MATH 172 PROBSET 2

1-A) A manufacturer's marginal-revenue function is $\frac{dr}{dq} = 275 - q - 0.3q^2$

If r is in TL, find the increase in the manufacturer's total revenue if production is increased from 10 to 20 units.

Solution:

$$r = \int_{10}^{20} \frac{dr}{dq} dq = \int_{10}^{20} \left(275 - q - 0.3q^2\right) dq = 275q - \frac{q^2}{2} - 0.3\frac{q^3}{3} + C\Big|_{10}^{20} = 1900TL$$

B) Evaluate the definite integrals

a) $\int_0^5 (x + x^2)dx = \frac{x^2}{2} + \frac{x^3}{3}\Big|_0^5 = \frac{25}{2} + \frac{125}{3} = \frac{325}{6}$

b) $\int_2^{10} \frac{dx}{x-1} = \left|\ln|x-1|\right|_2^{10} = \ln 9 - \ln 1 = 2\ln 3$

c) $\int_0^1 \frac{x^2 + x + \sqrt{x+1}}{x+1} dx = \int_0^1 \left(\frac{x^2 + x}{x+1} + \frac{\sqrt{x+1}}{x+1} \right) dx = \int_0^1 \left(x + (x+1)^{-\frac{1}{2}} \right) dx = \frac{x^2}{2} + 2(x+1)^{\frac{1}{2}}\Big|_0^1 = -\frac{3}{2} + 2\sqrt{2}$

d) $\int_0^2 x^2 e^{x^3} dx = \int_0^8 e^u \frac{du}{3} = \frac{1}{3}e^u\Big|_0^8 = \frac{1}{3}\left(e^8 - 1\right)$ $\quad \begin{cases} u = x^3 \\ du = 3x^2 dx \end{cases} \quad \therefore \quad \begin{cases} x = 2 \Rightarrow u = 2^3 \\ x = 0 \Rightarrow u = 0 \end{cases}$

e) $\int_{\sqrt{3}}^2 7x\sqrt{4 - x^2}\, dx = 7\int_1^0 \sqrt{u} \frac{du}{-2} = \frac{7}{2}\int_0^1 \sqrt{u}\, du = \frac{7}{2} \frac{u^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^1 = \frac{7}{3}$ $\quad \begin{cases} u = 4 - x^2 \\ du = -2x dx \end{cases} \quad \begin{cases} x = 2 \Rightarrow u = 0 \\ x = \sqrt{3} \Rightarrow u = 1 \end{cases}$

C) If $\int_1^5 f(x)dx = 6$ and $\int_5^3 f(x)dx = 2$, find $\int_1^3 f(x)dx$.

Solution:

$$\int_1^3 f(x)dx = \int_1^5 f(x)dx - \int_3^5 f(x)dx = \int_1^5 f(x)dx + \int_5^3 f(x)dx = 6 + 2 = 8$$

2) Find the area of the region bounded by the curve, lines and *x*-axis. Sketch the region on the *x-y* plane.

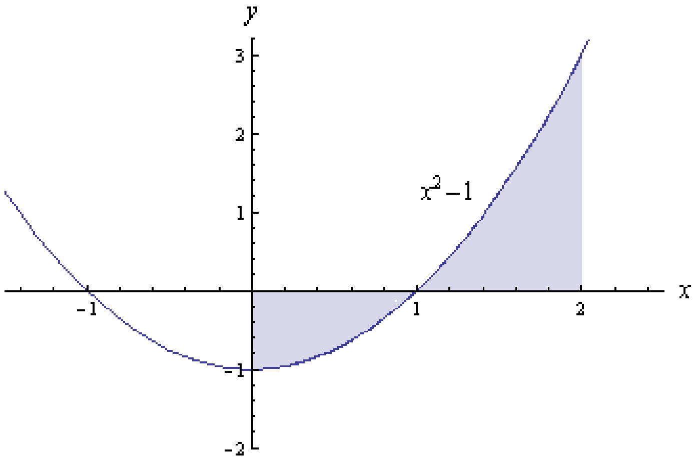
a) $y = x^2 - 1,\ x = 0,\ x = 2$

Solution:

The curve $y = x^2 - 1$ is just a parabola whose intercepts are given by

$x = 0 \quad \Rightarrow \quad y = 0^2 - 1 = -1$

$y = 0 \quad \Rightarrow \quad 0 = x^2 - 1 = (x-1)(x+1) \qquad x = 1 \text{ or } -1$

