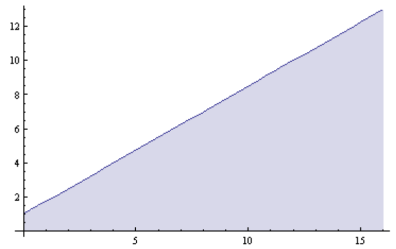
14.10 Area

Use a definite integral to find the area of the region bounded by the given curve, the x-axis, and the given lines. First sketch the region and then watch out for areas of regions that are below the x-axis.

2. $y = \frac{3}{4}x + 1, \qquad x = 0, \quad x = 16$

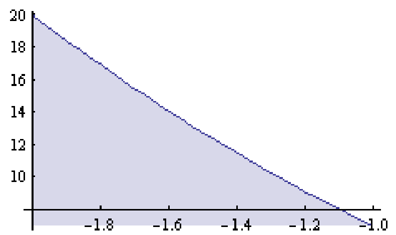
Answer:



$$area = \int_0^{16} \left(\frac{3}{4}x + 1 \right) dx = \frac{3}{8}x^2 + x \Big|_0^{16} = \frac{3}{8}16^2 + 16 = 400$$

12. $y = 3x^2 - 4x, \qquad x = -2, \qquad x = -1$

Answer:

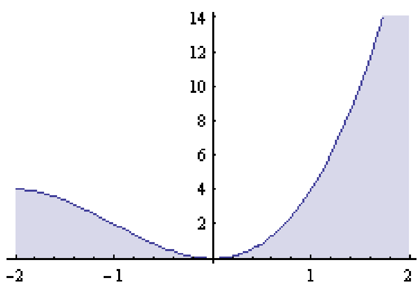


$$area = \int_{-2}^{-1} \left(3x^2 - 4x \right) dx = x^3 - 2x^2 \Big|_{-2}^{-1} = (-1)^3 - 2(-1)^2 - \left\{ (-2)^3 - 2(-2)^2 \right\} =$$

$$= -1 - 2 + 8 + 8 = 15$$

24. $y = x^3 + 3x^2, \qquad x = -2, \qquad x = +2$

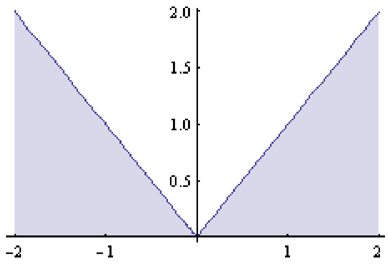
Answer:



$$area=\int_{-2}^{2}\left(x^3+3x^2\right)dx=\frac{x^4}{4}+x^3\bigg|_{-2}^{2}=\frac{\cancel{2^4}}{\cancel{4}}+2^3-\left\{\frac{\cancel{(-2)^4}}{\cancel{4}}+(-2)^3\right\}=8+8=16$$

28. $y=|x|,\qquad x=-2,\qquad x=+2$

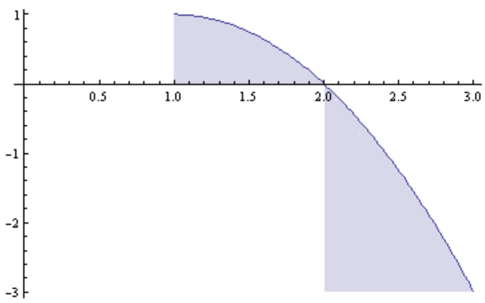
Answer:



$$area=\int_{-2}^{2}|x|dx=\int_{-2}^{0}(-x)dx+\int_{0}^{2}xdx=-\frac{x^2}{2}\bigg|_{-2}^{0}+\frac{x^2}{2}\bigg|_{0}^{2}=-0-\left(-\frac{(-2)^2}{2}\right)+\frac{2^2}{2}-0=4$$

33. $y=2x-x^2,\qquad x=1,\qquad x=3$

Answer:



The curve intersect the *x*-axis at $y=2x-x^2=x(2-x)=0 \quad\Rightarrow\quad x=0 \quad \& \quad x=2$.

