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Probability and Statistics

Chapter 1
Introduction to Probability and Statistics

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Contents

- Overview of
  - Statistical Inference
  - Samples,
  - Populations,
  - Role of Probability

Random Experiments

- Experimentation is useful to us because we can assume that if we perform certain experiments under very nearly identical conditions, we will arrive at results that are essentially the same.
- However, in some experiments, we are not able to ascertain or control the value of certain variables so that the results will vary from one performance of the experiment to the next, even though most of the conditions are the same. These experiments are described as random.
- **Example 1.1.** If we toss a die, the result of the experiment is that it will come up with one of the numbers in the set {1, 2, 3, 4, 5, 6}.

Definition

The set of all possible outcomes of a statistical experiment is called the **sample space** and is represented by the symbol S.

Sample Spaces

- A set *S* that consists of all possible outcomes of a random experiment is called a *sample space*, and each outcome is called a *sample point*. Often there will be more than one sample space that can describe outcomes of an experiment, but there is usually only one that will provide the most information.
- **Example 1.2.** If we toss a die, then one sample space is given by {1, 2, 3, 4, 5, 6} while another is {even, odd}. It is clear, however, that the latter would not be adequate to determine, for example, whether an outcome is divisible by 3.

Sample Spaces

- If a sample space has a finite number of points, it is called a *finite sample space*. If it has as many points as there are natural numbers 1, 2, 3, …. , it is called a *countably infinite sample space*. If it has as many points as there are in some interval on the *x* axis, such as $0 <= x <= 1$, it is called a *noncountably infinite sample space*. A sample space that is finite or countably finite is often called a *discrete sample space*, while one that is noncountably infinite is called a *nondiscrete sample space*.
- **Example 1.3.** The sample space resulting from tossing a die yields a discrete sample space. However, picking *any* number, not just integers, from 1 to 10, yields a nondiscrete sample space.

Definition

An **event** is a subset of a sample space.

Events

- An *event* is a subset *A* of the sample space *S*, i.e., it is a set of possible outcomes. If the outcome of an experiment is an element of *A*, we say that the event *A has occurred.* An event consisting of a single point of *S* is called a *simple* or *elementary event.*
- As particular events, we have *S* itself, which is the *sure* or *certain event* since an element of *S* must occur, and the empty set Ø, which is called the *impossible event* because an element of Ø cannot occur.

Definition

The **complement** of an event A with respect to S is the subset of all elements of S that are not in A. We denote the complement of A by the symbol A'.

Definition

The **intersection** of two events A and B, denoted by the symbol $A \cap B$, is the event containing all elements that are common to A and B.

Definition

Two events A and B are **mutually exclusive**, or **disjoint**, if $A \cap B = \phi$, that is, if A and B have no elements in common.

Definition

The **union** of the two events A and B, denoted by the symbol $A \cup B$, is the event containing all the elements that belong to A or B or both.

Events

- By using set operations on events in S, we can obtain other events in S. For example, if A and B are events, then;

1. $A \cup B$ is the event "either A or B or both." $A \cup B$ is called the *union* of A and B.

2. $A \cap B$ is the event "both A and B." $A \cap B$ is called the *intersection* of A and B.

3. A' is the event "not A." is called the *complement* of A.

Events

4. $A - B = A \cap B'$ is the event "A but not B."

   In particular, $A' = S - A$.

- If the sets corresponding to events A and B are disjoint, i.e., $A \cap B = \varnothing$, we often say that the events are *mutually exclusive.* This means that they cannot both occur. We say that a collection of events $A_1, A_2, \ldots, A_n$ is mutually exclusive if every pair in the collection is mutually exclusive.

The Concept of Probability

- In any random experiment there is always uncertainty as to whether a particular event will or will not occur. As a measure of the chance, or probability, with which we can expect the event to occur, it is convenient to assign a number between 0 and 1.

The Concept of Probability

- There are two important procedures by means of which we can estimate the probability of an event.

1. CLASSICAL APPROACH: If an event can occur in h different ways out of a total of n possible ways, all of which are equally likely, then the probability of the event is h/n. ***(Q)***

2. FREQUENCY APPROACH: If after n repetitions of an experiment, where n is very large, an event is observed to occur in h of these, then the probability of the event is h/n. This is also called the *empirical probability* of the event. ***(Q)***

The Axioms of Probability

- Suppose we have a sample space *S*. If *S* is discrete, all subsets correspond to events and conversely; if *S* is nondiscrete, only special subsets (called *measurable*) correspond to events. To each event *A* in the class *C* of events, we associate a real number *P(A)*. The *P* is called a *probability function*, and *P(A)* the *probability* of the event, if the following axioms are satisfied.

