MATH 172 PROBLEM SET 1

(A) Evaluate the integrals given below:

a) $\int \sqrt{x^4}\,dx = \int x^2 dx = \frac{x^3}{3} + C$

b) $\int \left(2x^3 - 5x^4 + 2\right)dx = 2\frac{x^4}{4} - \cancel{5}\frac{x^5}{\cancel{5}} + 2x + C = \frac{x^4}{2} - x^5 + 2x + C$

c) $\int \left(4x^4 - 2x^2 + 1\right)dx = \frac{4}{5}x^5 - \frac{2}{3}x^3 + x + C$

d) $\int (x+3)(2x-3)dx = \int \left(2x^2 + 3x - 9\right)dx = \frac{2}{3}x^3 + \frac{3}{2}x^2 - 9x + C$

e) $\int (x^2 + ax)^2 x dx = \int (x^5 + 2ax^4 + a^2x^3)dx = \frac{1}{6}x^6 + \frac{2}{5}ax^5 + \frac{1}{4}a^2x^4 + C$ (a is a constant)

f) $\int (2x^2 - \frac{1}{x^3})dx = \int (2x^2 - x^{-3})dx = \frac{2}{3}x^3 - \frac{x^{-2}}{-2} + C = \frac{2}{3}x^3 + \frac{1}{2x^2} + C$

g) $\int (\frac{1}{4}x^4 - \frac{4}{x^4})dx = \int (\frac{1}{4}x^4 - 4x^{-4})dx = \frac{1}{4}\frac{x^5}{5} - 4\frac{x^{-3}}{-3} + C = \frac{x^5}{20} + \frac{4}{3x^3} + C$

h) $\int \frac{3x^2 - 2x + 1}{6x^5}dx = \frac{1}{6}\int \left(3x^{-3} - 2x^{-4} + x^{-5}\right)dx = \frac{1}{6}\left(3\frac{x^{-2}}{-2} - 2\frac{x^{-3}}{-3} + \frac{x^{-4}}{-4}\right) + C = -\frac{1}{4x^2} + \frac{1}{9x^3} - \frac{1}{24x^4} + C$

i) $\int (\frac{3}{\sqrt{x}} - \frac{x\sqrt{x}}{4})dx = \int (3x^{-1/2} - \frac{1}{4}x^{3/2})dx = 3\frac{x^{1/2}}{1/2} - \frac{1}{4}\frac{x^{5/2}}{5/2} + C = 6x^{1/2} - \frac{1}{10}x^{5/2} + C$

j) $\int (\frac{1}{x^2} + \frac{4}{4\sqrt{x}} + 2)dx = \int (x^{-2} + x^{-1/2} + 2)dx = \frac{x^{-1}}{-1} + \frac{x^{1/2}}{1/2} + 2x + C = -\frac{1}{x} + 2x^{1/2} + 2x + C$

k) $\int (\frac{1}{\sqrt[4]{x}})dx = \int x^{-1/4}dx = \frac{x^{3/4}}{3/4} + C = \frac{4}{3}x^{3/4} + C$

l) $\int (x^2 + \frac{1}{\sqrt[3]{x}})dx = \int (x^2 + x^{-1/3})dx = \frac{x^3}{3} + \frac{3}{2}x^{2/3} + C =$

m) $\int (3x^5 + \frac{3}{\sqrt{x}} - \frac{x\sqrt{x}}{4})dx = \int (3x^5 + 3x^{-1/2} - \frac{1}{4}x^{3/2})dx = \frac{x^6}{2} + 6x^{1/2} - \frac{1}{10}x^{5/2} + C$

n) $\int 2^x 3^{2x} 5^{3x} dx = \int \left(2 \cdot 3^2 5^3\right)^x dx = \frac{1}{\ln(2 \cdot 3^2 5^3)} 2^x 3^{2x} 5^{3x} + C$

o) $\int \left(\sqrt{x} + \frac{1}{\sqrt[3]{x}}\right)dx = \int (x^{1/2} + x^{-1/3})dx = \frac{2}{3}x^{3/2} + \frac{3}{2}x^{2/3} + C =$

