ME 262 BASIC FLUID MECHANICS FALL 2010

MANNING EQUATION APPLIED TO PARTIALLY FILLED CIRCULAR PIPES

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BASIC EQUATIONS

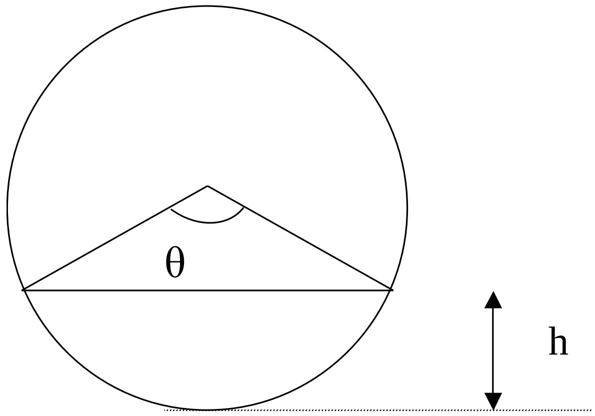
$$V = \frac{1}{n} R_h^{2/3} S^{1/2} \quad (1)$$

$$Q = \frac{A}{n} R_h^{2/3} S^{1/2} \quad (2)$$

$$\theta = 2\cos^{-1}(1 - \frac{2h}{D}) \quad (3a)$$

$$A = \frac{D^2}{8}(\theta - \sin\theta) \quad (4)$$

$$R_h = \frac{D}{4}\left(\frac{\theta - \sin\theta}{\theta}\right) \quad (5)$$



Note that the angle θ is expressed in radians. Equation (3a) can be rewritten as follows:

$$\frac{h}{D} = \frac{1}{2}\left(1 - \cos\left(\frac{\theta}{2}\right)\right) \quad (3b)$$

DEFINITION OF *K*

$$Q = \frac{K}{n} D^{8/3} S^{1/2} \quad (6)$$

$$K = \frac{Qn}{D^{8/3} S^{1/2}} \quad (7)$$

NOTE-1: Metcalf & Eddy (Table 2-5) use the symbol K' for what we denote by K here.

NOTE-2: Metcalf & Eddy use the symbol θ (in Figure 2-25) for half of the surface angle θ.

NOTE-3: In what follows, we will assume the use the metric system (m/sec for V, m for D, and m^3/sec for Q).

TYPES OF PROBLEM THAT REQUIRE ITERATIONS

Iterative calculations are required when h/D is not known in advance. This happens in the following four types of problem:

Type I : Given Q, D, S; find h/D and V.

Type II : Given Q, D, V; find h/D and S.

Type III: Given V, D, S; find h/D and Q.

Type IV: Given Q, V, S; find h/D and D.

SOLUTION METHODS

TYPE I : *Q, D, S* are known; find *h/D* and *V*

First calculate K using Eq.7. The following three possibilities need considering:

1. If $K > 0.335282$, the problem has no solution.
2. If $0 \leq K < 0.311686$, there is exactly one solution.
3. If $0.311686 \leq K < 0.335282$, there are two solutions, i.e. two possible flow depths.

If $K \leq 0.335282$, you can obtain the "exact" solution(s) by solving the following equation (derived by combining Equations 2, 4, and 5) for θ:

$$K = \frac{2^{-13/3}(\theta - \sin\theta)^{5/3}}{\theta^{2/3}} \tag{8}$$

In case there are two possible θ values, the root found by solving Equation 8 iteratively may depend on the initial guess for θ. To avoid iterations and to ensure that the smaller flow depth is found when there are two possible solutions, use the following approximate explicit equation:

$$\theta = 2\times 6^{5/13} K^{3/13}\left(1 + 0.431\sin^{-1}(2.98K)\right) \tag{9}$$

Next, use Eq.3b to calculate *h/D*. To calculate V, use the following equation:

$$K = \frac{\theta}{2} W^{5/2} \tag{10}$$

The definition of W is given by Equation 18.

