MATH 171 FINAL EXAM 22/05/2012

Name: **Student ID #:**

Q1 (20)	Q2(15)	Q3(15)	Q4(10)	Q5(15)	Q6(25)	Total (100)

ATTENTION: There are 6 questions on 6 pages. Solve all of them. Duration is 90 minutes.

1- a (10) Find $\lim\limits_{x\to 6}\left(\dfrac{\sqrt{x-2}-2}{x-6}\right)$. (Do not use L'Hospital)

Solution:

$$\lim_{x\to 6}\left(\frac{\sqrt{x-2}-2}{x-6}\right)=\lim_{x\to 6}\frac{\left(\sqrt{x-2}-2\right)\left(\sqrt{x-2}+2\right)}{(x-6)\left(\sqrt{x-2}+2\right)}$$

$$=\lim_{x\to 6}\frac{\left(x-2\right)-4}{(x-6)\sqrt{x-2}+2}=\lim_{x\to 6}\frac{x-6}{(x-6)\sqrt{x-2}+2}$$

$$=\lim_{x\to 6}\frac{1}{\sqrt{x-2}+2}=\frac{1}{4}$$

b (10) Find $\lim\limits_{x\to\infty}\dfrac{x^2-1}{(3x+2)^2}$. (Do not use L'Hospital)

Solution:

$$\lim_{x\to\infty}\frac{x^2-1}{(3x+2)^2}=\lim_{x\to\infty}\frac{x^2-1}{9x^2+12x+4}$$

$$=\lim_{x\to\infty}\frac{x^2}{9x^2}=\lim_{x\to\infty}\frac{1}{9}=\frac{1}{9}$$

2- a (10) Find an equation of the tangent line to the curve $y = 3x - \frac{2}{x}$ when $x_0 = -1$.

Solution:

$$y = 3x - 2x^{-1}$$

$$y' = 3 + \frac{2}{x^2} \quad \Rightarrow \quad m = y'\big|_{x_0=-1} = 3 + 2 = 5$$

$$y_0\big|_{x_0=-1} = -1$$

$$y - y_0 = m(x - x_0)$$

$$y - (-1) = 5(x - (-1))$$

$$y = 5x + 4$$

b (5) If $y = (2z+1)^2$ and $z = (x+1)^3$, find dy/dx when $x = 0$. (Hint: use the chain rule)

Solution:

$$y' = \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \left(2(2z+1)(2)\right)\left(3(x+1)^2\right)$$

$$y' = 12(x+1)^2(2z+1) = 12(x+1)^2\left(2(x+1)^3 + 1\right)$$

$$y'\big|_{x=0} = 12(0+1)^2\left(2(0+1)^3 + 1\right) = 12(3) = 36$$

3- (15) Find the derivative of $y = x^{\ln x}$. (Use logarithmic differentiation)

Solution:

$$\ln y = \ln x^{\ln x}$$

$$\ln y = \ln x \ln x$$

$$\frac{y'}{y} = \left(\frac{1}{x}\right)(\ln x) + (\ln x)\left(\frac{1}{x}\right)$$

$$y' = y\left(\frac{2}{x}\ln x\right)$$

$$y' = \frac{2x^{\ln x}}{x}\ln x$$

$$y' = 2x^{(\ln x)-1}\ln x$$

4- (10) If $\ln x + \ln y = e^{xy}$, find dy/dx by implicit differentiation.

Solution:

$$\frac{1}{x} + \frac{1}{y}y' = e^{xy}\left(y + xy'\right)$$

$$\frac{1}{y}y' - xy'e^{xy} = ye^{xy} - \frac{1}{x}$$

$$y'\left(\frac{1}{y} - xe^{xy}\right) = ye^{xy} - \frac{1}{x}$$

$$y' = \frac{ye^{xy} - \frac{1}{x}}{\frac{1}{y} - xe^{xy}}$$

5-(15) The demand function for ABC's product is $p = 700 - 2q$ and the average cost per unit for producing q units is $\bar{c} = q + 100 + \frac{1000}{q}$ where p and $\bar{c}$ are in TL per unit. Find the quantity (units) of production that maximizes the profit.

Solution:

Total Cost:

$$c = \bar{c}q = q^2 + 100q + 1000$$

Profit=Total Revenue - Total Cost

$$P = pq - c = (700 - 2q)q - (q^2 + 100q + 1000)$$

$$= -3q^2 + 600q - 1000$$

$$P' = -6q + 600$$

Setting $P' = 0$ yields $q = 100$. Since $P'' = -6 < 0$, P is maximum when $q = 100$.

6- (25) For the function $y = \frac{1}{x^2+1}$, (a) find the asymptotes, (b) determine the intervals in which the function is increasing and decreasing, (c) determine the intervals in which the function is concave up and concave down, (d) state the extrema, (e) find the inflection points. And then sketch the graph of function within the grid area on the back side of this page. (use symmetry and/or intercepts if necessary)

Solution:

(a) horizontal asymptote: $\lim_{x\to\infty} \frac{1}{x^2+1} = 0 = \lim_{x\to-\infty} \frac{1}{x^2+1}$, so $y=0$ is a horizontal asymptote.

(b) $y' = \frac{-2x}{\left(x^2+1\right)^2}$, CV: $x=0$, Increasing on $(-\infty,0)$, decreasing on $(0,\infty)$, (see sign chart)

(c) $y'' = \frac{2\left(3x^2-1\right)}{\left(x^2+1\right)^3}$. Possible inflection points are at $x = \pm\frac{1}{\sqrt{3}}$. Concave up on $\left(-\infty, -\frac{1}{\sqrt{3}}\right)$ and $\left(\infty, \frac{1}{\sqrt{3}}\right)$; concave down on $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ (see sign chart) ,

(d) Relative maximum at $(0,1)$,

(e) Inflection points are $\left(\pm\frac{1}{\sqrt{3}}, y\left(\pm\frac{1}{\sqrt{3}}\right) = \frac{3}{4}\right)$

Sign chart;

x	$-1/\sqrt{3}$	0	$1/\sqrt{3}$
f' Increase/decrease	+ ↗		- ↘
f'' concavity	+ ∪	- ∩	+ ∪

Symmetric about y axis: $y(-x) = y(x) = 1/(x^2+1)$, Intercept (0,1)

Grid area for skecthing graph of function