Individual Rationality and Budget Balance in VCG

Game Theory Course:
Jackson, Leyton-Brown & Shoham

Two definitions

Definition (Choice-set monotonicity)

An environment exhibits choice-set monotonicity if $\forall i,\ X_{-i} \subseteq X$.

- removing any agent weakly decreases—that is, never increases—the mechanism's set of possible choices X

Definition (No negative externalities)

An environment exhibits no negative externalities if $\forall i \forall x \in X_{-i},\ v_i(x) \geq 0$.

- every agent has zero or positive utility for any choice that can be made without his participation

Example: road referendum

Example

Consider the problem of holding a referendum to decide whether or not to build a road.

- The set of choices is independent of the number of agents, satisfying choice-set monotonicity.
- No agent negatively values the project, though some might value the situation in which the project is not undertaken more highly than the situation in which it is.

Example: simple exchange

Example

Consider a market setting consisting of agents interested in buying a single unit of a good such as a share of stock, and another set of agents interested in selling a single unit of this good. The choices in this environment are sets of buyer-seller pairings (prices are imposed through the payment function).

- If a new agent is introduced into the market, no previously-existing pairings become infeasible, but new ones become possible; thus choice-set monotonicity is satisfied.
- Because agents have zero utility both for choices that involve trades between other agents and no trades at all, there are no negative externalities.

VCG Individual Rationality

Theorem

The VCG mechanism is ex-post *individual rational when the choice set monotonicity and no negative externalities properties hold.*

Proof.

All agents truthfully declare their valuations in equilibrium. Then

$$u_i = v_i(\chi(v)) - \left(\sum_{j\neq i} v_j(\chi(v_{-i})) - \sum_{j\neq i} v_j(\chi(v))\right)$$
$$= \sum_{i} v_i(\chi(v)) - \sum_{j\neq i} v_j(\chi(v_{-i})) \qquad (1)$$

$\chi(v)$ is the outcome that maximizes social welfare, and so the optimization could have picked $\chi(v_{-i})$ instead (by choice set monotonicity). Thus,

$$\sum_{j} v_j(\chi(v)) \geq \sum_{j} v_j(\chi(v_{-i})).$$

VCG Individual Rationality

Theorem

The VCG mechanism is ex-post *individual rational when the choice set monotonicity and no negative externalities properties hold.*

Proof.

$$\sum_j v_j(\chi(v)) \geq \sum_j v_j(\chi(v_{-i})).$$

Furthermore, from no negative externalities,

$$v_i(\chi(v_{-i})) \geq 0.$$

Therefore,

$$\sum_i v_i(\chi(v)) \geq \sum_{j \neq i} v_j(\chi(v_{-i})),$$

and thus Equation (1) is non-negative.

Another property

Definition (No single-agent effect)

An environment exhibits no single-agent effect if $\forall i$, $\forall v_{-i}$, $\forall x \in \arg\max_y \sum_j v_j(y)$ there exists a choice x' that is feasible without i and that has $\sum_{j \neq i} v_j(x') \geq \sum_{j \neq i} v_j(x)$.

Welfare of agents other than i is weakly increased by dropping i.

Another property

Definition (No single-agent effect)

An environment exhibits no single-agent effect if $\forall i$, $\forall v_{-i}$, $\forall x \in \arg\max_y \sum_j v_j(y)$ there exists a choice x' that is feasible without i and that has $\sum_{j \neq i} v_j(x') \geq \sum_{j \neq i} v_j(x)$.

Welfare of agents other than i is weakly increased by dropping i.

Example

Consider a single-sided auction. Dropping an agent just reduces the amount of competition, making the other agents better off.

Good news

Theorem

The VCG mechanism is weakly budget-balanced when the no single-agent effect property holds.

Proof.

Assume truth-telling in equilibrium. We must show that the sum of transfers from agents to the center is greater than or equal to zero.

$$\sum_i p_i(v) = \sum_i \left(\sum_{j \neq i} v_j(\chi(v_{-i})) - \sum_{j \neq i} v_j(\chi(v)) \right)$$

From the no single-agent effect condition we have that

$$\forall i \ \sum_{j \neq i} v_j(\chi(v_{-i})) \geq \sum_{j \neq i} v_j(\chi(v)).$$

Thus the result follows directly.

More good news

Theorem (Krishna & Perry, 1998)

In any Bayesian game setting in which VCG is ex post *individually rational, VCG collects at least as much revenue as any other efficient and* ex interim *individually-rational mechanism.*

- This is somewhat surprising: does not require dominant strategies, and hence compares VCG to all Bayes–Nash mechanisms.
- A useful corollary: VCG is as budget balanced as any efficient mechanism can be
  - it satisfies weak budget balance in every case where *any* dominant strategy, efficient and *ex interim* IR mechanism is able to.