MATH 171 FINAL EXAM (03.01.2011)

Name: No:

Q1 (15)	Q2 (20)	Q3 (20)	Q4 (25)	Q5 (10)	Q6 (10)	Total (100)

ATTENTION: There are 6 questions on 6 pages. Solve all of them. Duration is 90 minutes. Show all your calculation.

1) (a) Find $\lim\limits_{h\to 0}\dfrac{\sqrt{4-h}-2}{h}$. Do not use L'Hospital. (10 pts.)

Solution:

$$\lim_{h\to 0}\frac{\sqrt{4-h}-2}{h}=\lim_{h\to 0}\frac{\left(\sqrt{4-h}-2\right)\left(\sqrt{4-h}+2\right)}{h\left(\sqrt{4-h}+2\right)}$$

$$=\lim_{h\to 0}\frac{\left(\sqrt{4-h}-2\right)\left(\sqrt{4-h}+2\right)}{h\left(\sqrt{4-h}+2\right)}$$

$$=\lim_{h\to 0}\frac{\cancel{4}-\cancel{h}-\cancel{4}}{\cancel{h}\left(\sqrt{4-h}+2\right)}$$

$$=\lim_{h\to 0}\frac{-1}{\left(\sqrt{4-h}+2\right)}=-\frac{1}{4}$$

(b) Find $\lim\limits_{x\to 2}\dfrac{x^4-16}{x^3-8}$. Do not use L'Hospital. (5 pts.)

Solution:

$$\lim_{x\to 2}\frac{x^4-16}{x^3-8}=\lim_{x\to 2}\frac{\left(x^2+4\right)\left(x^2-4\right)}{(x-2)\left(x^2+2x+4\right)}$$

$$=\lim_{x\to 2}\frac{\left(x^2+4\right)(x+2)\cancel{(x-2)}}{\cancel{(x-2)}\left(x^2+2x+4\right)}=\frac{(8)(4)}{(12)}=\frac{8}{3}$$

2) (a) Find an equation of the tangential line to the given curve at the point (1, 2):

$$y = x + \frac{1}{x}$$

(10 pts.)

Solution:

$$y' = 1 - \frac{1}{x^2} \qquad \Rightarrow \qquad m = y'\big|_{x=1} = 1 - 1 = 0$$

The equation of the straight line to the given curve at the point (1, 2) with the zero slope (m = 0) is

$$y - y_0 = m(x - x_0) \qquad \Rightarrow \qquad y - 2 = 0(x-1) \qquad \Rightarrow \qquad y = 2$$

b) $f(x) = x\ e^{x/2}$ find $f''(0)$. (10 pts.)

Solution:

$$f'(x) = e^{x/2} + x\frac{1}{2}e^{x/2} = \frac{1}{2}(2+x)e^{x/2}$$

$$f''(x) = \frac{1}{2}\left[(2+x)'e^{x/2} + (2+x)\left(e^{x/2}\right)'\right] = \frac{1}{2}\left[e^{x/2} + (2+x)\frac{1}{2}e^{x/2}\right]$$
$$= \frac{1}{2}\left[1 + (2+x)\frac{1}{2}\right]e^{x/2} = \frac{1}{4}\left(4+x\right)e^{x/2}$$

$$f''(0) = \frac{1}{4}\left(4+0\right)e^{0/2} = 1$$

3) (a) If $f(x)=\left(1+\sqrt{\frac{x-2}{3}}\right)^4$, find $f'(x)$. (10 pts.)

Solution:

$$
\begin{aligned}
f'(x) &= 4\left(1+\sqrt{\frac{x-2}{3}}\right)^3\left(1+\sqrt{\frac{x-2}{3}}\right)' \\
&= 4\left(1+\sqrt{\frac{x-2}{3}}\right)^3\frac{1}{2}\left(\frac{x-2}{3}\right)^{-1/2}\left(\frac{1}{3}\right) \\
&= \frac{2}{3}\left(1+\sqrt{\frac{x-2}{3}}\right)^3\left(\frac{x-2}{3}\right)^{-1/2}
\end{aligned}
$$

b) For the below equation, use implicit differentiation to find y' and evaluate y' at the point (1,1).

$$x^7 - xy = 5\ln y$$

(10 pts.)