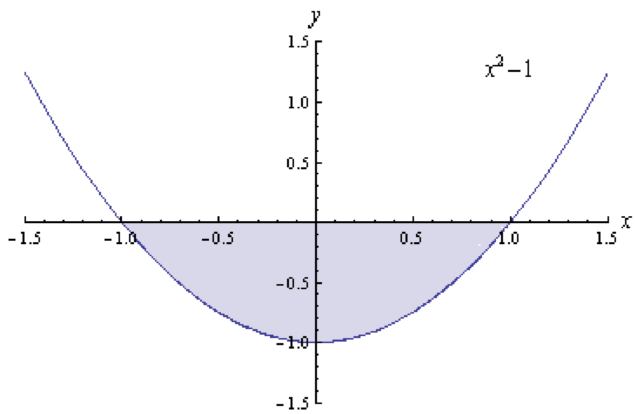


The area can be taken by splitting the integral from 0 to 1 and from 1 to 2 because the area from 0 to 1 is under the x-axis and the negative so we have to multiply by "-" sign to make it positive.

$$Area=-\int_0^1 \left(x^2-1\right)dx+\int_1^2 \left(x^2-1\right)dx=\left(-\frac{x^3}{3}+x\right)\Bigg|_0^1+\left(\frac{x^3}{3}-x\right)\Bigg|_1^2=-\frac{1}{3}+1+\frac{8}{3}-2-\frac{1}{3}+1=2$$

b) $y=x^2-1,\ y=0$

Solution:



The area can be taken from -1 to 1, but it is under the x-axis and the negative so we have to multiply by "-" sign to make it positive.

$$Area=-\int_{-1}^1 \left(x^2-1\right)dx=\left(-\frac{x^3}{3}+x\right)\Bigg|_{-1}^1=-\frac{1}{3}+1-\frac{1}{3}+1=\frac{4}{3}$$

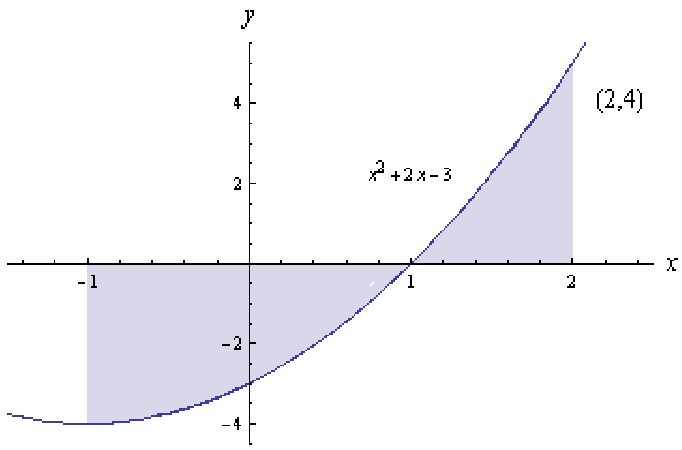
c) $y=x^2+2x-3,\ x=-1,\ x=2$

Solution:

The curve $y=x^2+2x-3$ is just a parabola whose intercepts are given by

$x=0 \quad \Rightarrow \quad y=-3$

$y=0 \quad \Rightarrow \quad 0=x^2+2x-3=(x+3)(x-1) \quad \Rightarrow \quad x=-3 \text{ or } 1$



The area can be taken by splitting the integral from -1 to 1 and from 1 to 2 because the area from -1 to 1 is under the x-axis and the negative so we have to multiply by "–" sign to make it positive.

$$Area = -\int_{-1}^{1}\left(x^2+2x-3\right)dx+\int_{1}^{2}\left(x^2+2x-3\right)dx = \underbrace{\left(-\frac{x^3}{3}-x^2+3x\right)\Big|_{-1}^{1}}_{-\frac{1}{3}-1+3-\frac{1}{3}+1+3} + \underbrace{\left(\frac{x^3}{3}+x^2-3x\right)\Big|_{1}^{2}}_{\frac{8}{3}+4-6-\frac{1}{3}-1+3}$$

$$=\frac{16}{3}+\frac{7}{3}=\frac{23}{3}$$

d) Evaluate $\int_{-1}^{2}(x^2+2x-3)dx$ and compare the results with the results of c

