Chapter 14. Integration

14.1 Differential

Find the differential of the function in terms of *x* and *dx*. $\left(dy = \dfrac{dy}{dx} dx \right)$

1. $y = 3x - 4 \quad \Rightarrow \quad dy = \dfrac{dy}{dx} dx = 3dx$

2. $y = 2 \quad \Rightarrow \quad dy = \dfrac{dy}{dx} dx = 0dx = 0$

3. $y = \sqrt{x^4 - 9} = \left(x^4 - 9\right)^{1/2} \quad \Rightarrow \quad dy = \dfrac{1}{2}\left(x^4 - 9\right)^{-1/2} 4x^3 dx = \dfrac{2x^3}{\left(x^4 - 9\right)^{1/2}} dx$

6. $p = \ln\left(x^2 + 7\right) \quad \Rightarrow \quad dp = \dfrac{dp}{dx} dx = \dfrac{\left(x^2 + 7\right)'}{x^2 + 7} dx = \dfrac{2x}{x^2 + 7} dx$

9.

$$y = (9x + 3)e^{2x^2+3} \quad \Rightarrow \quad dy = \left\{(9x + 3)' e^{2x^2+3} + (9x + 3)\left(e^{2x^2+3}\right)'\right\} dx$$

$$= \left\{9e^{2x^2+3} + (9x + 3)(4x)e^{2x^2+3}\right\} dx$$

$$= \left(36x^2 + 12x + 9\right)e^{2x^2+3} dx$$

10.

$$y = \ln\sqrt{x^2 + 12} \quad \Rightarrow \quad dy = \frac{\left(\sqrt{x^2 + 12}\right)'}{\sqrt{x^2 + 12}} dx = \frac{\frac{1}{\cancel{2}}\left(x^2 + 12\right)^{-1/2} \cancel{2}x}{\sqrt{x^2 + 12}} dx$$

$$= \frac{x}{x^2 + 12} dx$$

28. Find the dx/dy

$$y = 5x^2 + 3x + 2 \quad \Rightarrow \quad dy = \frac{dy}{dx} dx = (10x + 3) dx \quad \Rightarrow \quad \frac{dx}{dy} = \frac{1}{10x + 3}$$

30. Find the dp/dq

$$q = \sqrt{p + 5} \quad \Rightarrow \quad dq = \left[(p + 5)^{1/2}\right]' dp = (p + 5)^{-1/2} dp$$

$$\frac{dp}{dq} = \frac{1}{(p + 5)^{-1/2}} = (p + 5)^{1/2} = \sqrt{p + 5}$$

34. If $y=\ln x^2$, find the value of dx/dy when x = 3.

$$y=\ln x^2 \quad \Rightarrow \quad dy=\left[\ln x^2\right]' dx=\frac{2\not{x}}{x^{\not{2}}}dx$$

$$\frac{dx}{dy}=\frac{x}{2} \quad \Rightarrow \quad \left.\frac{dx}{dy}\right|_{x=3}=\frac{3}{2}$$

36. Find the rate of change of *q* with respect to *p* for the indicated value of *q*.

$$p=50-\sqrt{q}; \quad q=100$$

Answer: First differentiate the *p* equation and then pull the quantity dq/dp as below:

$$dp=\frac{dp}{dq}dq=-\frac{1}{2\sqrt{q}}dq \quad \Rightarrow \quad \frac{dq}{dp}=-2\sqrt{q} \quad \Rightarrow \quad \left.\frac{dq}{dp}\right|_{q=100}=-2\sqrt{100}=-20$$

40. Demand: Given the demand function in YTL

$$p=\frac{200}{\sqrt{q+8}}$$

use differentials to estimate the price per unit when 40 units are demanded.

Answer: First differentiate the *p* equation

$$dp=\frac{dp}{dq}dq=\left[200\left(q+8\right)^{-1/2}\right]' dq=200\left(-\frac{1}{2}\right)\left(q+8\right)^{-3/2}dq=-\frac{100}{\left(q+8\right)^{3/2}}dq$$

then obtain the derivative of dp/dq, just dividing both sides of the equation by *dq*:

$$\frac{dp}{dq}=-\frac{100}{\left(q+8\right)^{3/2}} \quad \Rightarrow \quad \left.\frac{dp}{dq}\right|_{q=40}=-\frac{100}{48^{3/2}}\simeq -0.30$$

which means that the price per unit decreases about 0.30 YTL when 40 units are demanded.