EXAMPLE 1 (Problem 6.2.6 in Hwang et al., 2010): A corrugated metal storm water pipe is not flowing full but is discharging 5.83 m^3/sec. Assuming uniform flow in the 2 m diameter pipe, determine the flow depth if the 100 m long pipe goes through a 2 m drop in elevation.

SOLUTION: In this problem, the following are given: $Q = 5.83\ m^3/sec$, $D = 2\ m$, $S = 2m/100m = 0.02$, $n = 0.024$ (from Table 6.2, corrugated metal). Since Q, D, and S are given, this is a Type I problem. Let us calculate K first:

$$K = \frac{Qn}{D^{8/3}S^{1/2}} = (5.83\times 0.024)/(2^{8/3}\times 0.02^{1/2}) = 0.1558$$

Thus, $K < 0.311686$ and there is only one solution. We next estimate this solution:

$$\theta = 2\times 6^{5/13} 0.1558^{3/13}\left(1 + 0.431\sin^{-1}(2.98\times 0.1558)\right) = 3.13 \text{ radians}$$

Then Eq.3b,

$$\frac{h}{D} = \frac{1}{2}\left(1 - \cos\left(\frac{3.13}{2}\right)\right) == 0.498$$

gives $h/D = 0.498 \approx 0.50$. Therefore, the pipe is 50 % full, and $h = 1\ m$.

EXAMPLE 2 (Example 2-4 in Metcalf & Eddy, 1981): Determine the depth of flow and velocity in a sewer with a diameter of 300 mm laid on a slope of 0.005 m/m with an *n* value of 0.015 when discharging 0.01 m^3/sec.
SOLUTION: Since *Q, D,* and *S* are given, this is a Type I problem. Calculate *K* first:

$$K = \frac{Qn}{D^{8/3}S^{1/2}} = (0.01\times 0.015)/(0.3^{8/3}\times 0.005^{1/2}) = 0.0526$$

Therefore, $K < 0.311686$ and there is only one solution. An acceptable estimate is:

$$\theta = 2\times 6^{5/13}0.0526^{3/13}\left(1+0.431\sin^{-1}(2.98\times 0.0526)\right) = 2.156 \text{ radians}$$

Then

$$\frac{h}{D} = \frac{1}{2}\left(1-\cos\left(\frac{2.156}{2}\right)\right) == 0.263$$

It is seen that the pipe is 26.3 % full and $h = 79$ *mm*. Velocity is calculated using Eq. 10:

$$V = \left(\frac{2K}{\theta}\right)^{2/5}\frac{D^{2/3}S^{1/2}}{n} = \left(\frac{2\times 0.0526}{2.156}\right)^{2/5}\frac{0.3^{2/3}0.005^{1/2}}{0.015} 0.631 \text{ m/sec.}$$

TYPE II : Given *Q, D, V*; find *h/D* and *S*

First calculate the quantity $Y = 8A/D^2 = 8(Q/V)/D^2$. You can find the exact value of θ by solving the following equation iteratively:

$$Y = (\theta - \sin\theta) \tag{11}$$

Alternatively, equations developed by Li (1994) can be used:

$$\theta = \frac{Y + \sin\theta_0 - \theta_0\cos\theta_0}{1-\cos\theta_0} \qquad (0 < Y < \pi) \tag{12}$$

$$\theta = 2\pi - \frac{(2\pi - Y) + \sin\theta_0' - \theta_0'\cos\theta_0'}{1-\cos\theta_0'} \qquad (\pi < Y < 2\pi) \tag{13}$$

where $\theta_0 = (6Y)^{1/3}$ and $\theta_0' = (6(2\pi - Y))^{1/3}$. R_h is next calculated using Equation 5. Next, the slope *S* can be calculated:

$$S = (n\,V)^2 R_h^{-4/3} \tag{14}$$

Simpler formulas suitable for quick manual calculations were developed (Akgiray, 2004):

$$\theta = 2.51Y^{-0.34}\left[\cos^{-1}(1-0.3183Y)\right]^{1.33} \tag{15}$$

Once θ is computed, Eq. 3b can next be used to calculate *h/D*. To calculate *h/D* directly (without calculating θ first), the following approximate relation may be used (Akgiray, 2004):

