'+vy")1+(v'y'+vy")2}+q{(vy')1+(vy')2}+r{(vy)1+(vy)2}=y'_p
- p{(v'y')1+(v'y')2}+v1**{py''+qy'+ry}**1+v2**{py''+qy'+ry}**1=g(x),
- {(v'y')1+(v'y')2}=g(x)/(p(x) (2)

- Practice second matlab code with this notes to make sense out of it. .

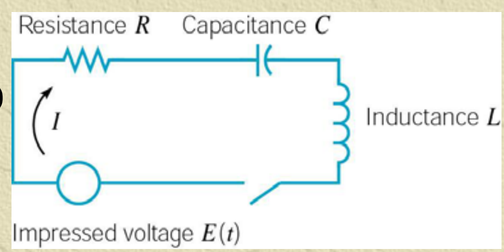
Ch 3.8: Behavior of vibrating systems. Unforced and forced cases.



$$m u''(t) + \gamma u'(t) + k u(t) = F_0 \cos \omega t$$

$$L I''(t) + r I'(t) + \frac{1}{C} I(t) = E'(t)$$

$$I(0) = I_0, \ I'(0) = I_0'$$

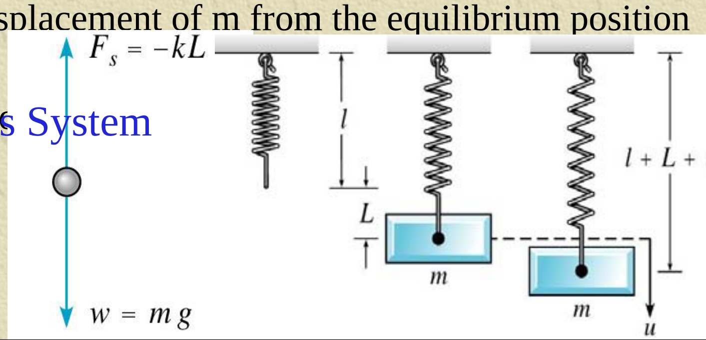


- Suppose a mass m hangs from vertical spring of original length l.
- The mass elongation L of the spring due to the F_G.
- $F_G = mg$ pulling downward.
- F_S of spring stiffness pulls mass up. For small elongations L and F are proportional. $F_s = kL$ (Hooke's Law).
- In equilibrium, the forces balancing each other: $mg = kL$
- $u(t)$ is the displacement of m from the equilibrium position at time t.

$mu''(t) = f(t)$

- f is the net f

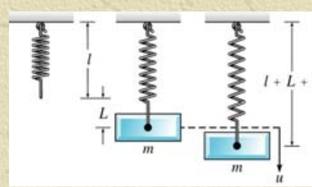
Spring – Mass System



- Newton's Law: f = Superposition of the acting forces:
  - Weight: $w = mg$ (downward force)
  - Spring force: $F_s = - k(L+ u)$ (up or down)
  - Damping force: $F_d(t) = - \gamma\, u'(t)$ (up or down)
  - External force: $F(t)$ (up or down force)

- where the parameters m, γ, and k are the positive constants and initial values to be considered $u(0)=u_0$, $u'(0)=v_0$.

Spring Model



- where the parameters m, γ, and k are the positive constants and initial values to be considered $u(0)=u_0$, $u'(0)=v_0$.

$mu''(t) = mf(t)$

$mu''(t) = mg + F_s(t) + F_d(t) + F(t)$

$= mg - k(L+ u) - \gamma\, u'(t) + F(t)$

$\mathbf{mu''(t) + \gamma\, u'(t) + ku = -kL + mg + F(t)}$

$mg = kL$, !!