14.2 The Indefinite Integral

Find the indefinite integral

2. $\int\frac{1}{2}dx=\frac{1}{2}\int dx=\frac{1}{2}x+C$

5. $\int 5x^{-7}dx=5\frac{x^{-7+1}}{-7+1}+C=-\frac{5}{6}x^{-6}+C=-\frac{5}{6x^6}+C$

10. $\int\frac{7}{2x^{9/4}}dx=\frac{7}{2}\int x^{-9/4}dx=\frac{7}{2}\frac{x^{-\frac{9}{4}+1}}{-\frac{9}{4}+1}=\frac{7}{2}\frac{x^{-\frac{5}{4}}}{-\frac{5}{4}}=-\frac{7}{2}\frac{4}{5}x^{-\frac{5}{4}}=-\frac{14}{5x^{\frac{5}{4}}}$

12. $\int\left(r^3+2r\right)dr=\frac{r^4}{4}+\not{2}\frac{r^2}{\not{2}}+C=\frac{r^4}{4}+r^2+C$

19. $\int\left(\frac{x}{2}-\frac{3}{4}x^4\right)dx=\frac{1}{2}\int xdx-\frac{3}{4}\int x^4dx=\frac{1}{2}\frac{x^2}{2}-\frac{3}{4}\frac{x^5}{5}+C=\frac{x^2}{4}-\frac{3x^5}{20}+C$

22. $\int\left(\frac{e^x}{3}+2x\right)dx=\frac{1}{3}\int e^xdx+2\int xdx=\frac{1}{3}e^x+2\frac{x^2}{2}+C=\frac{1}{3}e^x+x^2+C$

30. $\int\left(\frac{1}{2x^3}-\frac{1}{x^4}\right)dx=\frac{1}{2}\int x^{-3}dx-\int x^{-4}dx=\frac{1}{2}\frac{x^{-2}}{-2}-\frac{x^{-3}}{-3}+C=-\frac{1}{4x^2}+\frac{1}{3x^3}+C$

36. $\int\left(3y^3-2y^2+\frac{e^y}{6}\right)dy=3\int y^3dy-2\int y^2dy+\frac{1}{6}\int e^ydy=\frac{3}{4}y^4-\frac{2}{3}y^3+\frac{1}{6}e^y+C$

38. $\int 0dy=C$

40. $\int\left(\sqrt[3]{u}+\frac{1}{\sqrt{u}}\right)du=\int u^{\frac{1}{3}}du+\int u^{-\frac{1}{2}}du=\frac{u^{\frac{1}{3}+1}}{\frac{1}{3}+1}+\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+C=\frac{3}{4}u^{\frac{4}{3}}+2u^{\frac{1}{2}}+C$

44. $\int(z+2)^2\,dz=\int\left(z^2+4z+4\right)dz=\frac{z^3}{3}+2z^2+4z+C$

50.

$$\int\frac{x^4-3x^2+4x}{5x}dx=\frac{1}{5}\int\left(\frac{x^{\not{4}}}{\not{x}}-3\frac{x^{\not{2}}}{\not{x}}+4\frac{\not{x}}{\not{x}}\right)dx=\frac{1}{5}\int\left(x^3-3x+4\right)dx=\frac{1}{5}\left(\frac{x^4}{4}-\frac{3x^2}{2}+4x\right)+C$$

52. $\int\frac{\left(x^3+1\right)^2}{x^2}dx=\int\frac{x^6+2x^3+1}{x^2}dx=\int\left(x^4+2x+x^{-2}\right)dx=\frac{x^5}{5}+x^2-\frac{1}{x}+C$

55. $\int\frac{d}{dx}\left(\frac{1}{\sqrt{x^2+1}}\right)dx=\int d\left(\frac{1}{\sqrt{x^2+1}}\right)=\frac{1}{\sqrt{x^2+1}}+C$

14.3 Integration with Initial Conditions

Find the indefinite integral

2. If $\frac{dy}{dx} = x^2 - x; \quad y(3) = \frac{19}{2}$, then find y, subject to the given condition.

Answer: $\int dy = \int (x^2 - x)\,dx \quad \Rightarrow \quad y = \frac{x^3}{3} - \frac{x^2}{2} + C$

We have to determine the constant of integration *C* using the initial condition given in the question ($y(3) = \frac{19}{2}$) as follows:

$$y(3) = \frac{19}{2} = \frac{3^3}{3} - \frac{3^2}{2} + C = 9 - \frac{9}{2} + C \quad \Rightarrow \quad C = \frac{19}{2} - 9 + \frac{9}{2} = 5 \text{ s}$$

The result is then $y = \frac{x^3}{3} - \frac{x^2}{2} + 5$.