We are interested in the range of [1,3] so that we have to split up the integration from x = 1 to *x* = 2 and from x = 2 to *x* = 3. As seen on the figure above from x = 2 to *x* = 3, the area is under *x*-axis and negative. We measure the area as positive quantity so that we have to take the absolute value or negative of the area of this region.

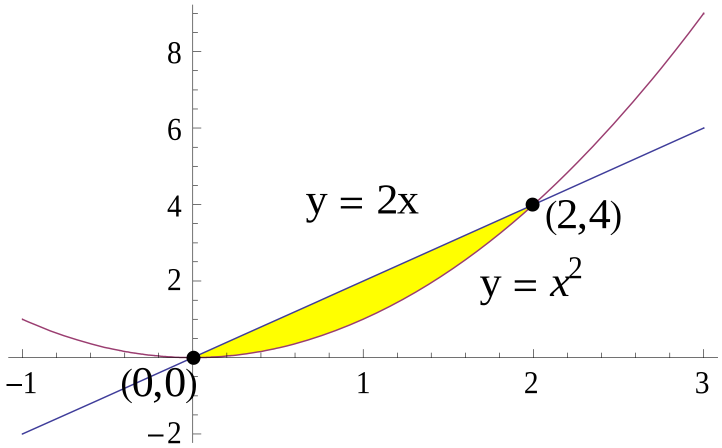
$$area=\int_{-2}^{2}\left(2x-x^2\right)dx=\int_{1}^{2}\left(2x-x^2\right)dx+\int_{2}^{3}-\left(2x-x^2\right)dx=\left(x^2-\frac{x^3}{3}\right)\bigg|_{0}^{1}-\left(x^2-\frac{x^3}{3}\right)\bigg|_{1}^{3}$$

$$=\left\{1^2-\frac{1^3}{3}-0\right\}-\left\{\left(\cancel{3^2}-\frac{\cancel{3^3}}{\cancel{3}}\right)-\left(1^2-\frac{1^3}{3}\right)\right\}=\frac{4}{3}$$

14.11 Area Between Curves

2. Express the area of the shaded region in terms of an integral (or integrals)

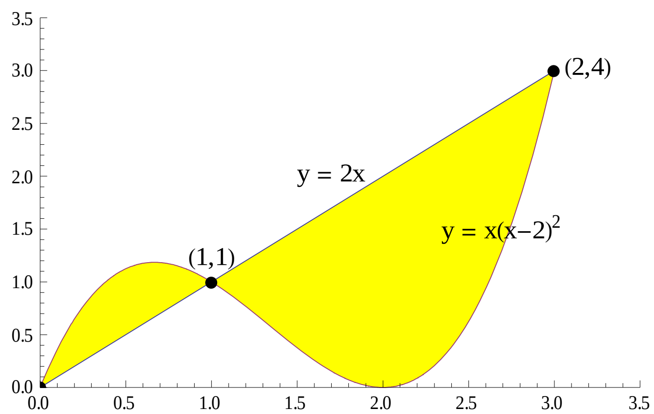
Answer:



$$area = \int_0^2 \left(2x - x^2\right)dx = \left(x^2 - \frac{x^3}{3}\right)\Bigg|_0^2 = \left\{2^2 - \frac{2^3}{3}\right\} - \left\{0^2 - \frac{0^3}{3}\right\} = \frac{4}{3}$$

4. Express the area of the shaded region in terms of an integral (or integrals)

Answer:



The curve intersects each other at

$$x = x(x-2)^2 \quad \Rightarrow \quad x\left[(x-2)^2 - 1\right] = x\left(x^2 - 4x + 3\right) = x(x-3)(x+1)0$$

$$\Rightarrow \quad x = 0 \quad \& \quad x = 2 \quad \& \quad x = 3$$

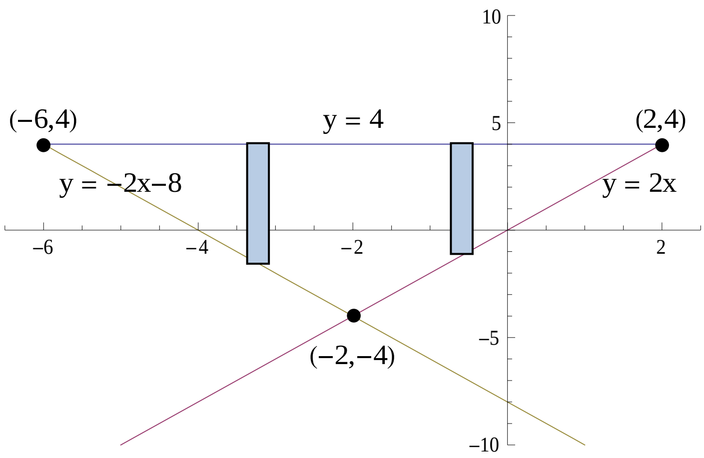
We have to find the area from (0,1) and (1,3) separately because upper and lower curve changes at x = 1.

$$area=\int_0^1\left(x(x-2)^2-x\right)dx+\int_1^3\left(x-x(x-2)^2\right)dx=\int_0^1\left(x^3-4x^2+3x\right)dx+\int_1^3\left(-x^3+4x^2-3x\right)dx$$

$$=\left(\frac{x^4}{4}-\frac{4}{3}x^3+\frac{3}{2}x^2\right)\Bigg|_0^1+\left(-\frac{x^4}{4}+\frac{4}{3}x^3-\frac{3}{2}x^2\right)\Bigg|_1^3=\frac{1}{4}-\cancel{\frac{4}{3}}+\frac{3}{2}-\frac{3^4}{4}+\frac{4}{3}3^3-\frac{3}{2}3^2+\frac{1}{4}-\cancel{\frac{4}{3}}+\frac{3}{2}3=\ldots$$

6. Express the area of the shaded (bounded by 3 curves) region in terms of an integral (or integrals)

Answer:



The curve intersects each other at

$$\begin{aligned} y&=-2x-8=4 &&\Rightarrow\quad x=-6,\quad y=4\\ y&=2x=4 &&\Rightarrow\quad x=2,\quad y=4\\ y&=2x=-2x-8 &&\Rightarrow\quad x=-2,\quad y=-4\end{aligned}$$

shown on the figure.

There are two methods to find the area between curves. The first one is *x*-axis method.

We have to find the area from *x* = -6 to *x* = -2 and from *x* = -2 to *x* = 2 separately because upper and lower curve changes at *x* = -2.

$$area=\int_{-6}^{-2}\underbrace{\left(4-(-2x-8)\right)}_{12+2x}dx+\int_{-2}^{2}\left(4-2x\right)dx=\left(12x+x^2\right)\Big|_{-6}^{-2}+\left(4x-x^2\right)\Big|_{-2}^{2}$$

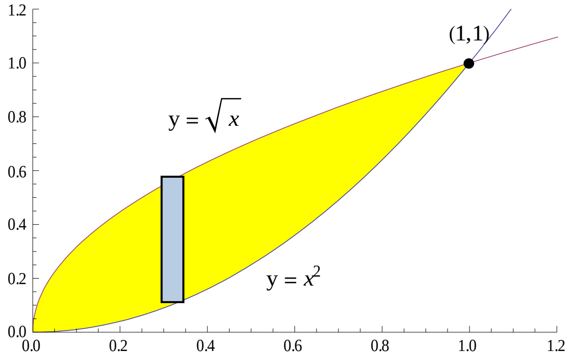
$$=\left(-24+4+72-36\right)+\left(8-\cancel{4}+8+\cancel{4}\right)=32$$