(B) Evaluate the integrals given below:

a) $\int \sqrt{2x+3}dx = \int u^{\frac{1}{2}} \frac{du}{2} = \frac{1}{2}\frac{u^{\frac{3}{2}}}{3/2} + C = \frac{1}{3}u^{\frac{3}{2}} + C = \frac{1}{3}(2x+3)^{\frac{3}{2}} + C$ $\quad \begin{cases} u = 2x+3 \\ du = 2dx \end{cases}$

b) $\int \frac{1}{3x-7}dx = \int \frac{1}{u}\frac{du}{3} = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|3x-7| + C$ $\quad \begin{cases} u = 3x-7 \\ du = 3dx \end{cases}$

c) $\int \frac{dx}{(2x-7)^2} = \int \frac{du}{2u^2} = \frac{1}{2}\int u^{-2}du = \frac{1}{2}\frac{u^{-1}}{-1} + C = -\frac{1}{2(2x-7)} + C$ $\quad \begin{cases} u = 2x-7 \\ du = 2dx \end{cases}$

d) $\int \frac{1}{1-x}dx = -\int \frac{1}{u}du = -\ln|u| + C = -\ln|1-x| + C$ $\quad \begin{cases} u = 1-x \\ du = -dx \end{cases}$

e) $\int \frac{1}{5-2x}dx = -\frac{1}{2}\int \frac{1}{u}du = -\frac{1}{2}\ln|u| + C = -\frac{1}{2}\ln|5-2x| + C$ $\quad \begin{cases} u = 5-2x \\ du = -2dx \end{cases}$

f) $\int \frac{xdx}{\sqrt{1-4x^2}} = \int \frac{1}{u^{\frac{1}{2}}}\left(\frac{du}{-8}\right) = -\frac{1}{8}\frac{u^{\frac{1}{2}}}{1/2} + C = -\frac{1}{4}\sqrt{1-4x^2} + C$ $\quad \begin{cases} u = 1-4x^2 \\ du = -8xdx \end{cases}$

g) $\int \frac{xdx}{(3x^2+4)^3} = \int \frac{1}{u^3}\left(\frac{du}{6}\right) = \frac{1}{6}\left(\frac{u^{-3+1}}{-3+1}\right) + C = -\frac{1}{12}(3x^2+4)^{-2} + C$ $\quad \begin{cases} u = 3x^2+4 \\ du = 6xdx \end{cases}$

h) $\int \frac{x^2dx}{\sqrt{x^3+5}} = \int \frac{1}{u^{\frac{1}{2}}}\left(\frac{du}{3}\right) = \frac{1}{3}\left(\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right) + C = \frac{2}{3}(x^3+5)^{\frac{1}{2}} + C$ $\quad \begin{cases} u = x^3+5 \\ du = 3x^2dx \end{cases}$

i) $\int e^{3x}dx = \int e^u \frac{du}{3} = \frac{1}{3}e^u + C = \frac{1}{3}e^{3x} + C$ $\quad \begin{cases} u = 3x \\ du = 3dx \end{cases}$

j) $\int xe^{-x^2}dx = \int e^u \frac{du}{-2} = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{-x^2} + C$ $\quad \begin{cases} u = -x^2 \\ du = -2xdx \end{cases}$

k) $\int 10^{2x}dx = \int e^{2x\ln 10}dx = \int e^u \frac{du}{2\ln 10}$ $\quad \left\{ 10^{2x} = e^{\ln 10^{2x}} = e^{2x\ln 10} \right.$

$= \frac{1}{2\ln 10}e^u + C = \frac{1}{2\ln 10}e^{2x\ln 10} + C = \frac{1}{2\ln 10}10^{2x} + C$ $\quad \begin{cases} u = 2x\ln 10 \\ du = 2\ln 10dx \end{cases}$

l) $\int \sqrt{x^2+1}\, xdx = \int u^{\frac{1}{2}}\left(\frac{du}{2}\right) = \frac{1}{2}\left(\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right) + C = \frac{1}{3}(x^2+1)^{\frac{3}{2}} + C$ $\quad \begin{cases} u = x^2+1 \\ du = 2xdx \end{cases}$

m) $\int \frac{x}{\sqrt{2x^2+3}}dx = \int u^{-\frac{1}{2}}\left(\frac{du}{4}\right) = \frac{1}{4}\left(\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) + C = \frac{1}{2}(2x^2+3)^{\frac{1}{2}} + C$ $\quad \begin{cases} u = 2x^2+3 \\ du = 4xdx \end{cases}$

