CHAPTER 17. MULTIVARIABLE CALCULUS

17.2 Partial Derivatives

$$f_x(x,y)=\frac{\partial f}{\partial x}=\lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}; \qquad f_y(x,y)=\frac{\partial f}{\partial y}=\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$

Exercise 17.2 For the given two or more variable functions, find the partial derivative of the function with respect to each of the variables.

2. $f(x,y)=2x^2+3xy$

Answer:

$$f_x=\frac{\partial f}{\partial x}=4x+3y; \qquad f_y=\frac{\partial f}{\partial y}=3x$$

6. $g(x,y)=(x+1)^2+(y-3)^3+5xy^3-2$

Answer:

$$g_x=\frac{\partial g}{\partial x}=2(x+1)+5y^3; \qquad g_y=\frac{\partial g}{\partial y}=3(y-3)^2+15xy^2$$

10. $h(u,v)=\dfrac{8uv^2}{u^2+v^2}$

Answer:

$$h_u=\frac{\partial h}{\partial u}=\frac{8v^2\cdot(u^2+v^2)-2u\cdot 8uv^2}{(u^2+v^2)^2}=\frac{8v^2\cdot(u^2+v^2-2u^2)}{(u^2+v^2)^2}=\frac{8v^2\cdot(v^2-u^2)}{(u^2+v^2)^2}$$

$$h_v=\frac{\partial h}{\partial v}=\frac{8uv^2}{u^2+v^2}=\frac{16uv\cdot(u^2+v^2)-2v\cdot 8uv^2}{(u^2+v^2)^2}=\frac{16uv\cdot(u^2+v^2-v^2)}{(u^2+v^2)^2}=\frac{16u^3v}{(u^2+v^2)^2}$$

16. $z=(x^2+y)e^{3x+4y}$

Answer:

$$z_x=\frac{\partial z}{\partial x}=2xe^{3x+4y}+3(x^2+y)e^{3x+4y}=(3x^2+3y+2x)e^{3x+4y};$$

$$z_y=\frac{\partial z}{\partial y}=e^{3x+4y}+4(x^2+y)e^{3x+4y}=(4x^2+4y+1)e^{3x+4y}$$

18. $z=\ln(5x^3y^2+2y^4)^4=4\ln\left[y^2(5x^3+2y^2)\right]=8\ln y+4\ln(5x^3+2y^2)$

Answer:

$$z_x=4\frac{\frac{\partial}{\partial x}(5x^3+2y^2)}{5x^3+2y^2}=\frac{60x^2}{5x^3+2y^2}; \qquad z_y=8\frac{1}{y}+4\frac{\frac{\partial}{\partial y}(5x^3+2y^2)}{5x^3+2y^2}=\frac{8}{y}+\frac{16y}{5x^3+2y^2};$$

20. $f(r,s)=\sqrt{rs}e^{2+r}$

Answer:

$$f_r=\frac{\partial f}{\partial r}=\frac{\sqrt{s}}{2\sqrt{r}}e^{2+r}+\sqrt{rs}e^{2+r}=\left(\frac{1}{2r}+1\right)\sqrt{rs}e^{2+r};\qquad f_s=\frac{\partial f}{\partial s}=\frac{\sqrt{r}}{2\sqrt{s}}e^{2+r}=\frac{1}{2}\sqrt{\frac{r}{s}}e^{2+r}$$

28. $z=\sqrt{5x^2+3xy+2y}$

Answer:

$$z_x=\frac{\partial z}{\partial x}=\frac{1}{2}\left(5x^2+3xy+2y\right)^{-1/2}\left(10x+3y\right)=\frac{10x+3y}{2\sqrt{5x^2+3xy+2y}}$$

$$\left.\frac{\partial z}{\partial x}\right|_{\substack{x=0\\y=2}}=\frac{10\cdot 0+3\cdot 2}{2\sqrt{5\cdot 0^2+3\cdot 0\cdot 2+2\cdot 2}}=\frac{6}{2\cdot 2}=\frac{3}{2}$$

$$z_y=\frac{\partial z}{\partial y}=\frac{1}{2}\left(5x^2+3xy+2y\right)^{-1/2}\left(3x+2\right)=\frac{3x+2}{2\sqrt{5x^2+3xy+2y}}$$

$$\left.\frac{\partial z}{\partial y}\right|_{\substack{x=0\\y=2}}=\frac{3\cdot 0+2}{2\sqrt{5\cdot 0^2+3\cdot 0\cdot 2+2\cdot 2}}=\frac{2}{2\cdot 2}=\frac{1}{2}$$

31. $h(r,s,t,u)=\left(s^2+tu\right)\ln\left(2r+7st\right);\qquad h_s(1,0,0,1)=?$

Answer:

$$h_s=\frac{\partial h}{\partial s}=\frac{\partial\left(s^2+tu\right)}{\partial s}\cdot\ln\left(2r+7st\right)+\left(s^2+tu\right)\cdot\frac{\partial\ln\left(2r+7st\right)}{\partial s}$$

$$=2s\ln\left(2r+7st\right)+\left(s^2+tu\right)\frac{7t}{2r+7st}=2s\ln\left(2r+7st\right)+\frac{7t\left(s^2+tu\right)}{2r+7st}$$

$$h_s(1,0,0,1)=\left.\frac{\partial h}{\partial s}\right|_{\substack{r=1\\s=0\\t=0\\u=1}}=2\cdot 0\cdot\ln\left(2\cdot 1+7\cdot 0\cdot 0\right)+\frac{7\cdot 0\cdot\left(0^2+0\cdot 1\right)}{2\cdot 1+7\cdot 0\cdot 0}=0$$