6. If $y'' = x + 1; \quad y'(0) = 0, \quad y(0) = 5$, then find y, subject to the given conditions.

Answer: First integrate the y''to obtain y':

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\frac{dy}{dx} = \frac{dy'}{dx} = x + 1 \quad \Rightarrow \quad dy' = (x+1)\,dx \quad \Rightarrow \quad y' = \frac{x^2}{2} + x + C_1$$

Applying the first initial condition $y'(0) = 0$ to the solution to obtain C_1:

$$y'(0) = 0 = \left.\left(\frac{x^2}{2} + x + C_1\right)\right|_{x=0} = C_1 \quad \Rightarrow \quad C_1 = 0$$

Second integrating the *y'* to obtain *y*:

$$y' = \frac{dy}{dx} = \frac{x^2}{2} + x \quad \Rightarrow \quad dy = \left(\frac{x^2}{2} + x\right)dx \quad \Rightarrow \quad y = \frac{x^3}{6} + \frac{x^2}{2} + C_2$$

Applying the second initial condition $y(0) = 5$ to the *y* solution to obtain C_2:

$$y(0) = 5 = \left(\frac{x^3}{6} + \frac{x^2}{2} + C_2\right)_{x=0} = C_2 \quad \Rightarrow \quad C_2 = 5.$$

The result is then $y = \frac{x^3}{6} + \frac{x^2}{2} + 5$.

12. The marginal-revenue function is $dr/dq = 5,000 - 3\left(2q + 2q^2\right)$. Find the *p* demand function.

Answer: The *r* revenue function can be obtained from the marginal-revenue function:

$$r = \int \frac{dr}{dq} dq = \int \left(5,000 - 6q - 6q^2\right) dq = 5,000q - 3q^2 - 2q^3 + C .$$

Here we have to find the *C* constant of integration by assuming that when no units are sold, total revenue is zero as

$$r_{q=0} = \left(5,000q - 3q^2 - 2q^3 + C\right)_{q=0} = C = 0 .$$

We know that the general relationship of revenue in term of *p* price per unit and *q* quantity *r* = *pq*. Hence the demand function (*p* price per unit) is

$$p = \frac{r}{q} = \frac{5,000q - 3q^2 - 2q^3}{q} = 5,000 - 3q - 2q^2$$

16. The marginal-cost function is $dc/dq = 0.000204q^2 - 0.046q + 6$ with a fixed cost 15,000. Find the total cost for the *q* = 200.

Answer: The total cost is just the integration of marginal-cost function with respect to q:

$$\begin{aligned} c &= \int \frac{dc}{dq} dq = \int \left(0.000204q^2 - 0.046q + 6\right) dq \\ &= 0.000204 \frac{q^3}{3} - 0.046 \frac{q^2}{2} + 6q + C \\ &= 0.00068q^3 - 0.023q^2 + 6q + C \end{aligned}$$

We know that fixed cost is just 15,000 when no units are sold (*q* = 0):

$$c_{q=0} = \left(0.00068q^3 - 0.023q^2 + 6q + C\right)_{q=0} = C = 15,000$$

Hence the total-cost function when *q* = 200 is just:

$$c(q = 200) = 0.00068 \times 200^3 - 0.023 \times 200^2 + 6 \times 200 + 15,000 = 20,720 .$$

14.4 More Integration Formulas

Find the indefinite integral

2. $\int 15\underbrace{(x+2)}_{u}^4 dx = 15\int u^4 du = 15\frac{u^5}{5} + C = 3(x+2)^5 + C \quad \Leftrightarrow \begin{cases} u = x+2 \\ du = dx \end{cases}$

4.
$$\int \underbrace{(x^3+7x^2+1)}_{u}\underbrace{(3x^2+14x)dx}_{du} = \int u\,du = \frac{u^2}{2} + C$$
$$= \frac{1}{2}(x^3+7x^2+1)^2 + C \quad \Leftrightarrow \begin{cases} u = x^3+7x^2+1 \\ du = (3x^2+14x)dx \end{cases}$$

8.
$$\int \frac{4x}{\underbrace{(2x^2-7)}_{u}^{10}} dx = \int \frac{du}{u^{10}} = \int u^{-10} du = \frac{u^{-9}}{-9} + C$$
$$= -\frac{1}{9(2x^2-7)^9} + C \quad \Leftrightarrow \begin{cases} u = 2x^2-7 \\ du = 4x\,dx \end{cases}$$

10. $\int \frac{1}{\sqrt{\underbrace{x-2}_{u}}} dx = \int \frac{du}{\sqrt{u}} = \int u^{-1/2} du = \frac{u^{1/2}}{1/2} + C = 2\sqrt{x-2} + C \quad \Leftrightarrow \begin{cases} u = x-2 \\ du = dx \end{cases}$