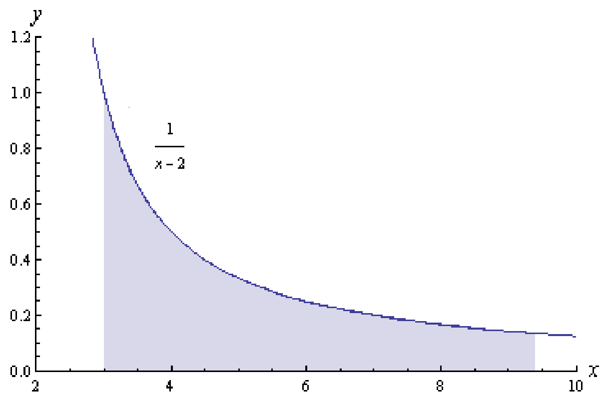
Solution:

$$\int_{-1}^{2}\left(x^2+2x-3\right)dx = \underbrace{\left(\frac{x^3}{3}+x^2-3x\right)\Big|_{-1}^{2}}_{\frac{8}{3}+4-6+\frac{1}{3}-1-3} = -3$$

We obtained the negative result which cannot be area from -1 to 2. Actually in this case, negative part from -1 to 1 is kept as negative so the area below the x-axis is subtracted from the area above x-axis from 1 to 2 such as $-\frac{16}{3}+\frac{7}{3}=-3$.

e) $y=\frac{1}{x-2}$, $x=3$, $x=e^2+2$

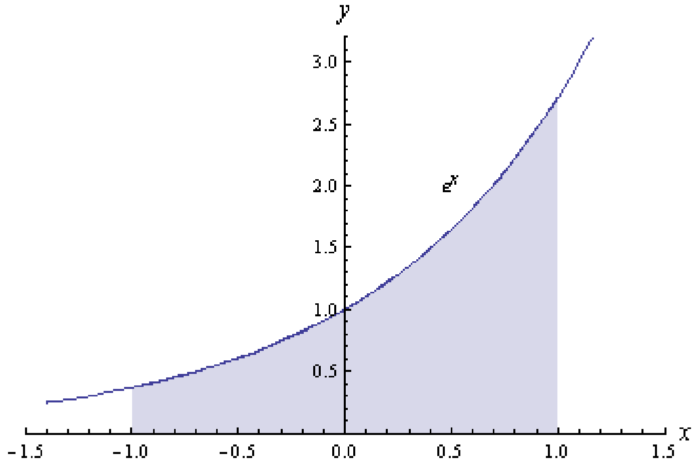
Solution:



$$Area=\int_{3}^{e^2+2}\frac{1}{x-2}dx=\ln(x-2)\Big|_{3}^{e^2+2}=\ln e^2-\ln 1=2$$

f) $y = e^x$, $x = -1$, $x = 1$

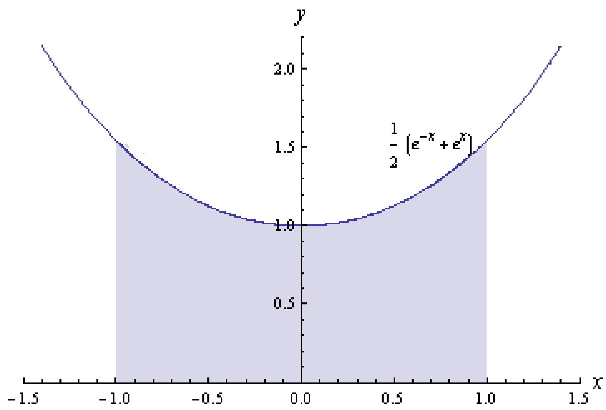
Solution:



$$Area = \int_{-1}^{1} e^x dx = e^x \Big|_{-1}^{1} = e - e^{-1}$$

g) $y = \frac{1}{2}(e^x + e^{-x})$, $x = -1$, $x = 1$

Solution:



$$Area = \frac{1}{2}\int_{-1}^{1}\left(e^x + e^{-x}\right)dx = \frac{1}{2}\left(e^x - e^{-x}\right)\Big|_{-1}^{1} = \frac{1}{2}\left(e - e^{-1} - e^{-1} + e\right) = e - e^{-1}$$

