Chapter 15 METHODS AND APPLICATIONS OF INTEGRATION

15.2 Integration by Partial Fractions

Exercise 15.2

Determine the integrals

10. $\int \frac{3x+8}{x^2+2x}dx = ?$

Answer: First we have to express the rational function in terms of partial fraction:

So we can solve the integral

$$\int \frac{3x+8}{x^2+2x}dx = \int \frac{4}{x}dx + \int \frac{-1}{x+2}dx = 4\ln x - \ln(x+2) + C = \ln\frac{x^4}{(x+2)} + C$$

12. $\int \frac{2x-1}{x^2-x-12}dx = ?$

Answer: First we have to express the rational function in terms of partial fraction:

$$\frac{2x-1}{x^2-x-12} = \frac{2x-1}{(x+3)(x-4)} = \frac{A}{x+3} + \frac{B}{x-4} = \frac{A(x-4)+B(x+3)}{x(x+2)}$$

$$2x-1 = A(x-4)+B(x+3)$$

$$x=-3 \quad \Rightarrow \quad 2\cdot(-3)-1 = A(-3-4)+B\cancel{(-3+3)} \Rightarrow \quad A = \frac{-7}{-7} = 1$$

$$x=4 \quad \Rightarrow \quad 2\cdot 4-1 = A\cancel{(4-4)} + B(4+3) \quad \Rightarrow \quad B = \frac{7}{7} = 1$$

So we can solve the integral

$$\int \frac{2x-1}{x^2-x-12}dx = \int \frac{1}{x+3}dx + \int \frac{1}{x-4}dx$$

$$= \ln(x+3) + \ln(x-4) + C = \ln\left[(x+3)(x-4)\right] + C$$

14. $\int \frac{7(4-x^2)}{(x-4)(x-2)(x+3)}dx = ?$

Answer: First we have to express the rational function in terms of partial fraction:

$$\frac{7(4-x^2)}{(x-4)(x-2)(x+3)} = \frac{A}{x-4} + \frac{B}{x-2} + \frac{C}{x+3}$$

$$= \frac{A(x-2)(x+3)+B(x-4)(x+3)+C(x-2)(x-4)}{(x-4)(x-2)(x+3)}$$

$7\left(4-x^2\right)=A(x-2)(x+3)+B(x-4)(x+3)+C(x-2)(x-4)$

$x=2 \quad \Rightarrow \quad 7\left(4-2^2\right)=0=A\cancel{(2-2)}(2+3)+B(2-4)(2+3)+C\cancel{(2-2)}(2-4) \quad \Rightarrow \quad B=\frac{0}{-10}=0$

$x=4 \quad \Rightarrow \quad 7\left(4-4^2\right)=A(4-2)(4+3)+B\cancel{(4-4)}(4+3)+C(4-2)\cancel{(4-4)} \quad \Rightarrow \quad A=\frac{7(-12)}{2\cdot 7}=-6$

$x=-3 \quad \Rightarrow \quad 7\left(4-(-3)^2\right)=A(-3-2)\cancel{(-3+3)}+B(-3-4)\cancel{(-3+3)}+C(-3-2)(-3-4) \Rightarrow \quad C=\frac{7\cdot(-5)}{5\cdot 7}=-1$

So we can solve the integral

$$\int \frac{7\left(4-x^2\right)}{(x-4)(x-2)(x+3)}dx=\int \frac{-6}{x-4}dx+\int \frac{0}{x-2}dx+\int \frac{-1}{x+3}dx$$

$$=-6\ln(x-4)-1\ln(x+3)+C=-\ln\left[(x+3)(x-4)^6\right]+C$$

20. $\int \frac{-3x^3+2x-3}{x^2\left(x^2-1\right)}dx=?$

Answer: First we have to express the rational function in terms of partial fraction:

$$\frac{-3x^3+2x-3}{x^2\left(x^2-1\right)}=\frac{-3x^3+2x-3}{x^2(x-1)(x+1)}=\frac{Ax+B}{x^2}+\frac{C}{x-1}+\frac{D}{x+1}$$

$$=\frac{(Ax+B)(x-1)(x+1)+Cx^2(x+1)+Dx^2(x-1)}{x^2\left(x^2-1\right)}$$

$-3x^3+2x-3=(Ax+B)(x-1)(x+1)+Cx^2(x+1)+Dx^2(x-1)$

$x=1 \quad \Rightarrow \quad -3+2-3=(A+B)\cancel{(1-1)}(1+1)+C\cdot 1^2(1+1)+D\cdot 1^2\cancel{(1-1)} \quad \Rightarrow \quad C=\frac{-4}{2}=-2$

$x=-1 \quad \Rightarrow \quad \cancel{-3(-1)^3}+2(-1)-\cancel{3}=(Ax+B)(-1-1)\cancel{(-1+1)}+C(-1)^2\cancel{(-1+1)}+D(-1)^2(-1-1) \quad \Rightarrow \quad D=\frac{-2}{-2}=1$

$x=0 \quad \Rightarrow \quad -3\cdot 0^3+2\cdot 0-3=(A\cdot 0+B)(0-1)(0+1)+C\cdot 0(0+1)+D\cdot 0(0-1) \quad \Rightarrow \quad B=\frac{-3}{-1}=3$

$x=2 \quad \Rightarrow \quad -3\cdot 2^3+2\cdot 2-3=(A\cdot 2+3)(2-1)(2+1)-2\cdot 2^2(2+1)+1\cdot 2^2(2-1) \quad \Rightarrow \quad A=\frac{-4}{2}=-2$

