PROBLEM SET I FOR MATH–171 (Fall 2006)

1) A company management would like to know the total sales units that are required for the company to earn a profit of $10,000. The following data are available; unit selling price of $20; variable cost per unit of $15; total fixed cost of $60,000. Determine the required sales units.

Solution:

Profit: P = $10,000
Unit Selling Price: S = $20
Variable Cost per Unit: VC = $15
Total Fixed Cost: FC = $60,000

Determine the required sales units?

Let u denote sales unit

Total Costs: TC = VC + FC = $15 u + $60,000
Total Sales: TS = $20 u
$P = TS - TC = 20u - 15u - 60{,}000 = 10{,}000 \Rightarrow 5u = 70{,}000$
$\Rightarrow u = 14{,}000$

2) A company manufacturer a product that has a unit selling price of $30 and a unit cost of $20. If fixed costs are $30,000, determine the least number of units that must be sold for the company to have a profit.

Solution:

Unit Selling Price: S = $30
Variable Unit Cost: VC = $20
Fixed Cost: FC = $30,000

Determine the least number of units sold to have a profit.

Let u denote sales unit

Total Costs: $TC = VC + FC = 20u + 30{,}000$
Total Sales: $TS = 30u$
Profit: $P = TS - TC = 30u - (20u + 30{,}000) = 10u - 30{,}000 > 0$ to have profit

$10u > 30{,}000 \Rightarrow u > 3{,}000$
The least number of units to be sold is more than 3,000 to have a profit.

3) Find the domain and range the following functions:

a) $y = f(x) = \dfrac{1-x}{\sqrt{x^2 - x - \frac{3}{4}}}$ b) $y = f(x) = \sqrt[4]{\dfrac{3x+2}{x-3}}$

c) $y = f(x) = \dfrac{3x^2+2}{x^2+1}$ d) $y = f(x) = \dfrac{x^2-3x+1}{x^2-9}$

Solution:

a. $y = f(x) = \dfrac{1-x}{\sqrt{x^2 - x - 3/4}}$

To be definable, the inside of the square root must be greater than zero. So

$$x^2 - x - \frac{3}{4} > 0 \quad \Rightarrow \quad \left(x+\frac{1}{2}\right)\left(x-\frac{3}{2}\right) > 0$$

$\frac{1}{2} \quad -\frac{3}{2}$

At $x = -\frac{1}{2}$ and $x = \frac{3}{2}$, the denominator is zero, so undefined.

Roots		-1/2		3/2	
$\left(x+\frac{1}{2}\right)$	-	0	+		+
$\left(x-\frac{3}{2}\right)$	-		-	0	+
$\left(x+\frac{1}{2}\right)\left(x-\frac{3}{2}\right)$	+	0	-	0	+

From the table only the parts in positive signs give the domain than makes nonnegative in the square root. So

Domain: $(-\infty, -\frac{1}{2}) \cup (\frac{3}{2}, \infty)$ **Forbidden Range:** $[-\frac{1}{2}, \frac{3}{2}]$

To find the range we have to find function values at the domain. First let's find the function values at the intervals of the domain.

$$x \to -\infty, \qquad y = f(x) = \frac{\cancel{x}(\frac{1}{x} - 1)}{-\cancel{x}\sqrt{1 - \frac{1}{x} - \frac{3}{4x^2}}} \to +1$$

$$x \to -\frac{1}{2}, \qquad y = f(x) = \frac{1-x}{\sqrt{x^2 - x - \frac{3}{4}}} \to \frac{1-(-\frac{1}{2})}{0} \to \frac{3/2}{0} \to +\infty$$

$$x \to \frac{3}{2}, \qquad y = f(x) = \frac{1-x}{\sqrt{x^2 - x - \frac{3}{4}}} \to \frac{1-\frac{3}{2}}{0} \to \frac{-1/2}{0} \to -\infty$$

$$x \to \infty, \qquad y = f(x) = \frac{\cancel{x}(\frac{1}{x} - 1)}{\cancel{x}\sqrt{1 - \frac{1}{x} - \frac{3}{4x^2}}} \to -1$$

Range: $(-\infty, -1) \cup (1, \infty)$

4) Using the absolute value symbol, express each fact.
   a) X is between -3 and 3, but is not equal to 3 or -3.
   b) The number x of hours that a machine will operate efficiently from 255 by less than 6
   c) The average monthly income x (in dollars) of a family differs 1050 by lless than 120
   d) x+4 is less than 5 units from 0.
   e) The distance between 7 and x is 4.