Solution:

$$
\begin{aligned}
&\frac{d}{dx}\left(x^7 - xy = 5\ln y\right) \\
&\frac{d}{dx}x^7 - \frac{d}{dx}\left(xy\right) = 5\frac{d}{dx}\ln y \\
&7x^6 - y - x\frac{dy}{dx} = 5\frac{1}{y}\frac{dy}{dx} \\
&7x^6 - y = \left(\frac{5}{y}+x\right)\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{7x^6-y}{\left(\frac{5}{y}+x\right)} = \frac{\left(7x^6-y\right)y}{\left(5+xy\right)}
\end{aligned}
$$

4) For the function $y = \dfrac{1}{x^2 - 1}$,

a) Determine the intervals in which the function is increasing and decreasing. State the extrema. (10 pts.)

Solution:

$$y' = -\frac{2x}{\left(x^2-1\right)^2}$$

Critical values come from $y' = 0$ and $x^2 - 1 = 0$ that makes y' undefined, as given:

$$y' = 0 \quad \Rightarrow \quad x = 0 \quad \text{and} \quad x^2 - 1 = 0 \quad \Rightarrow \quad x = \pm 1$$

So there are three critical values: x = -1, 0, 1.

x		-1		0		+1	
y'	+		+	0	-		-
y	↗ increasing		↗ increasing	max	↘ decreasing		↘ decreasing

b) Determine the intervals in which the function is concave up and concave down. Find the inflection points. (10 pts.)

Solution:

So the second derivative of y is

$$y'' = -\frac{2\left(x^2-1\right)^2 - 2x \cdot 2\left(x^2-1\right)(2x)}{\left(x^2-1\right)^4}$$

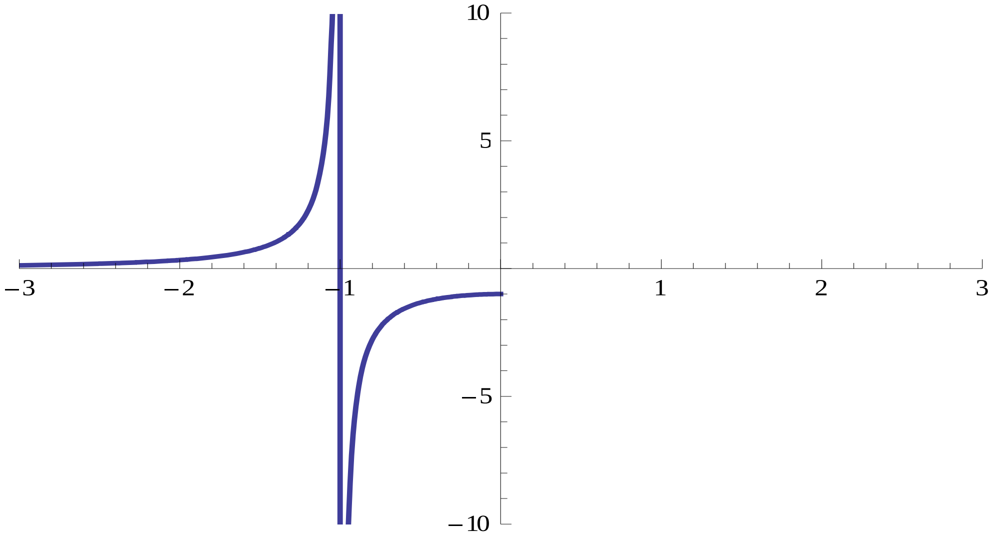
$$= -\frac{2\cancel{\left(x^2-1\right)}\left[x^2 - 1 - 4x^2\right]}{\left(x^2-1\right)^{\cancel{4}3}} = \frac{2\left(3x^2+1\right)}{\left(x^2-1\right)^3}$$