$$\frac{h}{D} = 0.0047Y^3 - 0.0453Y^2 + 0.2554Y \tag{16}$$

The equation giving the slope is the following:

$$S = \left((D/2Q)^{4/3}V^{10/3}n^2\right)\theta^{4/3} \tag{17}$$

EXAMPLE 3 (This example was used by Esen (1993) and Li (1994) to illustrate their calculation methods): Calculate the minimum slope of a sewer to carry a flow of 0.01 m^3/sec. The minimum velocity is specified as 0.5 m/sec and as a preliminary value the diameter of the sewer will be assumed to be 0.45 m. Assume n = 0.013.
SOLUTION: Since Q, D, and V are given, this is a Type II problem. Calculate Y first:
$Y = 8A/D^2 = 8Q/(VD^2) = 8(0.01)/(0.5\times0.45^2) = 0.7901$
Although it is not required in this problem, let us first calculate h/D using Eq.16:

$$\frac{h}{D} = 0.0047(0.7901)^3 - 0.0453(0.7901)^2 + 0.2554(0.7901) = 0.176$$

$h = (0.45)(0.176) = 0.079\ m = 79\ mm$
Surface angle is estimated as:

$$\theta = 2.51(0.7901)^{-0.34}\left[\cos^{-1}(1-0.3183(0.7901))\right]^{1.33} = 1.7730 \text{ radians}$$

We can now calculate the slope:

$$S = \left((0.45/2\times0.01)^{4/3}0.5^{10/3}0.013^2\right)1.773^{4/3} = 0.00229$$

TYPE III: Given *V, D, S*; find *h/D* and *Q*.

First, calculate the quantity W:

$$W = \frac{Vn}{D^{2/3}S^{1/2}} \tag{18}$$

The following three possibilities exist:

1. If $W > 0.452421$, the problem has no solution.
2. If $0 \leq W < 0.396850$, there is exactly one solution.
3. If $0.396850 \leq W < 0.452421$, there are two solutions, i.e. two possible flow depths.

If $W \leq 0.452421$, you can obtain the "exact" solution(s) by solving the following equation for θ:

$$W = \left(\frac{\theta - \sin\theta}{4\theta}\right)^{2/3} \tag{19}$$

In case there are two possible θ values, the root found by solving Equation 19 iteratively may depend on the initial guess for θ. To avoid iterations and to ensure that the smaller flow depth is found when there are two possible solutions, use the following approximate explicit equation:

$$\theta = 24^{1/2}W^{3/4}\left(1 + 0.255\left[\sin^{-1}(2.21W)\right]^{2.1}\right) \tag{20}$$

Next, use Eq.3b to calculate h/D and Equation 10 to calculate Q.

EXAMPLE 4 (Example 4-12 in Hammer & Hammer, 1996): An 18-in sewer pipe, n= 0.013, is placed on a slope of 0.0025. At what depth of flow does the velocity of flow equal 2.0 ft/sec?
SOLUTION: D, S, and V are given. This is a Type III problem. Let us first convert the values of D and V to the corresponding metric units:
D = (18in)(2.54cm/in)(1m/100cm) = 0.457 m

V = (2.0 ft/sec)(1m/3.2808 ft) = 0.6096 m/sec
Next, calculate W:

$$W = \frac{0.6096 \times 0.013}{0.457^{2/3} 0.0025^{1/2}} = 0.267$$

Since $W < 0.39685$, there is exactly one solution. A good estimate is:

$$\theta = 24^{1/2} 0.267^{3/4} \left(1 + 0.255 \left[\sin^{-1}(2.21 \times 0.267)\right]^{2.1}\right) = 2.0 \text{ radians}$$

$$\frac{h}{D} = \frac{1}{2}\left(1 - \cos\left(\frac{2.0}{2}\right)\right) == 0.230$$

Therefore, the pipe is 23 % full, and $h = 0.105$ *m* = *4.12 in*. Although it is not required in this problem, let us also calculate Q to illustrate the use of Eq. 10:

$$K = \frac{\theta}{2} W^{5/2} = \frac{2}{2} 0.267^{5/2} = 0.037 \Rightarrow$$

$$Q = \frac{K}{n} D^{8/3} S^{1/2} = \frac{0.037}{0.013} 0.457^{8/3} 0.0025^{1/2} = 0.0176\, m^3/\text{sec}$$

TYPE IV: Given *Q, V, S*; find *h/D* and *D*.