$\mathbf{mu''(t) + \gamma\, u'(t) + ku = -kL + mg + F(t)}$

w = 4 lb. m = w/g, g=32 ft/sec^2,

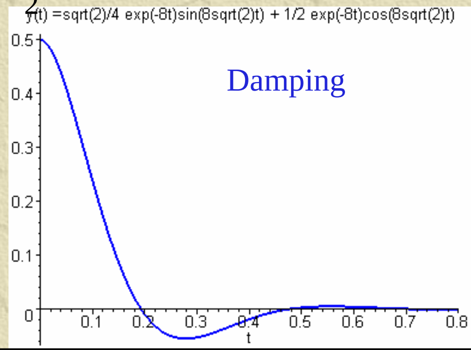
stretches a spring 2" =(1/6)ft→ -k (1/6) = -4 lb.

in a medium that exerts a viscous resistance

of 6 lb, and velocity is 3 ft/sec.

$\gamma u'(t) = \gamma\, 3 = 6$ lb → γ =2 lb sec/ft.

$$u''(t) + 16u'(t) + 192u(t) = 0,\ u(0) = \frac{1}{2},\quad u'(0) = 0$$



Complex case: Unforced and Undamped Free Vibrations

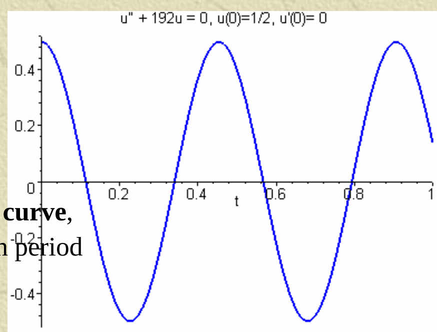
- Suppose no external driving force and no damping, $F(t) = 0$ and $\gamma = 0$, then the general solution ***mu"(t) + ku(t) = 0*** is $u(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$
- **where ω is angular frq and eql to sqrt(k/m). $mu''(t) + \omega^2 u(t) = 0$**

$$u(t) = A\cos\omega_0 t + B\sin\omega_0 t \quad \Leftrightarrow \quad u(t) = R\cos(\omega_0 t - \delta)$$

$$u(t) = R\cos\delta\cos\omega_0 t + R\sin\delta\sin\omega_0 t,$$

$$A = R\cos\delta,\ \ B = R\sin\delta$$

$$R = \sqrt{A^2 + B^2},\quad \tan\delta = \frac{B}{A}$$



- The solution **is a shifted cosine (or sine) curve**, that describes simple **harmonic motion**, with period

$$T = \frac{1}{f} = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{LC}$$

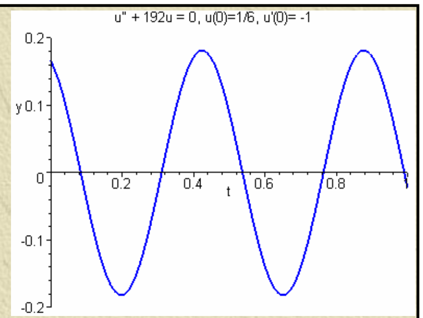
- The circular frequency ω_0 (radians/time) is **natural frequency** of the vibration.
- R is the **amplitude** of max displacement of mass from equilibrium.
- δ is the **phase** (dimensionless).

Example Exam question

$$u''(t)+192u(t)=0,\quad u(0)=1/6,\quad u'(0)=-1$$

$$u(t)=\frac{1}{6}\cos 8\sqrt{3}t-\frac{1}{8\sqrt{3}}\sin 8\sqrt{3}t$$

The natural frequency and the period and its amplitude and the phase δ...

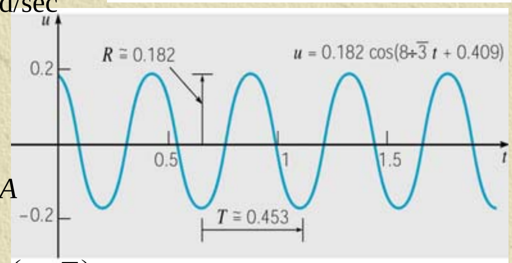


$$\omega_0=\sqrt{k/m}=\sqrt{192}=8\sqrt{3}\cong 13.856\text{ rad/sec}$$

$$T=2\pi/\omega_0\cong 0.45345\text{ sec}$$

$$R=\sqrt{A^2+B^2}\cong 0.18162\text{ ft}$$

$$A=R\cos\delta,\ \ B=R\sin\delta,\ \ \tan\delta=B/A$$



$$\tan\delta=\frac{B}{A}\Rightarrow\tan\delta=\frac{-\sqrt{3}}{4}\Rightarrow\delta=\tan^{-1}\left(\frac{-\sqrt{3}}{4}\right)\cong -0.40864\text{ rad}$$