n) $\int \frac{\ln^2 x}{x}dx = \int u^2du = \frac{u^3}{3} + C = \frac{1}{3}\ln^3 x + C$ $\quad \begin{cases} u = \ln x \\ du = \frac{1}{x}dx \end{cases}$

o) $\int \frac{x}{x^2+1}dx = \int u^{-1}\left(\frac{du}{2}\right) = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+1| + C$ $\quad \begin{cases} u = x^2+1 \\ du = 2xdx \end{cases}$

p) $\int \frac{1}{x\ln x}dx = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C$ $\quad \begin{cases} u = \ln x \\ du = \frac{1}{x}dx \end{cases}$

q) $\int 2x(x^2+1)^4dx = \int u^4du = \frac{1}{5}(x^2+1)^5 + C$ $\quad \begin{cases} u = x^2+1 \\ du = 2xdx \end{cases}$

r) $\int (x^3+1)^2 3x^2dx = \int u^2du = \frac{1}{3}\left(x^3+1\right)^3 + C$ $\begin{cases} u = x^3+1 \\ du = 3x^2dx \end{cases}$

s) $\int 6x\sqrt{1+2x^2}dx = \frac{6}{4}\int u^{\frac{1}{2}}du = \left(1+2x^2\right)^{\frac{3}{2}} + C$ $\begin{cases} u = 1+2x^2 \\ du = 4xdx \end{cases}$

t) $\int \frac{dx}{2x-3} = \int \frac{1}{u}\left(\frac{du}{2}\right) = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|2x-3| + C$ $\begin{cases} u = 2x-3 \\ du = 2dx \end{cases}$

(C) Find the following indefinite integrals.

a) $\int \frac{dx}{\sqrt[4]{x^5}} = \int x^{-\frac{5}{4}}dx = \frac{x^{-\frac{5}{4}+1}}{-\frac{5}{4}+1} + C = -4x^{-\frac{1}{4}} + C = -\frac{4}{\sqrt[4]{x}} + C$

b) $\int x^2(x^4+1)dx = \int (x^6+x^2)du = \frac{x^7}{7} + \frac{x^3}{3} + C$

c) $\int \frac{8}{(2x-1)^3}dx = 8\int \frac{1}{(2x-1)^3}dx \quad = 8\int \frac{1}{u^3}\frac{du}{2} = -2\left(2x-1\right)^{-2} + C$ $\begin{cases} u = 2x-1 \\ du = 2dx \end{cases}$

d) $\int (2x^3+x)(x^4+x^2)dx = \frac{1}{4}\left(x^4+x^2\right)^2 + C$

e) $\int x\sqrt{x}dx = \frac{2\sqrt{x^5}}{5} + C$

f) $\int \frac{e^{-x}}{1+2e^{-x}}dx = -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\ln\left|1+2e^{-x}\right| + C$ $\begin{cases} u = 1+2e^{-x} \\ du = -2e^{-x}dx \end{cases}$

g) $\int e^{\ln(x+1)}dx = \int \left(x+1\right)dx = \frac{x^2}{2} + x + C$

h) $\int \left(x^7 + \frac{1}{x^3} + \frac{1}{x} + e^x\right)dx = \frac{x^8}{8} - \frac{1}{2x^2} + \ln|x| + e^x + C$

i) $\int 2x\sqrt{3-2x^2}dx = 2\int u^{\frac{1}{2}}\frac{du}{-4} = -\frac{1}{3}\left(3-2x^2\right)^{\frac{3}{2}} + C$ $\begin{cases} u = 3-2x^2 \\ du = -4xdx \end{cases}$

j) $\int \frac{1}{x^2}\sqrt{\frac{1}{x}+1}\ dx = \int u^{\frac{1}{2}}(-du) = -\frac{2}{3}\left(\frac{1}{x}+1\right)^{\frac{3}{2}} + C$ $\begin{cases} u = \frac{1}{x}+1 \\ du = -\frac{1}{x^2}dx \end{cases}$

k) $\int \frac{\ln(x+1)}{x+1}dx = \int udu = \frac{1}{2}\left(\ln(x+1)\right)^2 + C$ $\begin{cases} u = \ln(x+1) \\ du = \frac{1}{x+1}dx \end{cases}$

l) $\int xe^{x^2}dx = \int e^u\frac{du}{2} = \frac{1}{2}e^{x^2} + C$ $\begin{cases} u = x^2 \\ du = 2xdx \end{cases}$

m) $\int \frac{2t^2}{3+2t^3}dt = \frac{1}{3}\ln\left|3+2t^3\right| + C$

n) $\int \frac{x+1}{x^2+2x}\ln(x^2+2x)dx = \int u\frac{du}{2} = \frac{1}{2}\frac{u^2}{2} + C = \frac{\left(\ln(x^2+2x)\right)^2}{4} + C$ $\begin{cases} u = \ln(x^2+2x) \\ du = \frac{2(x+1)}{x^2+2x}dx \end{cases}$