The second one is *y*-axis method. For the *y*-axis method you start from y = -4 to y = 4 with a single integral because the upper and lower curves are the same for the range of the integral. In this case integrand had to be expressed in terms of *y*.

$$area=\int_{-4}^{4}\underbrace{\left(\frac{y}{2}-(-\frac{y+8}{2})\right)}_{y+4}dy=\left(\frac{y^2}{2}+4y\right)\Bigg|_{-4}^{4}=8+16-8+16=32$$

22. Find the area of the region bounded by the graphs of the given equation: $y=\sqrt{x}$ and $y=x^2$

Answer:

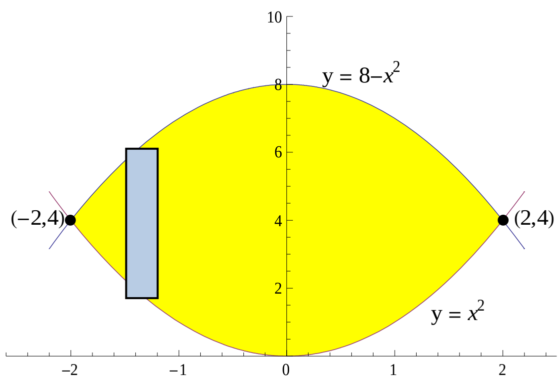


The curves intersect each other at $y=x^2=\sqrt{x} \quad \Rightarrow \quad x\left(x^3-1\right)=0 \quad \Rightarrow \quad x=0,1 \quad \& \quad y=0,1$ shown on the figure.

$$area=\int_0^1\left(\sqrt{x}-x^2\right)dx=\left(\frac{2}{3}x^{\frac{3}{2}}-\frac{x^3}{3}\right)\Bigg|_0^1=\frac{2}{3}-\frac{1}{3}=\frac{1}{3}$$

Soru: Find the area of the region bounded by the graphs of the given equation: $y=8-x^2$ and $y=x^2$.

Answer:



The curves intersect each other at $y=x^2=8-x^2 \quad \Rightarrow \quad x=\pm 2 \quad \& \quad y=4$ shown on the figure.

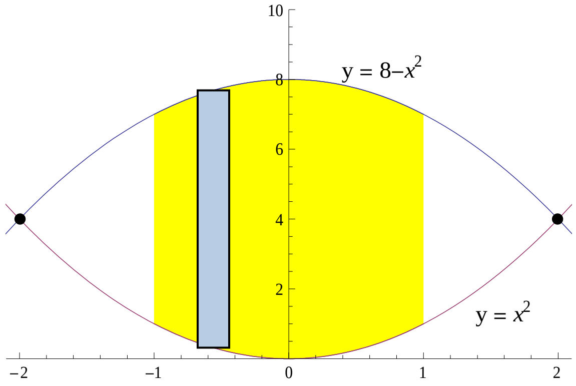
$$area=\int_{-2}^{2}\left(8-x^2-x^2\right)dx=\left(8x-2\frac{x^3}{3}\right)\Bigg|_{-2}^{2}=16-\frac{16}{3}+16-\frac{16}{3}=2\times 16\left(1-\frac{1}{3}\right)=\frac{64}{3}$$

There is an *x*-axis symmetry ($x\to -x \quad \Rightarrow \quad 8-2x^2=8-2\left(-x\right)^2$) so that the integral can be taken from 0 to 2 and then multiply it by 2.

$$area=\int_{-2}^{2}\left(8-2x^2\right)dx=2\int_0^2\left(8-2x^2\right)dx$$

25: Find the area of the region bounded by the graphs of the given equation: $y = 8 - x^2$ and $y = x^2$ $x = -1$ and $x = 1$.