14. $\int 9x\sqrt{1+2x^2}\,dx = 9\int u^{1/2}\frac{du}{2} = \frac{\cancel{9}^{3}}{\cancel{2}}\frac{u^{3/2}}{\cancel{3}/\cancel{2}} + C = 3(1+2x^2)^{3/2} + C \quad \Leftrightarrow \begin{cases} u = 1+2x^2 \\ du = 2x\,dx \end{cases}$

16. $\int (3-2x)\,dx = \int u\frac{du}{-2} = -\frac{1}{2}\frac{u^2}{2} + C = -\frac{1}{4}(3-2x)^2 + C \quad \Leftrightarrow \begin{cases} u = 3-2x \\ du = -2dx \end{cases}$

18. $\int 5e^{3t+7}dt = 5\int e^u \frac{du}{3} = \frac{5}{3}e^u + C = \frac{5}{3}e^{3t+7} + C \quad \Leftrightarrow \begin{cases} u = 3t+7 \\ du = 3dt \end{cases}$

20. $\int -3\omega^2 e^{-\omega^3} d\omega = \int e^u du = e^u + C = e^{-\omega^3} + C \quad \Leftrightarrow \begin{cases} u = -\omega^3 \\ du = -3\omega^2 d\omega \end{cases}$

24. $\int x^4 e^{-6x^5} dx = \int e^u \frac{du}{-30} = -\frac{1}{30}e^u + C = -\frac{1}{30}e^{-6x^5} + C \quad \Leftrightarrow \begin{cases} u = -6x^5 \\ du = -30x^4 dx \end{cases}$

25. $\int \frac{1}{x+5} dx = \int \frac{1}{u} du = \ln|u| + C = \ln|x+5| + C \quad \Leftrightarrow \begin{cases} u = x+5 \\ du = dx \end{cases}$

26.

$$\int \frac{12x^2+4x+2}{\underbrace{x+x^2+2x^3}_{u}}dx = \int \frac{2du}{u} = 2\ln|u| + C$$

$$= \ln\left(x+x^2+2x^3\right)^2 + C \quad \Leftrightarrow \begin{cases} u = x+x^2+2x^3 \\ du = \left(1+2x+6x^2\right)dx \end{cases}$$

28.

$$\int \frac{9x^2-2x}{\underbrace{1-x^2+3x^3}_{u}}dx = \int \frac{du}{u} = \ln|u| + C$$

$$= \ln\left|1-x^2+3x^3\right| + C \quad \Leftrightarrow \begin{cases} u = 1-x^2+3x^3 \\ du = \left(-2x+9x^2\right)dx \end{cases}$$

30. $$\int \frac{3}{\underbrace{(5\upsilon-1)}_{u}^4}d\upsilon = 3\int \frac{1}{u^4}\frac{du}{5} = \frac{3}{5}\int u^{-4}du = \frac{\cancel{3}}{5}\frac{u^{-3}}{-\cancel{3}} + C = \frac{1}{5(5\upsilon-1)^3} + C \quad \Leftrightarrow \begin{cases} u = 5\upsilon-1 \\ du = 5d\upsilon \end{cases}$$

32. $$\int \frac{1}{1+2y}dy = \int \frac{1}{u}\frac{du}{2} = \frac{1}{2}\ln|u| + C = \ln\sqrt{1+2y} + C \quad \Leftrightarrow \begin{cases} u = 1+2y \\ du = 2dy \end{cases}$$

38. $$\int \frac{1}{\underbrace{(3x)}_{u}^6}dx = \int \frac{1}{u^6}\frac{du}{3} = \frac{1}{3}\int u^{-6}du = \frac{1}{3}\frac{u^{-5}}{-5} + C = -\frac{1}{15(3x)^5} + C \quad \Leftrightarrow \begin{cases} u = 3x \\ du = 3dx \end{cases}$$

44. $$\int \frac{x^2}{\sqrt[3]{\underbrace{2x^3+9}_{u}}}dx = \int \frac{1}{u^{1/3}}\frac{du}{6} = \frac{1}{{}^2\cancel{6}}\frac{u^{2/3}}{2/\cancel{3}} + C = \frac{1}{4}\left(2x^3+9\right)^{-2/3} + C \quad \Leftrightarrow \begin{cases} u = 2x^3+9 \\ du = 6x^2dx \end{cases}$$

50.

$$\int\left(e^x+2e^{-3x}-e^{5x}\right)dx = \int e^x dx + 2\int e^{-3x}dx - \int e^{5x}dx \quad \Leftrightarrow \begin{cases} u = -3x, \ \ du = -3dx \\ \upsilon = 5x, \quad d\upsilon = 5dx \end{cases}$$

$$= \int e^x dx + 2\int e^u \frac{du}{-3} - \int e^\upsilon \frac{d\upsilon}{5} = e^x - \frac{2}{3}e^{-3x} - \frac{1}{5}e^{5x} + C$$