Definition

The **probability** of an event A is the sum of the weights of all sample points in A. Therefore,

$$0 \le P(A) \le 1, \quad P(\phi) = 0, \quad \text{and} \quad P(S) = 1.$$

Furthermore, if $A_1, A_2, A_3, \ldots$ is a sequence of mutually exclusive events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = P(A_1) + P(A_2) + P(A_3) + \cdots .$$

Definition

If an experiment can result in any one of N different equally likely outcomes, and if exactly n of these outcomes correspond to event A, then the probability of event A is

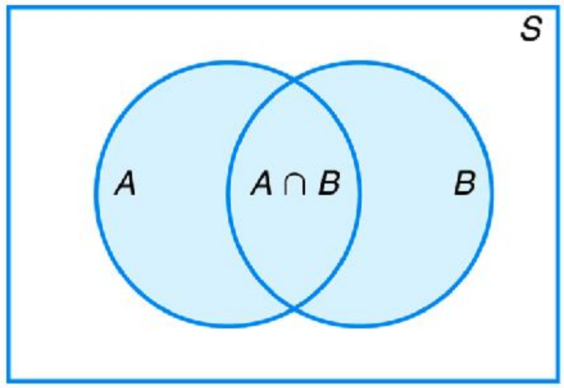
$$P(A) = \frac{n}{N}.$$

Theorem

If A and B are two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Additive rule of probability



Corollary

If A and B are mutually exclusive, then

$$P(A \cup B) = P(A) + P(B).$$

Corollary

If $A_1,\ A_2, \ldots, A_n$ are mutually exclusive, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n).$$

Corollary

If $A_1,\ A_2, \ldots, A_n$ is a partition of sample space S, then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n) = P(S) = 1.$$

Theorem

For three events A, B, and C,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

Theorem

If A and A' are complementary events, then

$$P(A) + P(A') = 1.$$

The Axioms of Probability

Axiom 1. For every event A in class C,
$P(A) \geq 0$

Axiom 2. For the sure or certain event S in the class C,
$P(S) = 1$

Axiom 3. For any number of mutually exclusive events $A_1, A_2, \ldots,$ in the class C,
$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$
In particular, for two mutually exclusive events A_1 and A_2,
$P(A_1 \cup A_2) = P(A_1) + P(A_2)$

Theorems of Probability

Theorem 1-1: If $A_1 \subset A_2$, then (1)
$$P(A_1) \leq P(A_2) \text{ and } P(A_2 - A_1) = P(A_1) - P(A_2)$$

Theorem 1-2: For every event A, (2)
$$0 \leq P(A) \leq 1,$$
i.e., a probability between 0 and 1.

Theorem 1-3: For $\varnothing$, the empty set, (3)
$$P(\varnothing) = 0$$
i.e., the impossible event has probability zero.

Theorems of Probability

Theorem 1-4: If A' is the complement of A, then (4)

$$P(A') = 1 - P(A)$$

Theorem 1-5: If $A = A_1 \cup A_2 \cup \ldots \cup A_n$, where $A_1, A_2, \ldots, A_n$ are mutually exclusive events, then

$$P(A) = P(A_1) + P(A_2) + \ldots + P(A_n) \quad (5)$$

Theorems of Probability

Theorem 1-6: If A and B are any two events, then (6)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

More generally, if A_1, A_2, A_3 are any three events, then

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_3 \cap A_1) + P(A_1 \cap A_2 \cap A_3).$$

Generalizations to n events can also be made.

Theorem 1-7: For any events A and B, (7)

$$P(A) = P(A \cap B) + P(A \cap B')$$

Assignment of Probabilities

If a sample space S consists of a finite number of outcomes $a_1, a_2, \ldots, a_n$, then by Theorem 1-5,

$$P(A_1) + P(A_2) + \ldots + P(A_n) = 1 \qquad (8)$$

where $A_1, A_2, \ldots, A_n$ are elementary events given by $A_i = \{a_i\}$.

Assignment of Probabilities

- if we assume *equal probabilities* for all simple events, then

$$P(A_k) = \frac{1}{n}, \qquad k = 1, 2, \ldots, n \qquad (9)$$

- And if A is any event made up of h such simple events;

$$P(A) = \frac{h}{n} \qquad (10)$$

This is equivalent to the classical approach to probability.