Answer:



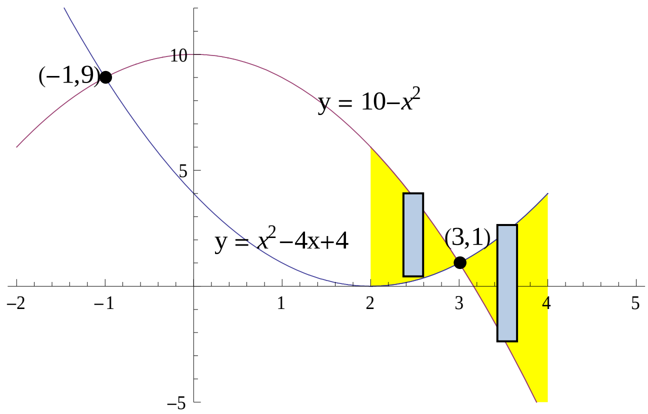
Here we take the integral from $x = -1$ to $x = 1$:

$$area = \int_{-1}^{1} \left(8 - 2x^2\right) dx = 2\int_{0}^{1} \left(8 - 2x^2\right) dx = 2\left(8x - 2\frac{x^3}{3}\right)\Bigg|_0^1 = \frac{44}{3}$$

There is an *x*-axis symmetry ($x \to -x \quad \Rightarrow \quad 8 - 2x^2 = 8 - 2\left(-x\right)^2$) so that the integral is taken from 0 to 1 and then multiplied by 2.

34. Find the area of the region that is between the curves $y = x^2 - 4x + 4$ and $y = 10 - x^2$ from *x* = 2 to *x* = 4.

Answer:



The curves intersect each other at

$$y = x^2 - 4x + 4 = 10 - x^2 \quad \Rightarrow \quad 2\left(x^2 - 2x - 3\right) = 2\left(x + 1\right)\left(x - 3\right) = 0$$

$$\Rightarrow \quad \left(x = -1, y = 9\right) \;\&\; \left(x = +3, y = 1\right)$$

shown on the figure.

The area of the region asked is from *x* = 2 to *x* = 4 shown as yellow. As seen from the figure integral should be taken from *x* = 2 to *x* = 3 and from *x* = 3 to *x* = 4 because in each part, upper and lower curves change.

$$area = \int_2^3 \underbrace{\left(10 - x^2 - x^2 + 4x - 4\right)}_{-2x^2+4x+6} dx + \int_3^4 \underbrace{\left(x^2 - 4x + 4 - 10 + x^2\right)}_{2x^2-4x-6} dx =$$

$$= 2\left(-\frac{x^3}{3} + x^2 + 3x\right)\Bigg|_2^3 + 2\left(\frac{x^3}{3} - x^2 - 3x\right)\Bigg|_3^4$$

$$= 2\left\{\cancel{-9+9} + 9 + \frac{8}{3} - 4 - 6\right\} + 2\left\{\frac{64}{3} - 16 - 12 - \cancel{9+9} + 9\right\} = \frac{10}{3} + \frac{14}{3} = 8$$

14.12 Consumers' and Producers' Surplus

Keep them in mind: $CS = \int_0^{q_0} [\,f(q) - p_0]dq,$ $\qquad PS = \int_0^{q_0} [\,p_0 - g(q)]dq$

where $f(q)$ is demand function of a product and $g(q)$ is supply function of a product.

2. The demand equation is $p = 2200 - q^2$ and a supply equation of a product is $p = 400 + q^2$. Determine the consumers' surplus (*CS*) and the producers' surplus (*PS*) under the market equilibrium.