52.

$$\int\left(t^2+4t\right)\left(t^3+6t^2\right)^6 dt = \int u^6\frac{du}{3} = \frac{1}{3}\frac{u^7}{7} + C \quad \Leftrightarrow \begin{cases} u = t^3+6t^2 \\ du = 3\left(t^2+4t\right)dt \end{cases}$$

$$= \frac{\left(t^3+6t^2\right)^7}{21} + C$$

56.

$$\int \frac{3}{5}(\upsilon-2)e^{2-4\upsilon+\upsilon^2}d\upsilon = \frac{3}{5}\int e^u\frac{du}{2} = \frac{3}{10}e^u + C$$

$$= \frac{3}{10}e^{2-4\upsilon+\upsilon^2} + C \quad \Leftrightarrow \begin{cases} u = 2-4\upsilon+\upsilon^2 \\ du = 2(-2+\upsilon)dx \end{cases}$$

60. $$\int\left(e^{x}+e^{-x}\right)^{2}dx=\int\left(e^{2x}+e^{-2x}+2\right)dx=\int e^{2x}dx+\int e^{-2x}dx+2\int dx$$
$$=\frac{1}{2}e^{2x}-\frac{1}{2}e^{-2x}+2x+C$$

62. $$\int\left(u^{3}-ue^{6-3u^{2}}\right)du=\int u^{3}du-\int ue^{6-3u^{2}}du=\frac{u^{4}}{4}-\int e^{z}\frac{dz}{-6}=\frac{u^{4}}{4}+\frac{1}{6}e^{z}+C$$
$$=\frac{u^{4}}{4}+\frac{1}{6}e^{6-3u^{2}}+C \qquad \Leftrightarrow \begin{cases} z=6-3u^{2} \\ dz=-6udu \end{cases}$$

70. $$\int\left[\frac{3}{x-1}+\frac{1}{\left(x-1\right)^{2}}\right]dx=3\int\frac{1}{x-1}dx+\int\frac{1}{\left(x-1\right)^{2}}dx \qquad \Leftrightarrow \begin{cases} u=x-1 \\ du=dx \end{cases}$$
$$=3\ln\left(x-1\right)+\int u^{-2}dx+C=3\ln\left(x-1\right)-\frac{1}{x-1}+C$$

76. $\int\left(e^{4}-2^{e}\right)dx=\left(e^{4}-2^{e}\right)\int dx=\left(e^{4}-2^{e}\right)x$

78. $$\int\frac{2}{t^{2}}\sqrt{\frac{1}{t}+9}dt=2\int\sqrt{u}(-du)=-2\int u^{\frac{1}{2}}du=-2\frac{u^{\frac{3}{2}}}{\frac{3}{2}}+C=-\frac{4}{3}u^{\frac{3}{2}}+C \qquad \Leftrightarrow \begin{cases} u=\frac{1}{t}+9 \\ du=-\frac{1}{t^{2}}dt \end{cases}$$

82. Find *y* in $y'=\frac{x}{x^{2}+6}$, subject to the given *y*(1) = 0 conditions.

Answer: First integrate the *y'* with respect to x to obtain *y*:

$$y=\int\frac{x}{x^{2}+6}dx=\int\frac{1}{u}\frac{du}{2}=\frac{1}{2}\ln\left|u\right|+C=\ln\sqrt{x^{2}+6}+C \qquad \Leftrightarrow \begin{cases} u=x^{2}+6 \\ du=2xdx \end{cases}$$

then apply the initial condition y(1) = 0 to determine the *C* constant of integration:

$$y_{x=1}=\ln\sqrt{1^{2}+6}+C=0 \quad \Rightarrow \quad C=-\ln 7$$

The result is: $y=\ln\sqrt{x^{2}+6}+7$

88. Find *f*(2) if *f*(1/2) = 1 and $f'(x) = e^{2x-1} - 6x$.

Answer: First integrate the *f′* with respect to x to obtain *f*(x):

$$f(x) = \int \left(e^{2x-1} - 6x\right) dx = \int e^{2x-1} dx - 6\int x dx = \frac{1}{2}\int e^{u} du - 6\int x dx \quad \Leftrightarrow \begin{cases} u = 2x-1 \\ du = 2dx \end{cases}$$

$$f(x) = \frac{1}{2}e^{2x-1} - 3x^2 + C$$

then apply the initial condition f(2) = 1 to determine the *C* constant of integration:

$$f(2) = \frac{1}{2}e^{2\cdot\frac{1}{2}-1} - 3\cdot\left(\tfrac{1}{2}\right)^2 + C = 1 \quad \Rightarrow \quad C = 1 - \frac{1}{2} + \frac{3}{4} = \frac{5}{4}$$