Conditional Probability

Let A and B be two events such that $P(A) > 0$. Denote $P(B \mid A)$ the probability of B *given that* A has occurred. Since A is known to have occurred, it becomes the new sample space replacing the original S. From this we are led to the definition

$$P(B \mid A) \equiv \frac{P(A \cap B)}{P(A)} \tag{11}$$

or

$$P(A \cap B) \equiv P(A)P(B \mid A) \tag{12}$$

Conditional Probability

Theorem 1-8: For any three events A_1, A_2, A_3, we have

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2 \mid A_1)P(A_3 \mid A_1 \cap A_2) \tag{13}$$

Theorem 1-9: If an event A must result in one of the mutually exclusive events $A_1, A_2, \ldots, A_n$, then $P(A)$

$$= P(A_1)P(A \mid A_1) + P(A_2)P(A \mid A_2) + \ldots + P(A_n)P(A \mid A_n) \tag{14}$$

Definition

Two events A and B are **independent** if and only if

$$P(B|A) = P(B) \quad \text{or} \quad P(A|B) = P(A),$$

assuming the existences of the conditional probabilities. Otherwise, A and B are **dependent**.

Theorem

If in an experiment the events A and B can both occur, then

$$P(A \cap B) = P(A)P(B|A), \text{ provided } P(A) > 0.$$

Theorem

Two events A and B are independent if and only if

$$P(A \cap B) = P(A)P(B).$$

Therefore, to obtain the probability that two independent events will both occur, we simply find the product of their individual probabilities.

Theorem

If, in an experiment, the events $A_1, A_2, \ldots, A_k$ can occur, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) \\ = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \cdots P(A_k|A_1 \cap A_2 \cap \cdots \cap A_{k-1}).$$

If the events $A_1, A_2, \ldots, A_k$ are independent, then

$$P(A_1 \cap A_2 \cap \cdots \cap A_k) = P(A_1)P(A_2) \cdots P(A_k).$$

Definition

A collection of events $\mathcal{A} = \{A_1, \ldots, A_n\}$ are mutually independent if for any subset of $\mathcal{A}$, $A_{i_1}, \ldots, A_{i_k}$, for $k \leq n$, we have

$$P(A_{i_1} \cap \cdots \cap A_{i_k}) = P(A_{i_1}) \cdots P(A_{i_k}).$$

Independent Events

If $P(B \mid A) = P(B)$, i.e., the probability of B occurring is not affected by the occurrence or nonoccurrence of A, then we say that A and B are *independent events*. This is equivalent to

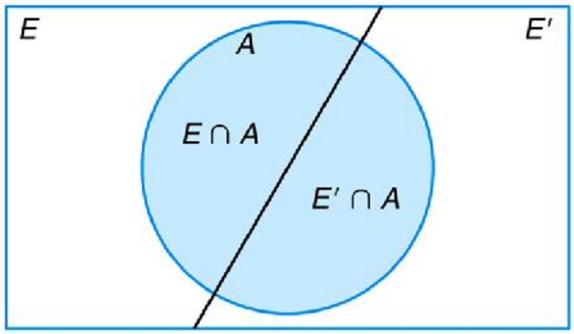
$$P(A \cap B) = P(A)P(B) \qquad (15)$$

$$P(A_j \cap A_k) = P(A_j)P(A_k) \; j \neq k \qquad \text{where} \qquad j,k = 1,2,3 \qquad (16)$$

and

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3) \qquad (17)$$

Bayes Rule

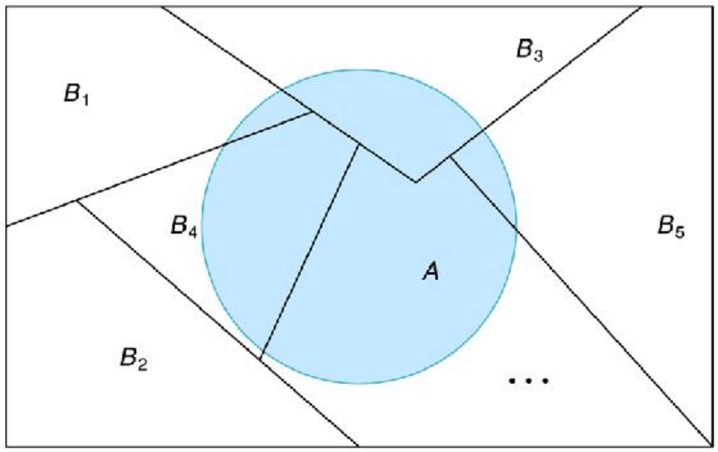


Theorem

If the events $B_1, B_2, \ldots, B_k$ constitute a partition of the sample space S such that $P(B_i) \neq 0$ for $i = 1, 2, \ldots, k$, then for any event A of S,