Answer:

First we have to find the market equilibrium where demand and supply functions are equal each other as follows:

$$f(q_0) = g(q_0)$$

$$2200 - q_0^2 = 400 + q_0^2 \;\Rightarrow\; 2q_0^2 = 1800 \;\Rightarrow\; q_0 = 30 \;\Rightarrow\; p_0 = 400 + q_0^2 = 1300.$$

Then we can find the consumers' surplus (*CS*):

$$CS = \int_0^{30} \underbrace{\left[2200 - q^2 - 1300\right]}_{900-q^2} dq = \left(900q - \frac{q^3}{3}\right)_0^{30} = 27,000 - 9,000 = 18,000$$

and the producers' surplus (*PS*):

$$PS = \int_0^{30} \underbrace{\left[1300 - 400 - q^2\right]}_{900-q^2} dq = \left(900q - \frac{q^3}{3}\right)_0^{30} = 27,000 - 9,000 = 18,000\,.$$

6. The demand equation is $q = \sqrt{100 - p}$ and a supply equation of a product is $q = \frac{p}{2} - 10$.

Determine the consumers' surplus (*CS*) and the producers' surplus (*PS*) under the market equilibrium.

Answer:

From the demand equation we may pull the *p* price per unit as $p = 100 - q^2$ and from the supply equation $p = 2q + 20$.

First we have to find the market equilibrium where demand and supply functions are equal each other as follows:

$$f(q_0) = g(q_0)$$

$$100 - q_0^2 = 2q_0 + 20 \quad \Rightarrow \quad q_0^2 + 2q_0 - 80 = (q_0 + 10)(q_0 - 8) = 0$$

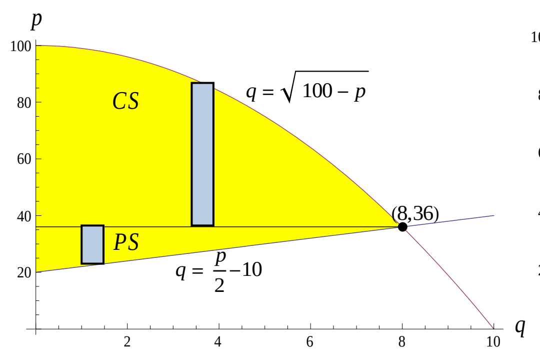
$$\Rightarrow \quad q_0 = 8 \quad \Rightarrow \quad p_0 = 2q_0 + 20 = 36$$

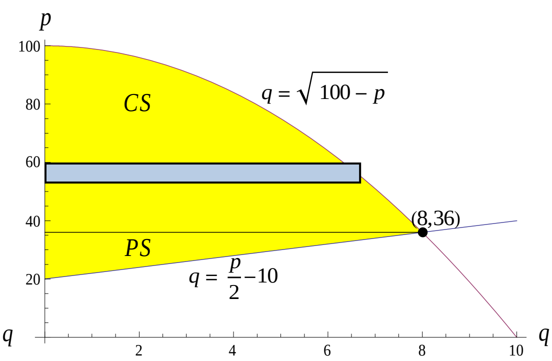
Then we can find the consumers' surplus (*CS*):

$$CS = \int_0^8 \underbrace{\left[100 - q^2 - 36\right]}_{64 - q^2} dq = \left(64q - \frac{q^3}{3}\right)_0^8 = \frac{1024}{3}$$

and the producers' surplus (*PS*):

$$PS = \int_0^8 \underbrace{\left[36 - 2q - 20\right]}_{16 - 2q} dq = \left(16q - q^2\right)_0^8 = 64$$





One may solve this problem using the integration over the *p* axis (the sample horizontal strip on the *CS* part is shown on the figure):

$$CS = \int_p f(p)dp = \int_{36}^{100} \left[\sqrt{100 - p}\right] dp, \quad \begin{cases} u = 100 - p, & du = -dp \\ p = \{100,\ 36\} \Rightarrow & u = \{0,\ 64\} \end{cases}$$

$$= \int_{64}^{0} \sqrt{u}\,(-du) = \frac{2}{3} u^{3/2} \Big|_0^{64} = \frac{1024}{3}$$

$$PS = \int_p g(p)dp = \int_{20}^{36} \left(\frac{p}{2} - 10\right) dp = \left(\frac{p^2}{4} - 10p\right)_{20}^{36} = 64.$$