$$u(t)=0.182\cos\left(8\sqrt{3}t+0.409\right)$$

Skip: Damped Free Vibrations **effect of damping coefficient γ**

- The characteristic equation is

$$r_1,r_2=\frac{-\gamma\pm\sqrt{\gamma^2-4mk}}{2m}=\frac{\gamma}{2m}\left[-1\pm\sqrt{1-\frac{4mk}{\gamma^2}}\right]$$

- Three cases for the solution (Fig for the complex case)

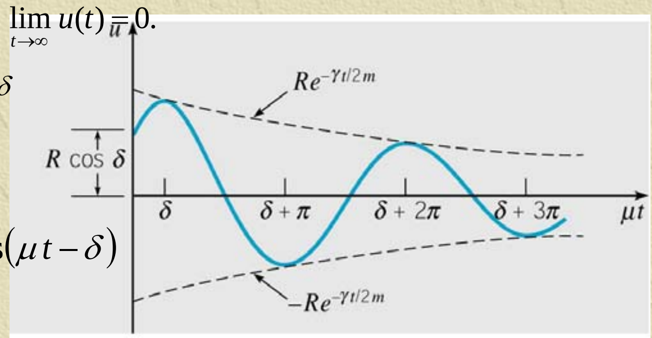
$$\gamma^2-4mk>0:\quad u(t)=Ae^{r_1t}+Be^{r_2t},\ \text{ where } r_1<0,\ r_2<0;$$

$$\gamma^2-4mk=0:\quad u(t)=(A+Bt)e^{-\gamma t/2m},\ \text{ where } \gamma/2m>0;$$

$$\gamma^2-4mk<0:\quad u(t)=e^{-\gamma t/2m}(A\cos\mu t+B\sin\mu t),\ \ \mu=\frac{\sqrt{4mk-\gamma^2}}{2m}>0.$$

Note: In all cases, $\lim_{t\to\infty}u(t)=0.$

$$A=R\cos\delta,\ \ B=R\sin\delta$$



$$u(t)=R\,e^{-\gamma t/2m}\cos(\mu t-\delta)$$

$$|u(t)|\le R\,e^{-\gamma t/2m}$$

Damped Free Vibrations effect of damping coefficient *r resistance*

- The characteristic equation is

$$r_1, r_2 = \frac{-r \pm \sqrt{r^2 - 4L/C}}{2L} = \frac{r}{2L}\left[-1 \pm \sqrt{1 - \frac{4L}{Cr^2}}\right]$$

- Three cases for the solution (Fig for the last case)

$$r^2 - 4L/C > 0: \quad u(t) = Ae^{r_1 t} + Be^{r_2 t}, \text{ where } r_1 < 0,\ r_2 < 0;$$

$$r^2 - 4L/C = 0: \quad u(t) = (A + Bt)e^{-rt/2L}, \text{ where } r/2L > 0;$$

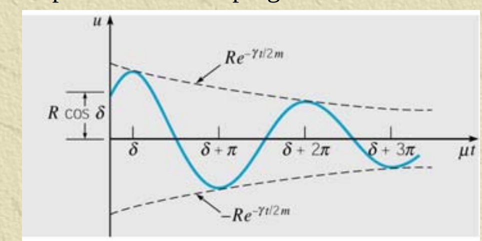
$$r^2 - 4L/C < 0: \quad u(t) = e^{-Rt/2L}(A\cos\mu t + B\sin\mu t),\ \ \mu = \frac{\sqrt{4L/C - r^2}}{2L} > 0.$$