Solution:

Using the absolute value symbol, express each fact.

f) x is between -3 and 3, but is not equal to 3 or -3.

$$-3 < x < 3 \quad \Rightarrow \quad |x| < 3$$

g) The number x of hours that a machine will operate efficiently from 255 by less than 6

$$|x - 255| < 6$$

h) The average monthly income x (in dollars) of a family differs 1050 by less than 120

$$|x - 1050| < 120$$

i) x+4 is less than 5 units from 0.

$$|x + 4| < 5$$

j) The distance between 7 and x is 4.

It can be $7 - x = 4 \quad \Rightarrow \quad x = 3$

or $x - 7 = 4 \quad \Rightarrow \quad x = 11$

$|x - 7| = 4 \quad or \quad |7 - x| = 4$ supplies the above results.

5) In functions is y a function of x? Is x a function of y?
   a) $x^2 + y = 0$ b) $y = 7x^2$ c) $x^2 + y^2 = 1$

Solution:

a. $y = -x^2$, y is a function of x.
b. $y = 7x^2$, y is a function of x.
c. $x^2 + y^2 = 1$, Neither y nor x is a function of x or y.

6) Solve the following inequalities:
   a) $2x - (7 + x) \le x$ b) $3p(1 - p) > 3(2 + p) - 3p^2$
   c) $4 < \left|\frac{2}{3}x + 5\right|$ d) $\frac{3y - 1}{3} > \frac{5(y + 1)}{4}$

Solution:

a. $2x - (7 + x) \le x$

$$\cancel{2x} - 7 - \cancel{x} - \cancel{x} \le 0 \quad \Rightarrow \quad -7 \le 0 \quad true$$

Solution: all reel numbers

b. $3p(1-p) > 3(2+p) - 3p^2$

$\cancel{3p} - 3p^2 > 6 + \cancel{3p} - 3p^2 \Rightarrow \quad 0 > 6 \quad \textit{false}$

no solution exist for this problem.

c. $4 < \left| \frac{2}{3}x + 5 \right|$

$$4 < \frac{2}{3}x + 5 \quad \Rightarrow \quad -1 < \frac{2}{3}x \quad \Rightarrow \quad -\frac{3}{2} < x \quad \left(\text{or } x > -\frac{3}{2} \right)$$

$$-4 > \frac{2}{3}x + 5 \quad \Rightarrow \quad -9 > \frac{2}{3}x \quad \Rightarrow \quad -\frac{27}{2} > x \quad \left(\text{or } x < -\frac{27}{2} \right)$$

Solution: $\left(-\infty, -27/2\right) \cup \left(-3/2, +\infty\right)$

d. $\frac{3y-1}{3} > \frac{5(y+1)}{4}$

$$\frac{3y-1}{3} > \frac{5(y+1)}{4} \quad \Rightarrow \quad 12y - 4 > 15y + 15$$

$$\Rightarrow \quad 12y - 15y > 15 + 4 \quad \Rightarrow \quad -3y > 19$$

$$\Rightarrow \quad y < -\frac{19}{3}$$

Solution: $\left(-\infty, -19/3\right)$

7) A manufacturer sells a product at $8 per unit, selling all produced. The fixed cost is $2,000 and the variable cost is $7 per unit.

a) At what level of production there will be a profit of $4,000.
b) At what level of production there will be a loss of $1,000.

