Revenue Equivalence

Game Theory Course:
Jackson, Leyton-Brown & Shoham

Revenue Equivalence

- Which auction? To some extent, it doesn't matter...

Theorem (Revenue Equivalence Theorem)

Assume that each of n risk-neutral agents has an independent private valuation for a single good at auction, each drawn from cumulative distribution F. Then any two auction mechanisms in which

- *in equilibrium, the good is always allocated in the same way; and*
- *any agent with valuation 0 has an expected utility of 0;*

both yield the same expected revenue, and both result in any bidder with valuation v making the same expected payment.

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In fact, this even holds beyond IPV and single good.

First and Second-Price Auctions

- The k^{th} order statistic of a distribution: the expected value of the k^{th}-largest of n draws.
- For n IID draws from $[0, v_{max}]$, the k^{th} order statistic is

$$\frac{n+1-k}{n+1} v_{max}.$$

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- First and second-price auctions satisfy the requirements of the revenue equivalence theorem
  - every symmetric game has a symmetric equilibrium
  - in a symmetric equilibrium of this auction game, higher bid $\Leftrightarrow$ higher valuation

Applying Revenue Equivalence

- Thus, a bidder in a FPA must bid his expected payment conditional on being the winner of a second-price auction
  - this conditioning will be correct if he does win the FPA; otherwise, his bid doesn't matter anyway
  - if v_i is the high value, there are then $n-1$ other values drawn from the uniform distribution on $[0, v_i]$
  - thus, the expected value of the second-highest bid is the first-order statistic of $n-1$ draws from $[0, v_i]$:

$$\frac{n+1-k}{n+1} v_{max} = \frac{(n-1)+1-(1)}{(n-1)+1}(v_i) = \frac{n-1}{n} v_i$$

- This shows how we derived our earlier claim about n-bidder first-price auctions.

Proving Revenue Equivalence

- $\chi_i(v_i|s)$: i's ex interim allocation probability given type v_i, everyone following equilibrium strategy s
- $p_i(v_i|s)$: i's ex interim expected payment

Theorem (Bayes–Nash Equilibrium Characterization)

When values are drawn from a continuous joint distribution F and agents are risk neutral, a strategy profile s is in Bayes–Nash equilibrium only if for all i:

1. *(monotonicity)* $\chi_i(v_i|s)$ *is monotone non-decreasing, and*
2. *(payment identity)* $p_i(v_i|s) = v_i\chi_i(v_i|s) - \int_0^{v_i} \chi_i(z|s)dz + p_i(0|s)$,

where often $p_i(0|s) = 0$. *If s is onto then the converse also holds.*

Proof

The proof[1] proceeds in three parts:

- s is a Bayes–Nash equilibrium if the characterization holds and s is onto;
- s is a Bayes–Nash equilibrium only if monotonicity holds; and
- s is a Bayes–Nash equilibrium only if the payment identity holds.

We consider the special case where the support of each agent's distribution is $[0, \infty)$. To reduce notation we assume $p_i(0|s) = 0$.

[1] The proof follows an elegant version by Jason Hartline; see `www.eecs.northwestern.edu/~hartline/amd.pdf`.
We also use figures adapted from that proof, with permission.

(1) s is BNE if characterization holds, s is onto

If i deviates from s and takes action $s_i(\widehat{v_i})$ rather than $s_i(v_i)$, i gets utility

$$u_{i,v_i}(\widehat{v_i}|s) = v_i x_i(\widehat{v_i}|s) - p_i(\widehat{v_i}|s).$$

Note that i can play any action in this way because s is onto. The strategy profile s is in equilibrium if for all i and all v_i and $\widehat{v_i}$,

$$u_{i,v_i}(v_i|s) \geq u_{i,v_i}(\widehat{v_i}|s).$$

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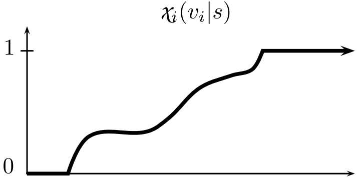

Consider some arbitrary, monotonic allocation rule.

