# The Myerson–Satterthwaite Theorem

Game Theory Course:
Jackson, Leyton-Brown & Shoham

# Efficient Trade

- People have private information about the utilities for various exchanges of goods at various prices.
- Can we design a mechanism that always results in efficient trade?
- Are strikes unavoidable?

# Simple Exchange Setting

- Exchange of a single unit of an indivisible good
- Seller initially has the item and has a value for it of $\theta_S \in [0, 1]$
- Buyer has need for the item and has a value for it of $\theta_B \in [0, 1]$

# An Example

- The buyer's value is equally likely to be either .1 or 1
- The seller's value for the good is equally likely to be 0 or .9
- Trade should take place for all combinations of values except (.1,.9)

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  - Better to set the high price.
  - *Inefficient trade:* (.1, 0) do not trade
- What about other mechanisms?

# Efficiency, Budget Balance and Individual Rationality

## Theorem (Myerson–Satterthwaite)

*There exist distributions on the buyer's and seller's valuations such that: There does not exist any Bayesian incentive-compatible mechanism is simultaneously efficient, weakly budget balanced and interim individual rational.*

# Proof

- Can get efficient trades for some distributions:
  - Suppose the buyers value is always above $v$ and the sellers value is always below $v$.
  - Mechanism: always exchange the good, and at the price $p_B(\theta_B) = v = -p_S(\theta_S)$.
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  - Satisfies all of the conditions.
- Let us show the proof based our example:
  - The buyer's value is equally likely to be either .1 or 1
  - The seller's value for the good is equally likely to be 0 or .9
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# Proof

- Show the proof for *fully* budget balanced trade that is *ex post* individually rational. Extension of the proof is easy (you can do it!)
  - Trade should take place for all combinations of values except $(\theta_B, \theta_S) = (.1, .9)$.
  - Budget balance: we can write payments as a single price $p(\theta_B, \theta_S)$ (payment made by buyer, received by the seller)
  - Weak budget balance: you can extend the proof - noting that the payment made by the buyer has to be at least that received by the seller.

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    - $1 - p(1,0) + .1 \geq .9 + 0$ or $.2 \geq p(1,0)$.
  - So: $.2 \geq p(1,0)$ and $p(1,0) \geq .8$ - impossible!

# Summary

- Private information about values:
  - necessitates some inefficiencies in voluntary trade
  - tension between incentives and efficiency
- Have you ever walked away from bargaining even when you initially thought a trade might have been possible?