Solution:

u: the number unit produced
s =$ 8 (selling price)
FC = $2,000
VC per unit = $7
Total Cost: TC = FC + VC = $2,000 + $7 u
Total Sales: TS = $ 8 u

a. TS-TC = $4,000 ⇒ $ 8 u - ($2,000 + $7 u) = $4,000
⇒ u = 6,000 units must be sold.

b. TS-TC = -$1,000 ⇒ $ 8 u - ($2,000 + $7 u) = -$1,000
⇒ u = 1,000 units will be sold.

8) If f(x) = 2x and g(x) = 6+x, find the following
a) (fog)(x) b) (gof)(x) c) (gof)(2)

Solution:

$a)\ (f\circ g)(x) = f(g(x)) = f(6+x) = f(u) = 2u = 2.(6+x) = 2x+12$

$b)\ (g\circ f)(x) = g(f(x)) = g(2x) = g(u) = 6+u = 6+2x$

$c)\ (g\circ f)(2) = 6+2.2 = 10$

9) Determine the x- and y-intercepts of the following functions. Graph them and give the domain and range of each function. a) y= 4 -x b) y= 4-x^2

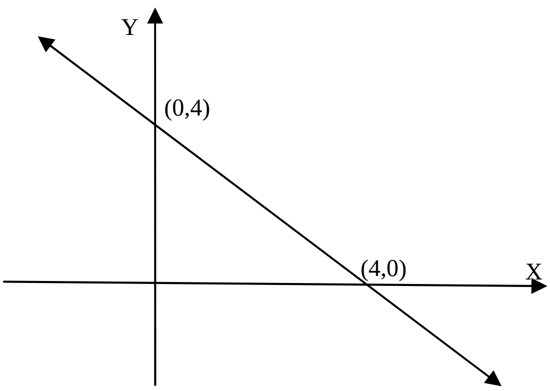
Solution:

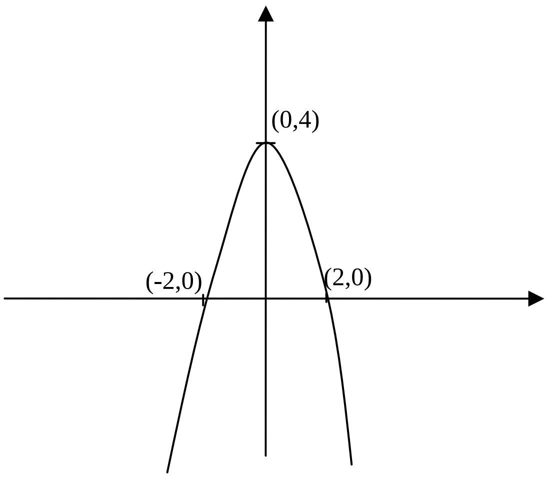
$a)\ y = 4-x$

$x-\text{intercept} \quad \rightarrow Give\ y=0, then\ \ find\ x \quad \rightarrow x = 4-y = 4; \quad x-\text{int}ercept\ is \quad (0,4)$

$y-\text{intercept} \quad \rightarrow Give\ x=0,\ then\ \ find\ y \quad \rightarrow y = 4-x = 4; \quad y-\text{int}ercept\ is\ (4,0)$

Since *y*=4-*x* is actually represents a line its range and domain has no restrictions therefore
Range= $(-\infty,+\infty)$ Domain= $(-\infty,+\infty)$





$b)\ y = 4-x^2$

$x-\text{int}ercept \ \rightarrow y=0,\ then\ 0 = 4-x^2 \rightarrow x = \pm 2 \quad \rightarrow \quad (-2,0)\ and\ (2,0)$

$y-\text{int}ercept \ \rightarrow x=0,\ \ then\ y = 4 \quad \rightarrow \quad (0,4)$