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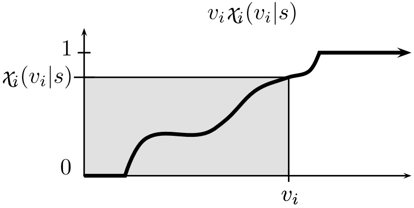
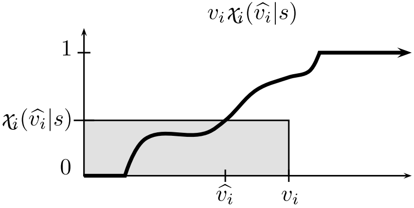
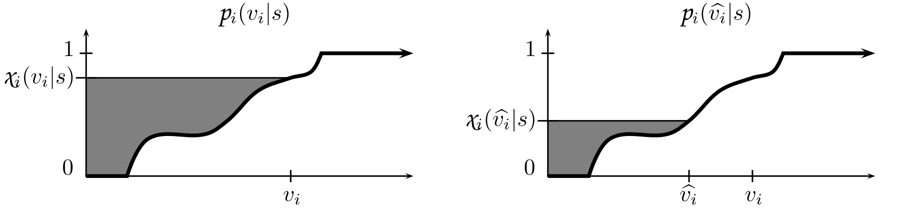




i's surplus for playing as type v_i and $\widehat{v}_i$. (We consider $\widehat{v}_i < v_i$; the opposite case follows from a similar argument.)

(I) s is BNE if characterization holds, s is onto



i's expected payment for playing as type v_i and $\widehat{v}_i$.
Recall: $p_i(v_i|s) = v_i x_i(v_i|s) - \int_0^{v_i} x_i(z|s)dz$.

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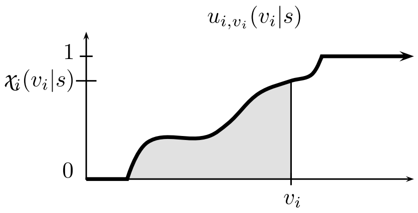

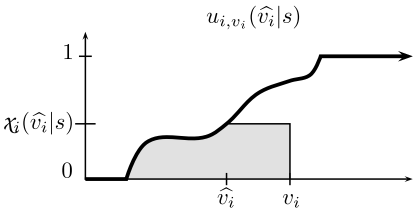

i's expected utility for playing as type v_i and $\widehat{v}_i$.

Recall: $u_{i,v_i}(\widehat{v}_i|s) = v_i \chi_i(\widehat{v}_i|s) - p_i(\widehat{v}_i|s)$.

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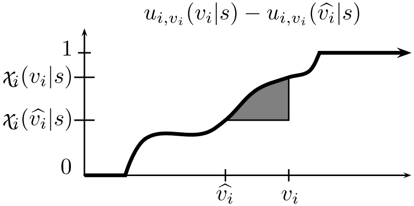

Difference in expected utility between following s and deviating.
This difference is nonnegative by monotonicity.

(2) s is BNE only if monotonicity holds

BNE implies $\forall v_i$ and v_i', $u_{i,v_i}(v_i|s) \geq u_{i,v_i}(v_i'|s)$. Expanding,

$$v_i x_i(v_i|s) - p_i(v_i|s) \geq v_i x_i(v_i'|s) - p_i(v_i'|s).$$

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$$v_i \chi_i(v_i|s) - p_i(v_i|s) \geq v_i \chi_i(v_i'|s) - p_i(v_i'|s).$$

Consider two values z_1 and z_2. Subbing in $v_i = z_1$, $v_i' = z_2$ and $v_i = z2$, $v_i' = z_1$, we obtain two inequalities:

$$\begin{aligned} v_i = z_1, v_i' = z_2 \quad &\Rightarrow \quad z_2 \chi_i(z_2|s) - p_i(z_2|s) &&\geq z_2 \chi_i(z_1|s) - p_i(z_1|s); \\ v_i = z_2, v_i' = z_1 \quad &\Rightarrow \quad z_1 \chi_i(z_1|s) - p_i(z_1|s) &&\geq z_1 \chi_i(z_2|s) - p_i(z_2|s). \end{aligned}$$

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v_i = z_2, v_i' = z_1 \quad &\Rightarrow \quad z_1 x_i(z_1|s) - p_i(z_1|s) &\geq z_1 x_i(z_2|s) - p_i(z_2|s).
\end{aligned}$$