35. If $z=xe^{x-y}-ye^{y-x}$, show that $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=e^{x-y}-e^{y-x}$.

Answer:

$$\left.\begin{aligned}\frac{\partial z}{\partial x}&=e^{x-y}+xe^{x-y}+ye^{y-x}\\\frac{\partial z}{\partial y}&=-xe^{x-y}-e^{y-x}-ye^{y-x}\end{aligned}\right\}\Rightarrow\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=e^{x-y}+\cancel{xe^{x-y}}+\cancel{ye^{y-x}}-\cancel{xe^{x-y}}-e^{y-x}-\cancel{ye^{y-x}}$$

$$=e^{x-y}-e^{y-x}$$

17.4 Implicit Partial Differentiation

Exercises 17.4 Find the indicated partial derivatives by the method of implicit partial differentiation.

2. $z^2 - 5x^2 + y^2 = 0;\qquad \frac{\partial z}{\partial x} = ?$

Answer:

$$2z\frac{\partial z}{\partial x} - 10x = 0 \quad\Rightarrow\quad \cancel{2}z\frac{\partial z}{\partial x} = \cancel{10}^{5}x \quad\Rightarrow\quad \frac{\partial z}{\partial x} = \frac{5x}{z}$$

6. $z^3 - xz - y = 0;\qquad \frac{\partial z}{\partial x} = ?$

Answer:

$$3z^2\frac{\partial z}{\partial x} - z - x\frac{\partial z}{\partial x} = 0 \quad\Rightarrow\quad \left(3z^2 - x\right)\frac{\partial z}{\partial x} = z \quad\Rightarrow\quad \frac{\partial z}{\partial x} = \frac{z}{3z^2 - x}$$

8. $xyz + 3y^3x^2 - \ln z^3 = 0;\qquad \frac{\partial z}{\partial y} = ?$

Answer:

$$xz + xy\frac{\partial z}{\partial y} + 9y^2x^2 - \frac{3}{z}\frac{\partial z}{\partial y} = 0 \quad\Rightarrow\quad \left(xy - \frac{3}{z}\right)\frac{\partial z}{\partial y} = -xz - 9y^2x^2$$

$$\Rightarrow\quad \frac{\partial z}{\partial y} = -\frac{xz + 9y^2x^2}{xy - \frac{3}{z}} = \frac{xz\left(z + 9y^2x\right)}{3 - xyz}$$

10. $\ln x + \ln y + \ln z = e^y;\qquad \frac{\partial z}{\partial x} = ?$

Answer:

$$\frac{1}{x} + \frac{1}{z}\frac{\partial z}{\partial x} = 0 \quad\Rightarrow\quad \frac{1}{z}\frac{\partial z}{\partial x} = -\frac{1}{x} \quad\Rightarrow\quad \frac{\partial z}{\partial x} = -\frac{z}{x}$$

10. $e^{zx} = xyz;\qquad \frac{\partial z}{\partial y} = ?\ \text{ at }\ x = 1, y = -e^{-1}, z = -1$

Answer:

$$\frac{\partial z}{\partial y}e^{zx} = xz + xy\frac{\partial z}{\partial y} \quad\Rightarrow\quad \frac{\partial z}{\partial x} = \frac{xz}{e^{zx} - xy}$$

$$\left.\frac{\partial z}{\partial x}\right|_{\substack{x=1\\ y=-e^{-1}\\ z=-1}} = \frac{1(-1)}{e^{-1\cdot 1} - 1\cdot(-e^{-1})} = -\frac{e}{2}$$

20. $\ln\left(x+y+z\right)+xyz=ze^{x+y+z};$ $\frac{\partial z}{\partial x}=?$ at $x=0, y=1, z=0$

Answer:

$$\frac{1}{x+y+z}\left(1+\frac{\partial z}{\partial x}\right)+yz+xy\frac{\partial z}{\partial x}=e^{x+y+z}\frac{\partial z}{\partial x}+z\left(1+\frac{\partial z}{\partial x}\right)e^{x+y+z}$$

$$\left(\frac{1}{x+y+z}+y(x+z)-(1+z)e^{x+y+z}\right)\frac{\partial z}{\partial x}=ze^{x+y+z}-\frac{1}{x+y+z}$$

$$\frac{\partial z}{\partial x}=\frac{ze^{x+y+z}-\frac{1}{x+y+z}}{\frac{1}{x+y+z}+y(x+z)-(1+z)e^{x+y+z}}$$

$$\left.\frac{\partial z}{\partial x}\right|_{\substack{x=0\\y=1\\z=0}}=\frac{0\cdot e^{0+1+0}-\frac{1}{0+1+0}}{\frac{1}{0+1+0}+1(0+0)-(1+0)e^{0+1+0}}=\frac{-1}{1-e}=\frac{1}{e-1}$$