From here, there are no inflection point where $y'' = 0$. To configure the concavity, we have to consider the points where y'' is not defined as follows:

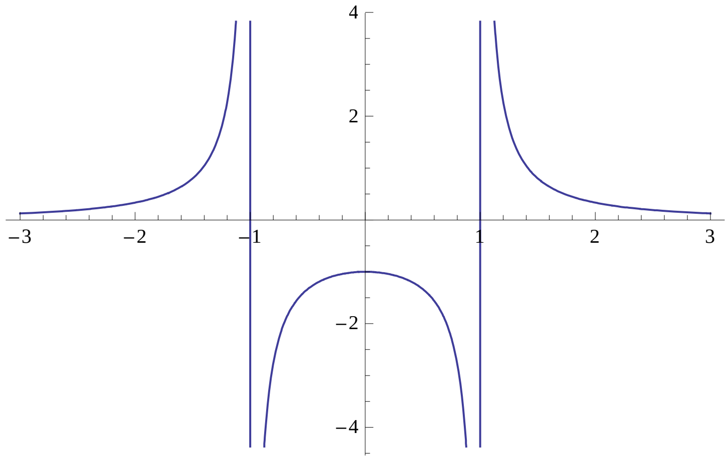
$$x^2 - 1 = 0 \quad \Rightarrow \quad x = \pm 1.$$

x	x < -1	-1	-1 < x < 1	1	x > 1
y''	$f''(x) > 0$		$f''(x) < 0$		$f''(x) > 0$
	Concave up	Vertical Asymptote	Concave down	Vertical Asymptote	Concave up

c) Complete the sketch of the graph below: (5 pts.)



Solution:
From the symmetry, we sketch the right hand side of the graph from the left:



5) Find all asymptotes of the function $y = \dfrac{x}{\sqrt{x^2 - 1}}$. (10 pts.)

Solution:

Vertical Asymptote

Let us find the x-values that the denominator is 0. $x^2 - 1 = 0 \quad \Rightarrow \quad x = \pm 1$

$$\lim_{x \to 1^+} \frac{x}{\sqrt{x^2 - 1}} = \frac{1}{+0} = +\infty \qquad \text{and} \qquad \lim_{x \to -1^-} \frac{x}{\sqrt{x^2 - 1}} = \frac{-1}{+0} = -\infty$$

The lines $x = 1$ and $x = -1$ are the vertical asymptotes.

Horizontal Asymptote

$$\lim_{x \to \pm\infty} \frac{x}{\sqrt{x^2 - \cancel{1}}} = \lim_{x \to \pm\infty} \frac{x}{\sqrt{x^2}} = \lim_{x \to \pm\infty} \frac{x}{|x|} = \pm 1$$

The lines $y = 1$ and $y = -1$ are the horizontal asymptotes.

6) A producer finds that the total revenue function is $r(q) = 100q\text{-}2q^2$ where q in thousands of units. If the producer's costs are given by $c(q) = 20+40q$, what should his level of production be to maximize profits? Show that the second derivative test is satisfied for maximization. (10 pts.)

Solution:

$$\text{Profit} = P = r(q) - c(q) = 100q - 2q^2 - 20 - 40q$$
$$= -2q^2 + 60q - 20$$

To find the extremum values, we take the derivative of profit P with respect to q and then equate it to zero to find the q extremum values:

$$\frac{dP}{dx} = -4q + 60 = 0 \qquad \Rightarrow \qquad q = 60/4 = 15$$

To test if it is maximum or not by using the second derivative test:

$$\frac{d^2P}{dx^2} = -4 < 0$$ so this proves that the level of production $q = 15$ is a maximum value.