The result is: $$f(x) = \frac{1}{2}e^{2x-1} - 3x^2 + \frac{5}{4}$$

14.5 Techniques of Integration

Keep in mind: $\int a^x dx = \frac{1}{\ln a}a^x + C$ (Proof in the book on page 714)

Determine the indefinite integrals below

2. $\int \frac{9x^2+5}{3x} dx = \int \left(\frac{9x^{\not 2}}{3\not x} + \frac{5}{3x}\right) dx = 3\int x dx + \frac{5}{3}\int \frac{1}{x} dx = \frac{3}{2}x^2 + \frac{5}{3}\ln x + C$

4. $\int \frac{x}{\sqrt[4]{x^2+1}} dx = \int \frac{1}{\sqrt[4]{u}} \frac{du}{2} = \frac{1}{2}\int u^{\frac{1}{4}} du = \frac{1}{2}\frac{u^{\frac{5}{4}}}{\frac{5}{4}} + C = \frac{2}{5}u^{\frac{5}{4}} + C \quad \Leftrightarrow \begin{cases} u = x^2+1 \\ du = 2x dx \end{cases}$

8. $\int 5^t dt = ?$

Answer: Another method to find the integration of *a*^{u} is as follows. Let us say

$$u = 5^t \quad \Rightarrow \quad \ln u = \ln 5^t = t\ln 5$$

Take the derivative of ln*u* with respect to *t*: (*u′* = *du*/*dt*)

$$\frac{u'}{u} = \ln 5 \quad \Rightarrow \quad u' = \frac{du}{dt} = u\ln 5 = 5^t \ln 5 \quad \Rightarrow \quad \frac{du}{\ln 5} = 5^t dt$$

Now we are ready to take the integration:

$$\int 5^t dt = \int \frac{du}{\ln 5} = \frac{1}{\ln 5}\int du = \frac{1}{\ln 5}u + C = \frac{1}{\ln 5}5^t + C$$

11. $\int \frac{6x^2 - 11x + 5}{3x-1} dx = ?$

Answer: First we have to find the ratio because the numerator's degree is greater than the denominator's. So we have then

$$\begin{array}{l} \qquad\qquad\qquad 3x-1 \\ 6x^2-11x+5\overline{\big)\,2x-3} \\ \mp 6x^2 \pm 2x \\ \hline \qquad -9x+5 \\ \qquad \pm 9x \mp 3 \\ \hline \qquad\quad 0x+2 \end{array} \qquad \Rightarrow \qquad \frac{6x^2-11x+5}{3x-1}=2x-3+\frac{2}{3x-1}$$

Hence we are ready to take integration:

$$\int \frac{6x^2-11x+5}{3x-1}\,dx=\int (2x-3)\,dx+\int \frac{2}{3x-1}\,dx=x^2-3x+\frac{2}{3}\ln|3x-1|+C$$

12. $\int \frac{(2x-1)(x+3)}{x-5}\,dx=\int \frac{2x^2+5x-3}{x-5}\,dx=?$

Answer: Let's divide the integrand fraction

$$\begin{array}{l} \qquad\qquad\qquad x-5 \\ 2x^2+5x-3\overline{\big)\,2x+15} \\ \mp 2x^2 \pm 10x \\ \hline \qquad +15x-3 \\ \qquad \mp 15x \pm 75 \\ \hline \qquad\qquad +72 \end{array} \qquad \Rightarrow \qquad \frac{2x^2+5x-3}{x-5}=2x+15+\frac{72}{x-5}$$

Hence we are ready to take integration:

$$\int \frac{6x^2-11x+5}{3x-1}\,dx=\int (2x+15)\,dx+\int \frac{72}{x-5}\,dx=x^2+15x+72\ln|x-5|+C$$

14. $\int 6\left(e^{4-3x}\right)^2 dx=?$

Answer: $\int 6\left(e^{4-3x}\right)^2 dx=6\int e^{8-6x}dx=\not{6}\underbrace{\int e^u \frac{du}{-\not{6}}}_{-e^u}=-e^{8-6x}+C \qquad \Leftrightarrow \begin{cases} u=8-6x \\ du=-6dx \end{cases}$

20. $\int \frac{5e^s}{1+3e^s}\,ds=?$

Answer:

$$5\int \frac{e^s}{1+3e^s}\,ds=5\int \frac{1}{u}\frac{du}{3}=\frac{5}{3}\ln|u|+C=\frac{5}{3}\ln\left(1+3e^s\right)+C \qquad \Leftrightarrow \begin{cases} u=1+3e^s \\ du=3e^s ds \end{cases}$$