Here we gave *x* = 0 and *x* = 2 values for convenience to find the *A* and *B* values. You may try any other value for *x* to obtain the desired values for *A* and *B*. Therefore we may write

$$\frac{-3x^3+2x-3}{x^2\left(x^2-1\right)}=\frac{-2x+3}{x^2}+\frac{-2}{x-1}+\frac{1}{x+1}=\frac{-2}{x}+\frac{3}{x^2}-\frac{2}{x-1}+\frac{1}{x+1}$$

So we can solve the integral

$$\int \frac{-3x^3+2x-3}{x^2\left(x^2-1\right)}dx = -2\int \frac{1}{x}dx + 3\int \frac{1}{x^2}dx - 2\int \frac{1}{x-1}dx + \int \frac{1}{x+1}dx$$

$$= -2\ln x + 3\frac{1}{-x} - 2\ln(x-1) + \ln(x+1) + C$$

$$= -\ln\left[\frac{(x+1)}{x^2(x-1)^2}\right] - \frac{3}{x} + C$$

24. $\int \frac{5x^4+9x^2+3}{x\left(x^2+1\right)^2}dx = ?$

Answer: First we have to express the rational function in terms of partial fraction:

$$\frac{5x^4+9x^2+3}{x\left(x^2+1\right)^2} = \underset{\left(x^2+1\right)^2}{\frac{A}{x}} + \underset{x\left(x^2+1\right)}{\frac{Bx+C}{x^2+1}} + \underset{x}{\frac{Dx+E}{\left(x^2+1\right)^2}} = \frac{A\left(x^2+1\right)^2 + (Bx+C)x\left(x^2+1\right) + x(Dx+E)}{x\left(x^2+1\right)^2}$$

Equating the denominator:

$$5x^4+9x^2+3 = A\left(x^2+1\right)^2 + (Bx+C)x\left(x^2+1\right) + x(Dx+E)$$

$$= A\left(x^4+2x^2+1\right) + \left(Bx^4+Bx^2+Cx^3+Cx\right) + Dx^2 + Ex$$

$$= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$$

Here we equate the coefficient of the same powers:

$A = 3$

$A + B = 5 \quad \Rightarrow \quad B = 2$

$C = 0$

$2A + B + D = 9 \Rightarrow \quad D = 1$

$C + E = 0 \quad \Rightarrow \quad E = 0$

$$\frac{5x^4+9x^2+3}{x\left(x^2+1\right)^2} = \frac{3}{x} + \frac{2x}{x^2+1} + \frac{x}{\left(x^2+1\right)^2}$$

So we can solve the integral

$$\int \frac{5x^4+9x^2+3}{x\left(x^2+1\right)^2}dx = 3\int \frac{1}{x}dx + \int \frac{2x}{x^2+1}dx + \frac{1}{2}\int \frac{2x}{\left(x^2+1\right)^2}dx$$

$$= 3\ln x + \ln(x^2+1) - \frac{1}{2\left(x^2+1\right)} + C$$

$$= \ln\left[x^3\left(x^2+1\right)\right] - \frac{1}{2\left(x^2+1\right)} + C$$

15.4 Average Value of a Function

Exercise 15.4 Find the average values of the function over the given integral.

$$\overline{f} = \frac{1}{b-a}\int_a^b f(x)dx$$

4. $f(x) = x^2 + x + 1; \qquad [1, 3]$

Answer:

$$\overline{f} = \frac{1}{3-1}\int_1^3 \left(x^2 + x + 1\right)dx = \frac{1}{2}\left(\frac{x^3}{3} + \frac{x^2}{2} + x\right)_1^3 = \frac{1}{2}\left(\frac{3^3}{3} + \frac{3^2}{2} + 3 - \frac{1^3}{3} - \frac{1^2}{2} - 1\right) = \frac{22}{3}$$

6. $f(t) = t\sqrt{t^2+9}; \qquad [0, 4]$

Answer:

$$\overline{f} = \frac{1}{4-0}\int_0^4 t\sqrt{t^2+9}dt = \frac{1}{4}\int_9^{25} \sqrt{u}\,\frac{du}{2} \qquad \begin{cases} t^2+9 = u & \Rightarrow \quad 2tdt = du \\ t = 0 & \Rightarrow \quad u = 9 \\ t = 4 & \Rightarrow \quad u = 25 \end{cases}$$

$$= \frac{1}{8}\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_9^{25} = \frac{3}{4}\left(25^{\frac{3}{2}} - 9^{\frac{3}{2}}\right) = \frac{3}{4}\left(5^3 - 3^3\right) = \frac{107}{2}$$

10. Cost Suppose (in dollars) of producing q units of a product is given by

$c = 4000 + 10q + 0.1q^2$

Find the average cost on the interval from q = 100 to q = 500.

Answer:

$$\overline{f} = \frac{1}{400}\int_{100}^{500} \left(4000 + 10q + 0.1q^2\right)dq = \frac{1}{400}\left(4000q + 5q^2 + \frac{1}{30}q^3\right)_{100}^{500}$$

$$= \frac{1}{400}\left(4000(500-100) + 5\left(500^2 - 100^2\right) + \frac{1}{30}\left(500^3 - 100^3\right)\right)$$

$$= \frac{1}{400}\left(1,600,000 + 1,200,000 + \frac{12,400,000}{3}\right) = 7000 + \frac{31,000}{3} = \frac{52,000}{3}$$