First, calculate $A = Q/V$ and then the quantity Z:

$$Z = \frac{Vn}{A^{1/3} S^{1/2}} \tag{21}$$

The following three possibilities exist:

1. If $Z > 0.541926$, the problem has no solution.
2. If $0 \le Z < 0.430127$, there is exactly one solution.
3. If $0.430127 \le Z < 0.541926$, there are two solutions, i.e. two possible flow depths.

If $Z \le 0.541926$, you can obtain the "exact" solution(s) by solving the following equation for θ:

$$Z = \left(\frac{\theta - \sin\theta}{2\theta^2}\right)^{1/3} \tag{22}$$

In case there are two possible θ values, the root found by solving Equation 22 iteratively may depend on the initial guess for θ. To avoid iterations and to ensure that the smaller flow depth is found when there are two possible solutions, use the following approximate explicit equation:

$$\theta = 12Z^3 \left(1 + 0.0765 \left[\sin^{-1}(1.84527Z)\right]^{4.72}\right) \tag{23}$$

Next, use Eq.3b to calculate *h/D*. To calculate *D*, use Equation 22:

$$D = \sqrt{\frac{8(Q/V)}{(\theta - \sin\theta)}} \tag{24}$$

EXAMPLE 5: A pipe with a roughness coefficient $n = 0.015$ is to be laid on a slope of 0.004 m/m. The velocity and discharge have been specified as 0.60 m/s and 0.02 m^3/s, respectively. Find the necessary pipe diameter and calculate the flow depth.
SOLUTION: This is a Type IV problem. First calculate

A = Q/V = 0.02/0.6 = 0.033 m^2
and then use Equations 21 and 23:
Z = Vn/(A$^{1/3}$*S*$^{1/2}$*) = 0.442*

$$\theta = 12(0.442)^3\left(1+0.0765\left[\sin^{-1}(1.84527\times 0.442)\right]^{4.72}\right) = 1.10 \text{ radians}$$

$$\frac{h}{D} = \frac{1}{2}\left(1-\cos\left(\frac{1.1}{2}\right)\right) == 0.074$$

Equation 24 is used next to calculate the diameter:

$$D = \sqrt{\frac{8(0.02/0.6)}{(1.1-\sin 1.1)}} = 1.13 \text{ m.}$$

EXAMPLE 6 (Example 2-5 in Metcalf & Eddy, 1981): Determine the diameter of a sewer required to handle a flow of 0.15 m^3/sec when flowing 65 percent full. The sewer is to be laid at a slope of 0.001 m/m and the *n* value is assumed to be 0.013.

SOLUTION: In this problem, the following are given: $Q = 0.15\ m^3/sec$, $S = 0.001$, $n = 0.013$, $h/D = 0.65$. Since *h/D* is given, this is not one of the four types of problem that require iterations. Calculations proceed in a straightforward manner:

$$\theta = 2\cos^{-1}(1-2\times 0.65) = 3.75 \text{ radians}$$

$$A = \frac{D^2}{8}(3.75-\sin 3.75) = 0.54D^2$$

$$R_h = \frac{D}{4}\left(\frac{3.75-\sin 3.75}{3.75}\right) = 0.288D$$

$$Q = \frac{A}{n}R_h^{2/3}S^{1/2} = \frac{0.54D^2}{0.013}(0.288D)^{2/3}0.001^{1/2} = 0.15 \text{ m}^3/\text{sec}$$

From this it follows that $D = 0.604\ m$.

EXERCISE: Re-work Examples 1 through 5 by solving the applicable equations (Eqs. 8, 11, 19, and 22) by means of manual iterations (i.e. with a hand calculator). Next, solve these equations using software such as the EXCEL Add-in *Solver*. How close are the approximate solutions found here to the "exact" answers you get?

REFERENCES

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