Note: In all three cases, $\lim_{t\to\infty} u(t) = 0$, as expected from damping term.

$$A = R\cos\delta,\ \ B = R\sin\delta$$

$$u(t) = Re^{-\gamma t/2m}\cos(\mu t - \delta)$$

$$|u(t)| \le Re^{-\gamma t/2m}$$



Frequency and period evaluations: Quasi Period

- Compare μ with ω_0, the frequency of undamped motion:

$$\frac{\mu}{\omega_0} = \frac{\sqrt{4km - \gamma^2}}{2m\sqrt{k/m}} = \frac{\sqrt{4km - \gamma^2}}{\sqrt{4m^2}\sqrt{k/m}} = \frac{\sqrt{4km - \gamma^2}}{\sqrt{4km}} = \sqrt{1 - \frac{\gamma^2}{4km}}$$

For small γ

$\gamma^2/8km <<$

$$\cong \sqrt{1 - \frac{\gamma^2}{4km} + \frac{\gamma^4}{64k^2m^2}} = \sqrt{\left(1 - \frac{\gamma^2}{8km}\right)^2} = 1 - \frac{\gamma^2}{8km}$$

- Thus, small damping reduces oscillation frequency slightly.
- Similarly, **quasi period** is defined as $T_d = 2\pi/\mu$. Then

$$\frac{T_d}{T} = \frac{2\pi/\mu}{2\pi/\omega_0} = \frac{\omega_0}{\mu} = \left(1 - \frac{\gamma^2}{4km}\right)^{-1/2} \cong \left(1 - \frac{\gamma^2}{8km}\right)^{-1} \cong 1 + \frac{\gamma^2}{8km}$$

- Thus, **small damping increases quasi period**.

Damped Free Vibrations: Neglecting Damping for Small $\gamma^2/4km$

- Comparing damped and undamped frqcies and periods: r1, r2<0

$$\frac{\mu}{\omega_0} = \left(1 - \frac{\gamma^2}{4km}\right)^{1/2}, \quad \frac{T_d}{T} = \left(1 - \frac{\gamma^2}{4km}\right)^{-1/2} \quad \lim_{\gamma \to 2\sqrt{km}} \mu = 0 \text{ and } \lim_{\gamma \to 2\sqrt{km}} T_d = \infty$$

$$\gamma^2 - 4mk > 0: \ u(t) = Ae^{r_1 t} + Be^{r_2 t}, \quad r_1 < 0, \ r_2 < 0 \qquad (1)$$

$$\gamma^2 - 4mk = 0: \ u(t) = (A + Bt)e^{-\gamma t/2m}, \quad \gamma / 2m > 0 \qquad (2)$$

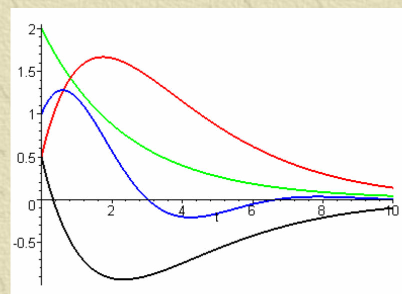
$$\gamma^2 - 4mk < 0: \ u(t) = e^{-\gamma t/2m}(A\cos \mu t + B\sin \mu t), \ \mu > 0 \qquad (3)$$

The nature of the solution changes as γ passes through the value sqrt(4*km*).

$\gamma^2 = 4km$ (2) critical damping. rd, blck

$\gamma^2 > 4km$ (1) overdamped, rd

$\gamma^2 < 4km$ underdamping, ble



Ch 3.9: Forced Vibrations: External forcing funcion $F(t)=F_0\cos\omega t$. With Damping

- The general solution = the homogeneous equation + the particular solution of the nonhomogeneous equation is

$$u(t) = c_1 u_1(t) + c_2 u_2(t) + A\cos(\omega t) + B\sin(\omega t) = u_C(t) + U(t)$$

