MATH 172FINAL EXAM 20/05/2014

Name and Surname: **Student ID # :** **Section #:**

Q1 (20)	Q2(20)	Q3(20)	Q4(10)	Q5(15)	Q6(15)	Total (100)

ATTENTION: Please show all your work in details. DO NOT USE calculators and cellphones. There are 6 questions on 6 pages. Solve all of them. Duration is 90 minutes.

1-(20 P) Solve the given system of equations by using **matrix reduction.**

$$\begin{cases} 2x + z = -5 \\ y + z = -1 \\ x + 2y + z = 0 \end{cases}$$

<u>Solution of Question 1</u>: Corresponding augmented matrix is $\left[\begin{array}{ccc|c} 2 & 0 & 1 & -5 \\ 0 & 1 & 1 & -1 \\ 1 & 2 & 1 & 0 \end{array}\right]$

$$\left[\begin{array}{ccc|c} 2 & 0 & 1 & -5 \\ 0 & 1 & 1 & -1 \\ 1 & 2 & 1 & 0 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 2 & 0 & 1 & -5 \end{array}\right] \xrightarrow{-2R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & -4 & -1 & -5 \end{array}\right]$$

$$\xrightarrow{4R_2 + R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 3 & -9 \end{array}\right] \xrightarrow{\frac{1}{3}R_3} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -3 \end{array}\right] \xrightarrow[-R_3 + R_1]{-R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

$$\xrightarrow{-2R_2 + R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

Hence, we have the reduced matrix is $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right]$.

The number of equation is the same as the unknowns; therefore it has unique solution as $x = -1, y = 2\ and\ z = -3$.

2-(20 P) Use the matrix reduction to find the inverse of **coefficient matrix**. Solve the system of equations by using the **inverse of its coefficient matrix**.

$$\begin{cases} X_1 - X_2 = 1 \\ X_2 - X_3 = -2 \\ 2X_3 - X_1 = 2 \end{cases}$$

<u>**Solution of Question 2:**</u> The equation can be written in the matrix form **AX**= **B** where

$$\mathbf{A}=\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 2 \end{bmatrix}, \mathbf{X}=\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \text{ and } \mathbf{B}=\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}.$$

The inverse of the coefficient matrix **A** can be found as follows: **For the** inverse of **coefficient matrix**, corresponding augmented matrix is

$$[\mathbf{A} \mid \mathbf{I}]=\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ -1 & 0 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1+R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2+R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right] \xrightarrow{R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 2 & 1 \\ 0 & 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right] = [\mathbf{I} \mid \mathbf{A}^{-1}]$$

where $\mathbf{A}^{-1} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Hence, we may solve the system of equations using the inverse of matrix **A** by

$$\mathbf{X}=\mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

The set of solutions is $\mathbf{X}=\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}=\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

3- (20P) Find the critical points of the following function and classify each critical point as a relative maximum, a relative minimum, or neither. If its extrema exist, then find it.

$$f(x,y) = x^3 + y^3 - 3xy + 4.$$

Solution of Question 3:

We have for critical points $f_x(x,y) = 3x^2 - 3y = 3(x^2 - y) = 0$ and $f_y(x,y) = 3(y^2 - x) = 0$.Then $= x^2$, $x = y^2$ and $x^4 = x$ or $x(x^3 - 1) = 0$ so $x = 0, x = 1$ and $y = 0, y = 1$. Then the critical points are $(0, 0)$and $(1, 1)$. Then

$f_{xx}(x,y) = 6x, f_{yy}(x,y) = 6y, f_{xy}(x,y) = -3$ *and* $D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - {f_{xy}}^2(x,y)$

$$D(x,y) = (6x)(6y) - (-3)^2 = 36xy - 9.$$

At (0,0) we therefore have $D(0,0) = -9 < 0$ so (0,0)is a saddle point. At (1,1) we therefore have $D(1,1) = 36 - 9 = 27 > 0$ and $f_{xx}(1,1) = 6 > 0$ so f must have a local minimum at (1,1). The minimum value of f at (1,1) is $f(1,1) = 3$.