Adding them and canceling p_i terms we have

$$z_2 x_i(z_2|s) + z_1 x_i(z_1|s) \geq z_2 x_i(z_1|s) + z_1 x_i(z_2|s)$$

$$(z_2 - z_1)(x_i(z_2|s) - x_i(z_1|s)) \geq 0$$

(2) s is BNE only if monotonicity holds

BNE implies $\forall v_i$ and v_i', $u_{i,v_i}(v_i|s) \geq u_{i,v_i}(v_i'|s)$. Expanding,

$$v_i \chi_i(v_i|s) - p_i(v_i|s) \geq v_i \chi_i(v_i'|s) - p_i(v_i'|s).$$

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$$z_2 \chi_i(z_2|s) + z_1 \chi_i(z_1|s) \geq z_2 \chi_i(z_1|s) + z_1 \chi_i(z_2|s)$$

$$(z_2 - z_1)(\chi_i(z_2|s) - \chi_i(z_1|s)) \geq 0$$

Thus, $z_2 - z_1 > 0$ implies $\chi_i(z_2|s) - \chi_i(z_1|s) \geq 0$. In other words, $\chi_i(\cdot|s)$ must be monotone non-decreasing.

(3) s is BNE only if payment identity holds

Recall our two inequalities from Step (2):

$$v_i = z_1, v_i' = z_2 \quad \Rightarrow \quad z_2 \chi_i(z_2|s) - p_i(z_2|s) \geq z_2 \chi_i(z_1|s) - p_i(z_1|s);$$
$$v_i = z_2, v_i' = z_1 \quad \Rightarrow \quad z_1 \chi_i(z_1|s) - p_i(z_1|s) \geq z_1 \chi_i(z_2|s) - p_i(z_2|s).$$

Solve each for $p_i(z_2|s) - p_i(z_1|s)$:

$$z_2(\chi_i(z_2|s) - \chi_i(z_1|s)) \geq p_i(z_2|s) - p_i(z_1|s) \geq z_1(\chi_i(z_2|s) - \chi_i(z_1|s))$$

We now have an upper bound and a lower bound on the difference in expected payments for types z_2 and z_1.

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We can visualize this…

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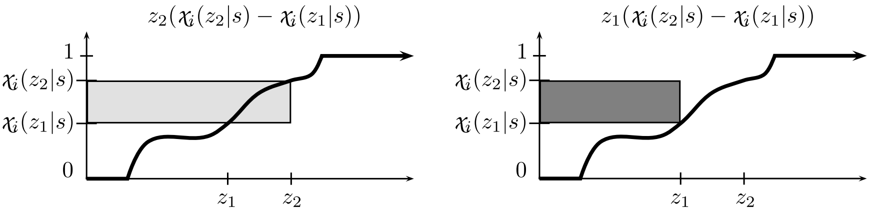

The upper bound and lower bound on the payment difference.

$$z_2(x_i(z_2|s) - x_i(z_1|s)) \geq p_i(z_2|s) - p_i(z_1|s) \geq z_1(x_i(z_2|s) - x_i(z_1|s))$$

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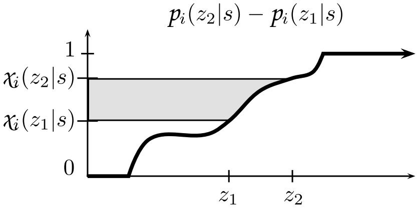

The only payment rule that satisfies these upper and lower bounds for all pairs of types z_2 and z_1 has payment difference exactly equal to the area to the left of the allocation rule between $\chi_i(z_1|s)$ and $\chi_i(z_2|s)$.

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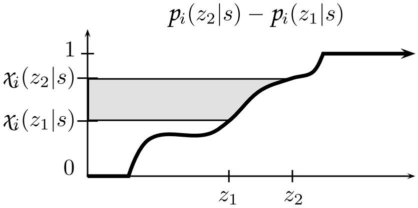

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Conclusions

- If two mechanisms have the same allocation rule, they need to have (essentially) the same payment rule too.
- A key corollary: all efficient auctions yield the same revenue in equilibrium.
- This applies to some pretty strange auction types: 3rd price, auctions in which losers have to pay, etc.
- Do note: we need risk neutrality for revenue equivalence.