

Hardness Beyond 2×2 Games

Game Theory Course:
Jackson, Leyton-Brown & Shoham

Algorithms



- LCP (Linear Complementarity) formulation
 - [Lemke-Howson '64]
- Support Enumeration Method
 - [Porter et al. '04]

Hardness beyond 2×2 games

Early History

- **1928 von Neumann:** existence of Equilibrium in 2-player, zero-sum games
 - proof uses Brouwer's fixed point theorem;
 - led directly to algorithms:
 - Danzig '57: equivalent to LP duality
 - Khachiyan'79: polynomial-time solvable



[illegible]

- **1928 von Neumann:** existence of Equilibrium in 2-player, zero-sum games

- proof also uses Brouwer's fixed point theorem;
- intense effort on equilibrium algorithms:
 - Kuhn '61, Mangasarian '64, **Lemke-Howson '64**, Rosenmüller '71, Wilson '71, Scarf '67, Eaves '72, Laan-Talman '79, **Porter et al. '04**, ...
- ... all exponential in the worst case

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The Lemke-Howson Algorithm

- LCP (Linear Complementarity) formulation



$$\sum_{k \in A_2} u_1(a_1^j, a_2^k) \cdot s_2^k + r_1^j = U_1^* \quad \forall j \in A_1$$

$$\sum_{j \in A_1} u_2(a_1^j, a_2^k) \cdot s_1^j + r_2^k = U_2^* \quad \forall k \in A_2$$

$$\sum_{j \in A_1} s_1^j = 1, \quad \sum_{k \in A_2} s_2^k = 1$$

$$s_1^j \geq 0, \quad s_2^k \geq 0 \quad \forall j \in A_1, \quad \forall k \in A_2$$

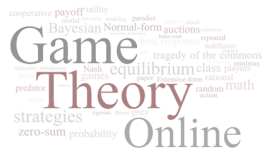
$$r_1^j \geq 0, \quad r_2^k \geq 0 \quad \forall j \in A_1, \quad \forall k \in A_2$$

$$r_1^j \cdot s_1^j = 0, \quad r_2^k \cdot s_2^k = 0 \quad \forall j \in A_1, \quad \forall k \in A_2$$

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Support Enumeration Method: Porter et al. 2004

- Step 1: Finding a NE with a specific support



$$\sum_{a_{-i} \in \sigma_{-i}} p(a_{-i}) u_i(a_i, a_{-i}) = v_i \quad \forall i \in \{1, 2\}, a_i \in \sigma_i$$

$$\sum_{a_{-i} \in \sigma_{-i}} p(a_{-i}) u_i(a_i, a_{-i}) \leq v_i \quad \forall i \in \{1, 2\}, a_i \notin \sigma_i$$

$$p_i(a_i) \geq 0 \quad \forall i \in \{1, 2\}, a_i \in \sigma_i$$

$$p_i(a_i) = 0 \quad \forall i \in \{1, 2\}, a_i \notin \sigma_i$$

$$\sum_{a_i \in \sigma_i} p_i(a_i) = 1 \quad \forall i \in \{1, 2\}$$

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Support Enumeration Method: Porter et al. 2004



- Step 2: Smart heuristic search through all sets of support

Hardness beyond 2×2 games

From Algorithms to Complexity Analysis

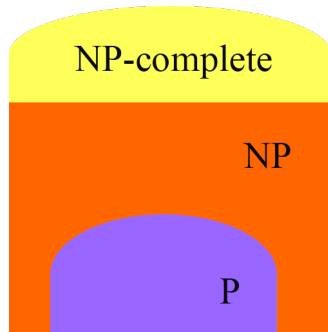


- These algorithms have exponential worst-case time complexity.
- So do all known others.
- Can we do better?

Hardness beyond 2×2 games

From Algorithms to Complexity Analysis

- Reminder of a (small part) of the complexity hierarchy.



Game Theory Online

cooperative payoff utility should increase seeking paradox cooperative equal Bayesian Normal-form auctions economic repeated tragedy of the commons Nash equilibrium class players rational math random action strategies zero-sum probability

Hardness beyond 2×2 games

From Algorithms to Complexity Analysis



So, is it NP-complete to find a Nash equilibrium?

- Strictly speaking, no, since a solution is guaranteed to exist...
- However, it is NP-complete to find a “tiny” bit more info than a Nash equilibrium; e.g., the following are NP-complete:
 1. **(Uniqueness)** Given a game G , does there exist a unique equilibrium in G ?
 2. **(Pareto optimality)** Given a game G , does there exist a strictly Pareto efficient equilibrium in G ?
 3. **(Guaranteed payoff)** Given a game G and a value v , does there exist an equilibrium in G in which some player i obtains an expected payoff of at least v ?
 4. **(Guaranteed social welfare)** Given a game G , does there exist an equilibrium in which the sum of agents' utilities is at least k ?
 5. **(Action inclusion)** Given a game G and an action $a_i \in A_i$ for some player $i \in N$, does there exist an equilibrium of G in which player i plays action a_i with strictly positive probability?
 6. **(Action exclusion)** Given a game G and an action $a_i \in A_i$ for some player $i \in N$, does there exist an equilibrium of G in which player i plays action a_i with zero probability?

Game Theory

Bayesian Normal-form auctions economic

cooperative payoff utility paradox prisoner's dilemma

tragedy of the commons repeated adulterous antitrust

equilibrium class rational math

Nash equilibria paper Extensive-form rational

predator strategies zero-sum probability Online

agencies Brock (1997) game form auctions economic best first last second

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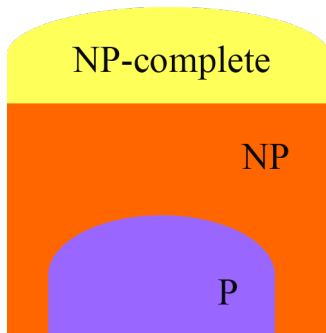
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From Algorithms to Complexity Analysis

Where is PPAD?

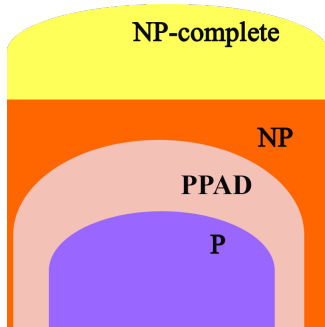
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Hardness beyond 2×2 games

From Algorithms to Complexity Analysis



Where is PPAD?



From Algorithms to Complexity Analysis

The Complexity of the Nash Equilibrium

- for games with ≥ 4 players;
[Daskalakis, Goldberg, Papadimitriou '05]
- for games with 3 players; [Chen, Deng '05] &
[Daskalakis, Papadimitriou '05]
- for games with 2 players. [Chen, Deng '06]