- The roots r_1 and r_2 of the characteristic equation of homogeneous DE, for u_1 and u_2:

$$mr^2 + \gamma r + kr = 0 \ \Rightarrow \ r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

- Since m, γ, and k are are all positive constants, it follows that r_1 and r_2 are either real and negative, or complex conjugates with negative real roots r_1 and $r_2 < 0$.

$$\lim_{t\to\infty} u_C(t) = \lim_{t\to\infty}\left(c_1 e^{r_1 t} + c_2 e^{r_2 t}\right) = 0,$$

while in the second case

$$\lim_{t\to\infty} u_C(t) = \lim_{t\to\infty}\left(c_1 e^{\lambda t}\cos \mu t + c_2 e^{\lambda t}\sin \mu t\right) = 0.$$

- Thus in either case, $\lim_{t\to\infty} u_C(t) = 0$

Transient and Steady-State Solutions

$$mu''(t) + \gamma u'(t) + ku(t) = F_0 \cos \omega t$$

$$u(t) = \underbrace{c_1 u_1(t) + c_2 u_2(t)}_{u_C(t)} + \underbrace{A\cos(\omega t) + B\sin(\omega t)}_{U(t)},$$

$$\lim_{t\to\infty} u_C(t) = \lim_{t\to\infty}\left(c_1 u_1(t) + c_2 u_2(t)\right) = 0 \qquad U(t) = A\cos(\omega t) + B\sin(\omega t)$$

- Thus $u_C(t)$ is called the **transient solution**.
- For this reason, $U(t)$ is called the **steady-state solution**, or **forced response**.
- With increasing time, the energy put into system by initial displacement and velocity is dissipated through damping force. The motion then becomes the response $U(t)$ of the system to the external force $F_0 \cos \omega t$.
- Without damping, the effect of the initial conditions would persist for all time.

Rewriting Forced Response and Amplitude Analysis of SS response

$$U(t) = A\cos(\omega t) + B\sin(\omega t) \qquad U(t) = R\cos(\omega t - \delta)$$

$$R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}, \qquad \omega_0^2 = k/m$$

$$\cos\delta = \frac{m(\omega_0^2 - \omega^2)}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}, \ \sin\delta = \frac{\gamma\omega}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}$$

The driving frequency ω.

For low-frequency excitation lim $\omega \to 0$

Recall $(\omega_0)^2 = k/m$.

$$\lim_{\omega\to 0} R = \lim_{\omega\to 0} \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} = \frac{F_0}{m\omega_0^2} = \frac{F_0}{k}$$

Note that F_0/k is the static displacement of the spring produced by force F_0.

For high frequency excitation lim $\omega \to$ inf

$$\lim_{\omega\to\infty} R = \lim_{\omega\to\infty} \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} = 0$$

Maximum Amplitude of Forced Response

- Thus $\lim_{\omega \to 0} R = F_0 / k, \quad \lim_{\omega \to \infty} R = 0$
- At an intermediate value of ω, the amplitude R may have a maximum value. To find this frequency ω, differentiate R and set the result equal to zero. Solving for ω_{max}, we obtain

$$\omega_{\max}^2 = \omega_0^2 - \frac{\gamma^2}{2m^2} = \omega_0^2 \left(1 - \frac{\gamma^2}{2mk}\right)$$

where $(\omega_0)^2 = k/m$. Note $\omega_{max} < \omega_0$, and ω_{max} is close to ω_0 for small γ. The maximum value of R is

$$R_{\max} = \frac{F_0}{\gamma \omega_0 \sqrt{1 - (\gamma^2 / 4mk)}}$$

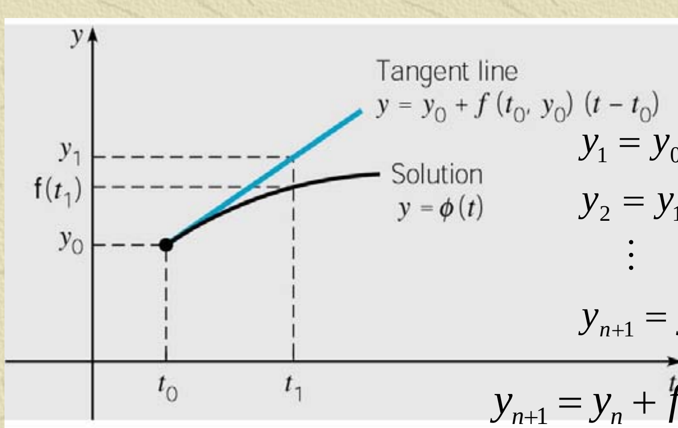
Numerical approximation..

