





# Simple Exchange Setting



- Exchange of a single unit of an indivisible good
- Seller initially has the item and has a value for it of  $\theta_S \in [0, 1]$
- Buyer has need for the item and has a value for it of  $\theta_B \in [0, 1]$

# An Example



- The buyer's value is equally likely to be either  $.1$  or  $1$
- The seller's value for the good is equally likely to be  $0$  or  $.9$
- Trade should take place for all combinations of values except  $(.1, .9)$





# An Example of a Mechanism

- The seller announces a price in  $[0, 1]$
- The buyer either buys or not at that price.
- The seller should say a price of either  $.1$  or  $1$  (presume that buyer says yes when indifferent)
- When the seller's value is  $0$ :
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  - Better to set the high price.
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  - Better to set the high price.
  - *Inefficient trade*:  $(.1, 0)$  do not trade
- What about other mechanisms?



# Efficiency, Budget Balance and Individual Rationality

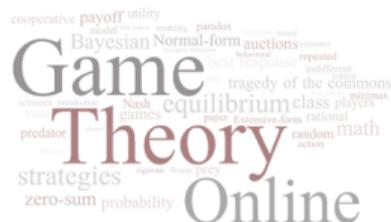


## Theorem (Myerson–Satterthwaite)

*There exist distributions on the buyer's and seller's valuations such that: There does not exist any Bayesian incentive-compatible mechanism is simultaneously efficient, weakly budget balanced and interim individual rational.*

- Can get efficient trades for some distributions:
  - Suppose the buyers value is always above  $v$  and the sellers value is always below  $v$ .
  - Mechanism: always exchange the good, and at the price  $p_B(\theta_B) = v = -p_S(\theta_S)$ .
  - Satisfies all of the conditions.

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  - Satisfies all of the conditions.
- Let us show the proof based our example:
  - The buyer's value is equally likely to be either .1 or 1
  - The seller's value for the good is equally likely to be 0 or .9
  - Trade should take place for all combinations of values except (.1,.9)

# Proof



- Show the proof for *fully* budget balanced trade that is *ex post* individually rational. Extension of the proof is easy (you can do it!)
  - Trade should take place for all combinations of values except  $(\theta_B, \theta_S) = (.1, .9)$ .
  - Budget balance: we can write payments as a single price  $p(\theta_B, \theta_S)$  (payment made by buyer, received by the seller)
  - Weak budget balance: you can extend the proof - noting that the payment made by the buyer has to be at least that received by the seller.



# Proof

- (1)  $p(1, .9) \geq .9$  by individual rationality of the seller.
- (2)  $p(.1, 0) \leq .1$  by individual rationality of the buyer.











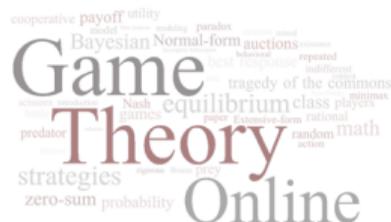
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- (3)  $p(.1, .9) = 0$  by individual rationality of both the buyer and the seller.
- (4)  $p(1, 0) = ?$ 
  - incentive compatibility for seller of type  $\theta_S = 0$  not wanting to pretend to be  $\theta_S = .9$ :
    - $\frac{p(1,0)}{2} + \frac{p(.1,0)}{2} \geq \frac{p(1,.9)}{2} + \frac{p(.1,.9)}{2}$ , which implies by (1), (2), (3) that
    - $p(1, 0) + .1 \geq .9 + 0$  or  $p(1, 0) \geq .8$ .



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  - incentive compatibility for buyer of type  $\theta_B = 1$  not wanting to pretend to be  $\theta_B = .1$ :



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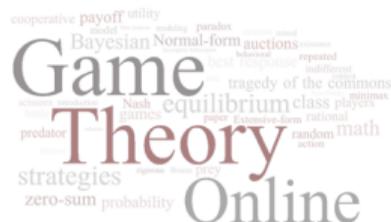


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    - $1 - p(1, 0) + .1 \geq .9 + 0$  or  $.2 \geq p(1, 0)$ .

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    - $1 - p(1, 0) + .1 \geq .9 + 0$  or  $.2 \geq p(1, 0)$ .
  - So:  $.2 \geq p(1, 0)$  and  $p(1, 0) \geq .8$  - impossible!