24. $\int \sqrt{t}\left(5-t\sqrt{t}\right)dt=?$

Answer:

$$\int \sqrt{t}\left(5-t\sqrt{t}\right)^{0.4} dt = \frac{2}{3}\int u^{\frac{2}{5}} du \qquad \Leftrightarrow \begin{cases} u = 5-t\sqrt{t} = 5-t^{\frac{1}{2}} \\ du = \frac{3}{2}t^{\frac{1}{2}}dt = \frac{3}{2}\sqrt{t}dt \end{cases}$$

$$= \frac{2}{3}\frac{5}{7}u^{\frac{7}{5}} + C = \frac{10}{21}\left(5-t\sqrt{t}\right)^{\frac{7}{5}} + C$$

26. $\int \frac{9x^5 - 6x^4 - ex^3}{7x^2} dx = \frac{1}{7}\int \left(9x^3 - 6x^2 - ex\right) dx = \frac{1}{7}\left(\frac{9}{4}x^4 - 2x^3 - \frac{e}{2}x^2\right)$

30. $\int \frac{x+3}{x+6} dx = ?$

Answer: First let's divide the integrand for a partial fraction:

$$\begin{array}{l} \quad\;\; x+6 \\ \not{x}+3 \,\big)\;\; 1 \\ \mp\not{x}\mp 6 \\ \hline \quad -3 \end{array} \qquad \Rightarrow \quad \frac{x+3}{x+6} = 1 - \frac{3}{x+6}$$

Hence we are ready to take integration:

$$\int \frac{x+3}{x+6} dx = \int 1 dx - 3\int \frac{1}{x+6} dx = x - 3\ln\left|x+6\right| + C$$

34.

$$\int \frac{4x\ln\sqrt{1+x^2}}{1+x^2} dx = 4\int u du = \frac{1}{2}u^2 + C$$

$$= \frac{1}{2}\left(\ln\sqrt{1+x^2}\right)^2 + C \qquad \Leftrightarrow \begin{cases} u = \ln\sqrt{1+x^2} = \frac{1}{2}\ln\left(1+x^2\right) \\ du = \frac{1}{\not{2}}\frac{\not{2}x}{1+x^2} dx = \frac{x}{1+x^2} dx \end{cases}$$

40. $\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx = \int \frac{du}{u} = \ln\left|u\right| + C = \ln\left|e^x - e^{-x}\right| + C \qquad \Leftrightarrow \begin{cases} u = e^x - e^{-x} \\ du = \left(e^x + e^{-x}\right) dx \end{cases}$

41. $\int \frac{x}{x-1} dx = ?$

Answer: First let's divide the integrand for a partial fraction:

$$\begin{array}{l} \quad x-1 \\ \not{x}\,\big)\;\; 1 \\ \mp\not{x}\pm 1 \\ \hline \quad\;\; 1 \end{array} \qquad \Rightarrow \quad \frac{x}{x-1} = 1 + \frac{1}{x-1}$$

Hence we are ready to take integration:

$$\int \frac{x}{x-1} dx = \int 1 dx + \int \frac{1}{x-1} dx = x + \ln\left|x-1\right| + C$$

43. $\int \frac{xe^{x^2}}{\sqrt{e^{x^2}+2}}dx = \int \frac{1}{\sqrt{u}}\frac{du}{2} = \frac{1}{2}\int u^{-\frac{1}{2}}du = \left(e^{x^2}+2\right)^{\frac{1}{2}} + C \quad \Leftrightarrow \begin{cases} u = e^{x^2}+2 \\ du = 2xe^{x^2}dx \end{cases}$

45. $\int \frac{\left(e^{-x}+6\right)^2}{e^x}dx = \int \frac{e^{-2x}+12e^{-x}+36}{e^x}dx = \int \left(e^{-3x}+12e^{-2x}+36e^{-x}\right)dx$

$= -\frac{1}{3}e^{-3x} - 6e^{-2x} - 36e^{-x} + C$

48. $\int 2^{x\ln x}(1+\ln x)dx = \int 2^u du = \frac{1}{\ln 2}2^{x\ln x} + C \quad \Leftrightarrow \begin{cases} u = x\ln x \\ du = (1+\ln x)dx \end{cases}$

50. $\int \frac{2}{x\underbrace{(\ln x)}_{u}{}^{2/3}}dx = 2\int u^{-\frac{2}{3}}du = 2u^{\frac{1}{3}} + C = (\ln x)^{\frac{2}{3}} + C \quad \Leftrightarrow \begin{cases} u = \ln x \\ du = \frac{1}{x}dx \end{cases}$