Ch 2.7: Numerical Approximations of 1st ord DE, IVP: Euler's Method

- If f and $\partial f/\partial y$ are **continuous, then this IVP has a unique solution** $y = \phi(t)$ in some interval about t_0.
- When the DE is linear, separable or exact, solution is possible by analytic.
- However, the solutions for most DE cannot be found by analytical means.
- Numerical Methods
  - compute approximate values of the solution $y = \phi(t)$ at a selected set of t-values. Ideally, the approximate solution values will be accompanied by error bounds that ensure the level of accuracy.
  - the **tangent line method**, which is also called **Euler's**.

Euler's Method: Tangent Line Approximation

- $y = \phi(t)$ at initial point t_0.
- The solution passes through initial point (t_0, y_0) with slope $f(t_0,y_0)$.
- good with an interval short enough.
- Thus if t_1 is close enough to t_0, we can approximate $\phi(t_1)$ by



$$y_1 = y_0 + f_0 \cdot (t_1 - t_0)$$
$$y_2 = y_1 + f_1 \cdot (t_2 - t_1)$$
$$\vdots$$
$$y_{n+1} = y_n + f_n \cdot (t_{n+1} - t_n)$$
$$y_{n+1} = y_n + f_n h, \quad n = 0,1,2,\ldots$$

Example: Euler's approximation Method

$$y' = 9.8 - 0.2y, \quad y(0) = 0$$

$$y_1 = y_0 + f_0 \cdot h = 0 + 9.8(0.1) = .98$$
$$y_2 = y_1 + f_1 \cdot h = .98 + (9.8 - (0.2)(.98))(0.1) \approx 1.94$$
$$y_3 = y_2 + f_2 \cdot h = 1.94 + (9.8 - (0.2)(1.94))(0.1) \approx 2.88$$
$$y_4 = y_3 + f_3 \cdot h = 2.88 + (9.8 - (0.2)(2.88))(0.1) \approx 3.80$$

$$y' = 9.8 - 0.2y, \quad y(0) = 0$$
$$y' = -0.2(y - 49)$$
$$\frac{dy}{y-49} = -0.2dt$$
$$y(0) = 1 \Rightarrow k = -49$$
$$\Rightarrow y = 49\left(1 - e^{-0.2t}\right)$$

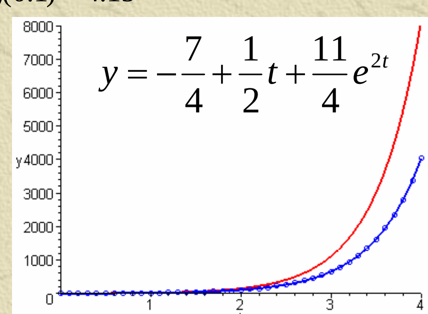
$$\text{Relative Err } \% = \frac{y_{exact} - y_{approx}}{y_{exact}} \times 100$$

t	Exact y	Approx y	Error	% Rel Error
0.00	0	0.00	0.00	0.00
0.10	0.97	0.98	-0.01	-1.03
0.20	1.92	1.94	-0.02	-1.04
0.30	2.85	2.88	-0.03	-1.05
0.40	3.77	3.8	-0.03	-0.80