When you attempt to sketch a graph of a quadratic function which is a parabole. first thing to do is to determine whether it is “upward opening or downward opening type”. To do this we look at the sign of the coefficient of x^2 term. Here it is “-“, so our parabole is “downward opening. Second thing to do is find x-intercept and y-intercept points. Third to find “vertex” position which is given as

$$x_{Vertex} = \frac{-b}{2a} = \frac{0}{2} = 0 \quad then\ to\ find\ y_{vertex}\ \ simply\ use\ x_{vertex}\ \text{in}\ the\ equation$$

$$y_{vertex} = y(x_{vertex}) = 4-(x_{vertex})^2 = 4$$

$$(x_{vertex}, y_{vertex}) = (0,4)$$

The graph is shown above (on the right hand side).

10) Find the x- and y-intercepts of the following functions. Also test for symmetry about the x-axis, the y-axis, and the origin. a) y= f(x)= $2x^3$-8x b) y=$5x^2$-10

Solution:

$a)$

$y = f(x) = 2x^3 - 8x$

$x - intercepts \;\rightarrow\; y = 0 \;\rightarrow 0 = 2x^3 - 8x = 2x(x^2 - 4) = 2x(x-2)(x+2) \;\rightarrow x = 0, \pm 2$

$There\ are\ 3\ points\ for\ x - intercepts.\ (0,0);\ (2,0);\ (-2,0)$

$y - intercept \rightarrow \; x = 0 \;\rightarrow y = 2.0 - 8.0 = 0 \;\rightarrow (0,0)$

$y - axis\ symmetry \;\rightarrow (-a,b) \leftrightarrow (a,b)\;\; so\;\; for\ x - axis\ symmetry\ f(x) = f(-x)$

$f(-a) = 2.(-a)^3 - 8(-a) = -2a^3 + 8a$

$\neq f(a) \rightarrow NOT\ \ y - axis\, symmetry$

$x - axis\ symmetry \;\rightarrow (a,-b) \leftrightarrow (a,b)$

$Let\ us\ try\;\; x = 1 \rightarrow y = -6\;\; if\;\; (1,-6)\ is\ a\ point\ then\ (1,6)\;\; MUST\ also\ be\ a\ point\ on\ the\ graph.$

$x = -1 \rightarrow y = +6 \;\rightarrow Therefore\;\; NOT\;\; x - axis\ symmetry$

$symmetry\ about\ origin\ requires\;\; (a,b) \rightarrow (-a,-b)$

$We\ found\ that\;\; (1,-6) \leftrightarrow (-1,+6)\;\;\; Therefore\;\; \dot{I}t\ is\ symmetric\ about\ origin.$

11) Find the equation of the straight line that has the following properties:
   a) Passes through (4, -2) and (-6, 3)
   b) Passes through (-2, 5) and has a slope 4
   c) Perpendicular to y=x+5 and passes through (1, 1)

Solution:

$a)\ slope = m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{3-(-2)}{-6-4} = \dfrac{5}{-10} = -0.5$

$y - y_1 = m(x - x_1) = -0.5(x-4) = -0.5x + 2 = y - (-2) \;\;\rightarrow y = -0.5x$

$c)\ For\;\; perpendicular\ lines,\ their\ slopes\ must\ satisfy\ the\ condition\;\; m_1 m_2 = -1$

$m_1 = +1 \;\;\rightarrow m_2 = -1\;\; and\ the\ point\ is\;\; (1,1)$

$y - y_1 = m(x - x_1)$

$y - 1 = -1(x-1) \;\;\rightarrow y = -x + 2$

12) For the following find a) the vertex b) x-intercepts, c) y-intercept, d) sketch the graph.
   a) y=12- 8s+ s^2 b) y=x^2-4 c) y=-4x^2

Solution:

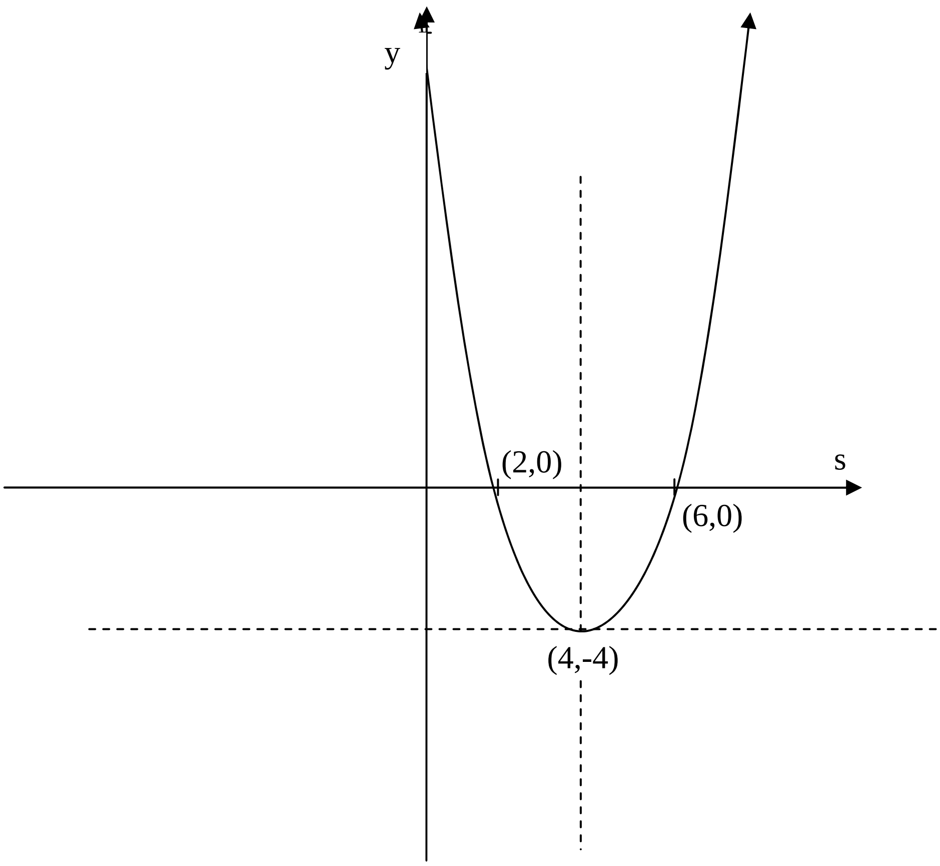
$a)$

$y = 12 - 8s + s^2 = s^2 - 8s + 12$

$s_{vertex} = \dfrac{-b}{2a} = \dfrac{-(-8)}{2.1} = 4 \qquad y_{vertex} = 4^2 - 8.4 + 12 = -4$

$s - intercepts$

$y = 0, \qquad 0 = s^2 - 8s + 12 \qquad \rightarrow s = +2, +6$



13) The demand function for an electronic company's computer line is p=1,200-3q, where p is the price per unit when q units are demanded by consumers. Find the level of production that will maximize the manufacturer's total revenue, and determine this revenue.

Solution:

$$Total\ Revenue = p.q = (1200 - 3q)q = -3q^2 + 1200q$$

Maximum revenue would occur at the vertex of the parabole. Therefore

$$q_{vertex} = \frac{-b}{2a} = \frac{-1200}{2.(-3)} = 200 \qquad q = 200\,units$$

Maximum revenue at this production level is

$$r_{vertex} = -3(200)^2 + 1200.200 = 120,000$$

14) Suppose consumers will demand 40 units a product when the price is $12 per unit and 26 units when the price is $19 each. Find the demand equation assuming that it is linear. Find the price per unit when 30 units are demanded.

Solution:

Two points are

$$(q,p) = (12,40)$$
$$= (19,26)$$

The equation of the line passing through two points

$$m = \frac{p_2 - p_1}{q_2 - q_1} = \frac{40-26}{12-19} = -2$$

$$p - p_1 = m(q - q_1) = -2(q - 12) = p - 40 \qquad \rightarrow p = -2q + 64$$