$$P(A) = \sum_{i=1}^{k} P(B_i \cap A) = \sum_{i=1}^{k} P(B_i)P(A|B_i).$$

Partitioning the sample space *S*



Bayes' Theorem

Suppose that $A_1, A_2, \ldots, A_n$ are mutually exclusive events whose union is the sample space S, i.e., one of the events must occur. Then if A is any event, we have the important theorem:

Theorem 1-10 (Bayes' Rule):

$$P(A_k \mid A) = \frac{P(A_k)P(A \mid A_k)}{\sum_{j=1}^{n} P(A_j)P(A \mid A_j)} \qquad (18)$$

Combinatorial Analysis

- In many cases the number of sample points in a sample space is not very large, and so direct enumeration or counting of sample points needed to obtain probabilities is not difficult. However, problems arise where direct counting becomes a practical impossibility. In such cases use is made of *combinatorial analysis*, which could also be called a *sophisticated way of counting*.

Fundamental Principle of Counting

- If one thing can be accomplished n_1 different ways and after this a second thing can be accomplished n_2 different ways, … , and finally a kth thing can be accomplished in n_k different ways, then all k things can be accomplished in the specified order in $n_1 n_2 \ldots n_k$ different ways.

Permutations

Suppose that we are given n distinct objects and wish to *arrange* r of these objects in a line. Since there are n ways of choosing the first object, and after this is done, $n - 1$ ways of choosing the second object, … , and finally $n - r + 1$ ways of choosing the rth object, it follows by the fundamental principle of counting that the number of different *arrangements*, or *permutations* as they are often called, is given by

$$_nP_r = n(n-1)\ldots(n-r+1) \qquad (19)$$

where it is noted that the product has r factors. We call $_nP_r$ the *number of permutations of n objects taken r at a time.*

Definition

A **permutation** is an arrangement of all or part of a set of objects.

Theorem

The number of permutations of n objects is $n!$.

Theorem

The number of permutations of n distinct objects taken r at a time is

$$_nP_r = \frac{n!}{(n-r)!}.$$

Theorem

The number of permutations of n objects arranged in a circle is $(n-1)!$.

Permutations

Example 1.4. It is required to seat 5 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible?

The men may be seated in ${}_5P_5$ ways, and the women ${}_4P_4$ ways. Each arrangement of the men may be associated with each arrangement of the women. Hence,

$$\text{Number of arrangements} = {}_5P_5 \cdot {}_4P_4 = 5!\ 4! = (120)(24) = 2880$$

Permutations

In the particular case when $r = n$, this becomes

$$_nP_n = n(n-1)(n-2)...1 = n! \qquad (20)$$

which is called *n factorial*. We can write this formula in terms of factorials as

$$_nP_r = \frac{n!}{(n-r)!} \qquad (21)$$

Permutations

Suppose that a set consists of n objects of which n_1 are of one type (i.e., indistinguishable from each other), n_2 are of a second type, … , n_k are of a kth type. Here, of course, $n = n_1 + n_2 + \ldots + n_k$. Then the number of different permutations of the objects is

$$ {}_nP_{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! n_2! \cdots n_k!} \qquad (22) $$

Combinations

In a permutation we are interested in the order of arrangements of the objects. For example, *abc* is a different permutation from *bca*. In many problems, however, we are only interested in selecting or choosing objects without regard to order. Such selections are called *combinations*. For example, *abc* and *bca* are the same combination.

Combinations

The total number of combinations of r objects selected from n (also called the *combinations of n things taken r at a time*) is denoted by ${}_nC_r$ or $\binom{n}{r}$. We have

$$\binom{n}{r} = {}_nC_r = \frac{n!}{r!(n-r)!} \tag{23}$$

It can also be written

$$\binom{n}{r} = \frac{n(n-1)\cdots(n-r+1)}{r!} = \frac{{}_nP_r}{r!} \tag{24}$$