Example 2: $h = 0.1$, blue is approx $\quad y' = 4 - t + 2y, \quad y(0) = 1$

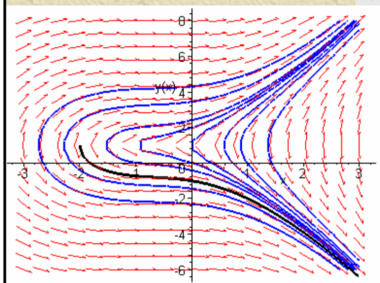
$$y_1 = y_0 + f_0 \cdot h = 1 + (4 - 0 + (2)(1))(0.1) = 1.6$$
$$y_2 = y_1 + f_1 \cdot h = 1.6 + (4 - 0.1 + (2)(1.6))(0.1) = 2.31$$
$$y_3 = y_2 + f_2 \cdot h = 2.31 + (4 - 0.2 + (2)(2.31))(0.1) \approx 3.15$$
$$y_4 = y_3 + f_3 \cdot h = 3.15 + (4 - 0.3 + (2)(3.15))(0.1) \approx 4.15$$

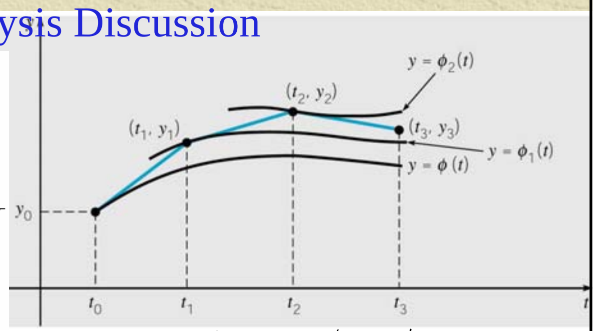
t :	Exact y	Approx y	Error	% Rel Error
0.00	1.00	1.00	0.00	0.00
0.10	1.66	1.60	0.06	3.55
0.20	2.45	2.31	0.14	5.81
0.30	3.41	3.15	0.26	7.59
0.40	4.57	4.15	0.42	9.14
0.50	5.98	5.34	0.63	10.58
0.60	7.68	6.76	0.92	11.96
0.70	9.75	8.45	1.30	13.31
0.80	12.27	10.47	1.80	14.64
0.90	15.34	12.89	2.45	15.96
1.00	19.07	15.78	3.29	17.27



t	Exact y	Approx y	Error	% Rel Error
0.00	1.00	1.00	0.00	0.00
1.00	19.07	15.78	3.29	17.27
2.00	149.39	104.68	44.72	29.93
3.00	1109.18	652.53	456.64	41.17
4.00	8197.88	4042.12	4155.76	50.69

General Error Analysis Discussion

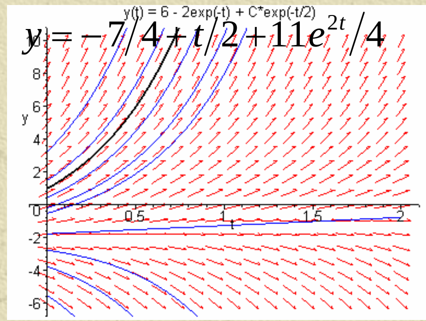




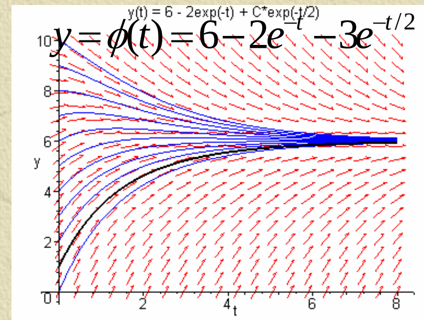
$y' = 4 - t + 2y, \;\; y(0) = 1$

$y' = 3 + e^{-t} - y/2, \;\; y(0) = 1$

$y = -7/4 + t/2 + 11e^{2t}/4$



$y = \phi(t) = 6 - 2e^{-t} - 3e^{-t/2}$



Error Bounds and Numerical Methods

- In using a numerical procedure, keep in mind the question of whether the results are accurate enough to be useful.
- Truncation (precision) and rounding errors (local)
- Approximation error, local but accumulates to global
- Taylor analysis.