52. $\int \frac{\ln^3 x}{3x}dx = \frac{1}{3}\int u^3 du = \frac{1}{3}\frac{u^4}{4} + C = \frac{\ln^4 x}{12} + C \quad \Leftrightarrow \begin{cases} u = \ln x \\ du = \frac{1}{x}dx \end{cases}$

<u>14.6 Summation</u>

Keep in mind: $\sum_{k=1}^{n} k = 1+2+\cdots+n = \frac{n(n+1)}{2}$ (page 718)

$\sum_{k=1}^{n} k^2 = 1+4+9+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$ (page 719)

2. Evaluate the given sum below

$\sum_{k=4}^{10}(5-3k) = -7-10-13-16-19-22-25 = -112$

$\sum_{k=4}^{10}(5-3k) = \sum_{k=1}^{10}(5-3k) - \sum_{k=1}^{3}(5-3k) = 5\underbrace{\sum_{k=1}^{10}1}_{=10} - 5\underbrace{\sum_{k=1}^{3}1}_{=3} - 3\sum_{k=1}^{10}k + 3\sum_{k=1}^{3}k$

$= 5(10-3) - 3\left(\frac{\overset{5}{\cancel{10}}\times 11}{\cancel{2}} - \frac{3\times \cancel{4}^{2}}{\cancel{2}}\right) = 35 - 3\times 49 = 112$

Express the given sums in sigma notation

12. $7+8+9+10 = \sum_{k=7}^{10} k$

14. $3+6+9+\cdots+36 = 3(1+2+3+\cdots+12) = 3\sum_{k=1}^{12} k = \sum_{k=1}^{12} 3k$

20. $\sum_{i=1}^{40} \frac{i}{2} = \frac{1}{2}\sum_{i=1}^{40} i = \frac{1}{\cancel{2}} \frac{\cancel{40}^{10} \times 41}{\cancel{2}} = 410$

22. $\sum_{n=1}^{8} \left(\frac{2n}{3}\right)^2 = \sum_{n=1}^{8} \frac{4n^2}{9} = \frac{4}{9}\sum_{n=1}^{8} n^2 = \frac{4}{\cancel{9}} \frac{8 \times \cancel{9} \times 17}{6} = \frac{272}{3}$

14.8 The Fundamental Theorem of Integral Calculus

Keep in mind: $\int_a^b f(x)dx = F(b) - F(a)$ (page 730)

Evaluate the definite integral for the problems below:

2. $\int_2^8 -5xdx = -5\int_2^8 xdx = -5\left.\frac{x^2}{2}\right|_2^8 = -\frac{5}{2}(64-4) = -150$

8. $\int_3^2 \left(2t - t^2\right)dt = -\int_2^3 \left(2t - t^2\right)dt = -\left(\cancel{2}\frac{t^2}{\cancel{2}} - \frac{t^3}{3}\right)_2^3 = -9+9+4-\frac{8}{3} = \frac{4}{3}$

12. $\int_1^2 \frac{x^{-2}}{2}dx = \frac{1}{2}\left.\frac{x^{-1}}{-1}\right|_1^2 = -\left.\frac{1}{2x}\right|_1^2 = -\frac{1}{2\times 2} + \frac{1}{2} = \frac{1}{4}$

20. $\int_1^3 (x+3)^3 dx = \int_4^6 u^3 du = \left.\frac{u^4}{4}\right|_4^6 = \frac{1}{4}\left(6^4 - 4^4\right) = 260 \quad \Leftrightarrow \begin{cases} u = x+3, & du = dx \\ x = 1 \Rightarrow & u = 4 \\ x = 3 \Rightarrow & u = 6 \end{cases}$

24. $\int_2^{e+1} \frac{1}{x-1}dx = \ln(x-1)\Big|_2^{e+1} = \ln e - \ln 1 = 1$

30. $\int_{-1}^{1} q\sqrt{q^2+3}dq = \frac{1}{2}\int u^{\frac{1}{2}}dq = \frac{1}{\cancel{2}}\left.\frac{\left(q^2+3\right)^{\frac{3}{2}}}{\frac{3}{\cancel{2}}}\right|_{-1}^{1} = \frac{1}{3}\left(4^{\frac{3}{2}} - 4^{\frac{3}{2}}\right) = 0 \quad \Leftrightarrow \begin{cases} u = q^2+3 \\ du = 2qdq \end{cases}$

34. $\int_a^b (m+ny)dy == \left.\left(my + \frac{n}{2}y^2\right)\right|_a^b = m(b-a) + \frac{n}{2}\left(b^2 - a^2\right)$