(D) Find $y(x)$ subject to the given conditions

a) $y' = 0.5e^{x} - 2x$ and $y(0) = 0.5$

Solution:

$$y(x) = \int y'dx = \int \left(0.5e^{x} - 2x\right) dx = 0.5e^{x} - x^{2} + C$$

Applying the initial condition to find the integration constant C:

$$y(0) = 0.5 = 0.5e^{x} - x^{2} + C\Big|_{x=0} = 0.5e^{0} - 0^{2} + C = 0.5 + C \qquad \Rightarrow \qquad C = 0\,,$$

The solution would be:

$$y = 0.5e^{x} - x^{2}$$

b) $x > 0, \quad y(1) = 0, \quad y'(1) = \frac{1}{4}, \quad y''(x) = x^{-2} + x^{3} + 2$

Solution:

First we have to integrate to find y':

$$y' = \int y''dx = \int \left(x^{-2} + x^{3} + 2\right) dx = -x^{-1} + \frac{1}{4}x^{4} + 2x + C_{1}$$

Applying the initial condition $y'(1) = \frac{1}{4}$ to find the integration constant C_1:

$$y'(1) = \frac{1}{4} = -x^{-1} + \frac{1}{4}x^{4} + 2x + C_{1}\Bigg|_{x=1} = -1 + \frac{1}{4} + 2 + C_{1} \quad \Rightarrow \quad C_{1} = -1$$

Then to integrate y' to find y:

$$y(x) = \int y'dx = \int \left(-x^{-1} + \frac{1}{4}x^{4} + 2x - 1\right) dx = -\ln|x| + \frac{1}{20}x^{5} + x^{2} - x + C_{2}$$

Applying the initial condition $y(1) = 0$ to find the integration constant C_2:

$$y(1) = 0 = -\ln|x| + \frac{1}{20}x^{5} + x^{2} - x + C_{2}\Bigg|_{x=1} = -\ln 1 + \frac{1}{20} + 1 - 1 + C_{2} \qquad \Rightarrow \qquad C_{2} = -\frac{1}{20}$$

Hence the solution:

$$y(x) = -\ln|x| + \frac{1}{20}x^{5} + x^{2} - x - \frac{1}{20}$$

(E) Find demand function if the marginal revenue function is given as,

$$\frac{dr}{dq} = \frac{150}{(q+1)^{2}} \qquad \text{(answer : } p = -\frac{150}{q(q+1)} + \frac{150}{q} = \frac{150}{q+1}\text{)}$$

Solution:

$$r = \int \frac{dr}{dq}dq = \int \frac{150}{(q+1)^{2}}dq = -\frac{150}{q+1} + C$$

When the quantity is zero (no product is sold), the total revenue would be zero:

$$r\big|_{q=0} = 0 = -\frac{150}{0+1} + C \quad \Rightarrow \qquad C = 150$$

The revenue is then given by

$$r = 150\left(1 - \frac{1}{q+1}\right) = 150\frac{q}{q+1}$$

The demand function is

$$r = pq \;\Rightarrow \qquad p = \frac{r}{q} = \frac{150}{q+1}$$

(F) Marginal cost function is given, and fixed cost is 250$/per week. Find the cost function per week

$$\frac{dc}{dq} = 10 - \frac{100}{q+e}$$

Solution:

$$c = \int \frac{dc}{dq} dq = \int \left(10 - \frac{100}{q+e} \right) dq = 10q - 100\ln\left(q+e\right) + C$$

Fixed cost 250$/per week is constant regardless of output so that

$$c\big|_{q=0} = 250 = 10 \times 0 - 100\ln\left(0+e\right) + C \quad \Rightarrow \quad C = 350$$

The cost function per week would be:

$$c = 10q - 100\ln\left(q+e\right) + 350$$