4-(10P) If $q_A = 400 - \ln(2^{p_A}) + 3\,e^{p_A} p_B$ and $q_B = 125 + \ln(3^{p_B}) - 5e^{-p_A} p_B - 2p_B$, where q_A and q_B are the number of units demanded of products A and B, respectively, and p_A and p_B are their respective prices per unit, determine whether A and B are competitive products, complementary products, or neither.

Solution of Question 4: We have the partial derivatives of q_A and q_B with respect to p_B and p_A *are* respectively

$$\frac{\partial q_A}{\partial p_B} = 3\,e^{p_A} > 0$$
$$\frac{\partial q_B}{\partial p_A} = 5e^{-p_A} p_B > 0$$

Therefore, the products A and B are competitive products or substitutes.

5 − (08 P) − (a) If $f(u,v,w)$ is differentiable and $u = x - y, v = y - z \; and \; w = z - x,$ then find $f_x(u,v,w) + f_y(u,v,w) + f_z(u,v,w)$.

Solution of Question 5 − a: $f_x(u,v,w) = \frac{\partial f}{\partial u}\frac{du}{dx} + \frac{\partial f}{\partial v}\frac{dv}{dx} + \frac{\partial f}{\partial w}\frac{dw}{dx} = \frac{\partial f}{\partial u} - \frac{\partial f}{\partial w}$

$$f_y(u,v,w) = \frac{\partial f}{\partial u}\frac{du}{dy} + \frac{\partial f}{\partial v}\frac{dv}{dy} + \frac{\partial f}{\partial w}\frac{dw}{dy} = -\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$$

$$f_z(u,v,w) = \frac{\partial f}{\partial u}\frac{du}{dz} + \frac{\partial f}{\partial v}\frac{dv}{dz} + \frac{\partial f}{\partial w}\frac{dw}{dz} = -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}$$

$$f_x(u,v,w) + f_y(u,v,w) + f_z(u,v,w) = 0.$$

5 − (07 P) − (b) If $w = x^2 + y^2$ and $x = r - s, y = r + s,$ then find $s\frac{\partial w}{\partial r} - r\frac{\partial w}{\partial s}$.

Solution of Question 5 − b: $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{dx}{dr} + \frac{\partial w}{\partial y}\frac{dy}{dr} = 2x + 2y = 2(x+y) = 4r$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{dx}{ds} + \frac{\partial w}{\partial y}\frac{dy}{ds} = -2x + 2y = 2(-x+y) = 4s$$

$$s\frac{\partial w}{\partial r} - r\frac{\partial w}{\partial s} = s(4r) - r(4s) = 0.$$

6 − (07 P) − (a) If $f(x,y) = \ln\sqrt{x^2 + y^2}$, then find $f_{xx}(x,y) + f_{yy}(x,y)$.

Solution of Question 6 − a: $f_x(x,y) = \frac{x}{x^2 + y^2}, f_y(x,y) = \frac{y}{x^2 + y^2}$

$$f_{xx}(x,y) = \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}, f_{yy}(x,y) = \frac{x^2 + y^2 - 2y^2}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$f_{xx}(x,y) + f_{yy}(x,y) = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2} = 0.$$

6 − (08 P) − (b) If $z^3 + \ln(xyz) + \ln(z) - x^2 + e^{z^2} = 0$, then find $\frac{\partial z}{\partial x}$.

Solution of Question 6 − b: $3z^2\frac{\partial z}{\partial x} + \frac{yz + xy\frac{\partial z}{\partial x}}{xyz} + \frac{\frac{\partial z}{\partial x}}{z} - 2x + 2z\frac{\partial z}{\partial x}e^{z^2} = 0$

$$\frac{\partial z}{\partial x}\left[3z^2 + \frac{2}{z} + 2ze^{z^2}\right] = 2x - \frac{1}{x} \text{ or } \frac{\partial z}{\partial x} = \left(\frac{2x - \frac{1}{x}}{3z^2 + \frac{2}{z} + 2ze^{z^2}}\right)$$