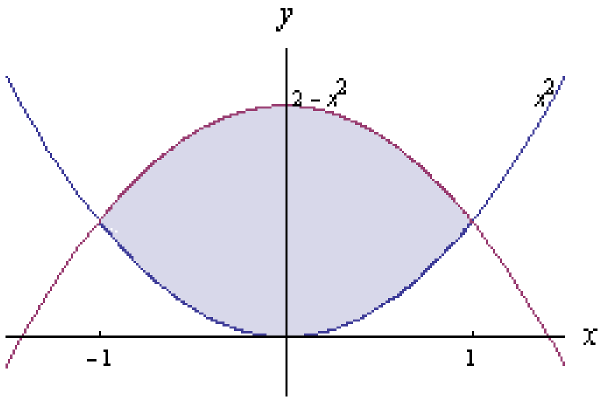
3) Find the area of the region bounded by the given curves and lines. Sketch the region on the x-y plane.

a) $y = x^2$, $y = -x^2 + 2$

Solution:

The intercepts of the two plots are:

$$x^2 = -x^2 + 2 \quad \Rightarrow \quad x^2 = 1 \quad \Rightarrow \quad x = \pm 1$$



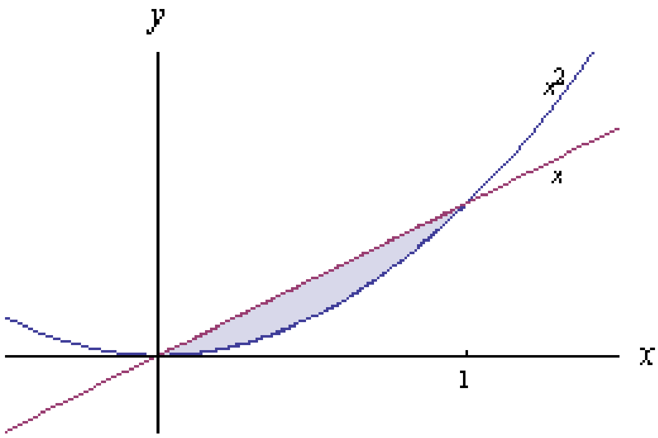
$$Area = \int_{-1}^{1} \left(2 - x^2 - x^2\right) dx = 2x - 2\frac{x^3}{3}\Big|_{-1}^{1} = e - e^{-1}$$

b) $y = x^2,\ y = x$

Solution:

The intercepts of the two plots are:

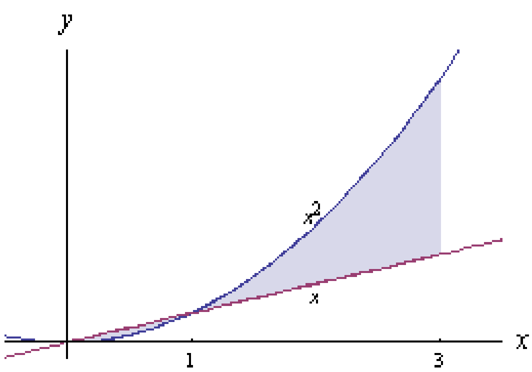
$x^2 = x \quad \Rightarrow \quad x(x-1) = 0 \quad \Rightarrow x = 0 \text{ or } 1$



$$Area = \int_{0}^{1} \left(x - x^2\right) dx = \frac{x^2}{2} - \frac{x^3}{3}\Big|_{0}^{1} = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

c) $y = x^2,\ y = x,\ x = 3$

Solution:



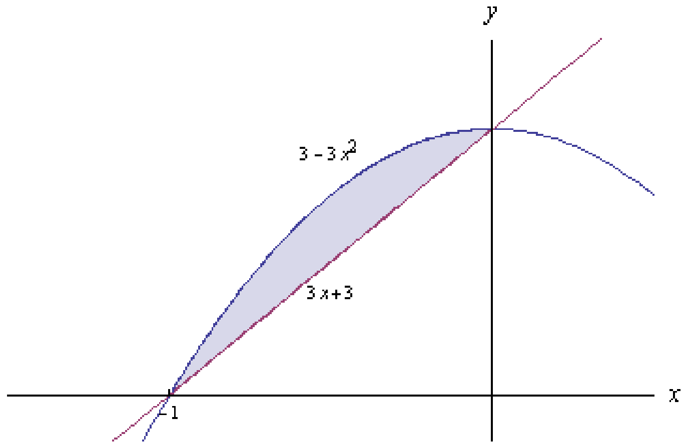
$$Area = \int_0^1 \left(x - x^2\right)dx + \int_1^3 \left(x^2 - x\right)dx = \underbrace{\left.\frac{x^2}{2} - \frac{x^3}{3}\right|_0^1}_{\frac{1}{2}-\frac{1}{3}} + \underbrace{\left.\frac{x^3}{3} - \frac{x^2}{2}\right|_1^3}_{\frac{3^3}{3}-\frac{3^2}{2}+\frac{1}{2}-\frac{1}{3}} = \frac{1}{6} + \frac{14}{3} = \frac{29}{6}$$

d) $y = -3x^2 + 3,\ y = 3x + 3$

Solution:

The intercepts of the two plots are:

$-3x^2 + \cancel{3} = 3x + \cancel{3} \quad \Rightarrow \quad 3x(x+1) = 0 \quad \Rightarrow \quad x = 0 \text{ or -1}$



$$Area = \int_{-1}^{0} \left(-3x^2 + \cancel{3} - 3x - \cancel{3}\right)dx = -\cancel{3}\frac{x^3}{\cancel{3}} - 3\left.\frac{x^2}{2}\right|_{-1}^{0} = -1 + \frac{3}{2} = \frac{1}{2}$$

e) $y = -3x^2 + 3,\ y = 3x + 3,\ y = 1$

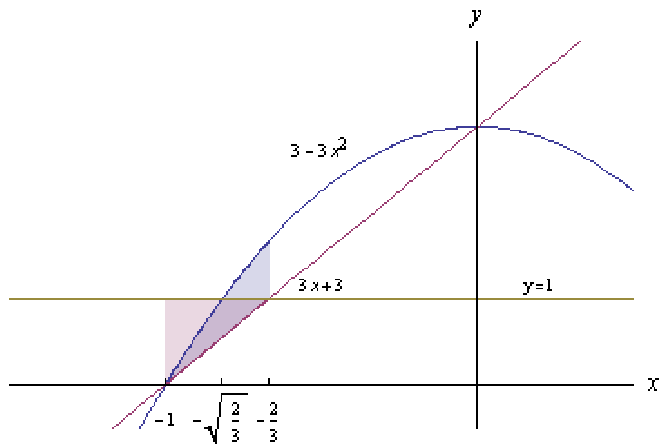
Solution:

The intercepts of the line $y = 1$ with $y = -3x^2 + 3$ *and* $y = 3x + 3$ are:

$y = -3x^2 + 3 = 1 \quad \Rightarrow \quad x = \pm\sqrt{\frac{2}{3}}$

$y = 3x + 3 = 1 \quad \Rightarrow \quad x = -\frac{2}{3}$

The darker area on the plot is the area to be found.



The area can be calculated from x = -1 to $x=-\sqrt{2/3}$ and $x=-\sqrt{2/3}$ to x = -2/3. So

$$Area=\int_{-1}^{-\sqrt{2/3}}\left(-3x^2+\not{3}-3x-\not{3}\right)dx+\int_{-\sqrt{2/3}}^{-2/3}\left(1-3x-3\right)dx$$

$$=\left(-\not{3}\frac{x^3}{\not{3}}-3\frac{x^2}{2}\right)\Bigg|_{-1}^{-\sqrt{2/3}}+\left(-3\frac{x^2}{2}-2x\right)\Bigg|_{-\sqrt{2/3}}^{-2/3}=0.05+2.63=2.68$$

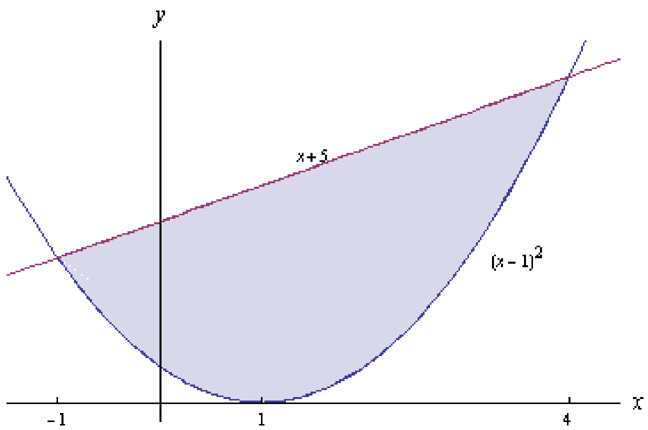
4) Express the area of the region bounded by the given curves and lines in terms of definite integral or integrals.

a) $y=(x-1)^2,\ y=x+5$

Solution:

The intercepts of the two plots are:

$$y=(x-1)^2=x+5\quad\Rightarrow\quad(x+1)(x-4)=0\qquad\Rightarrow\quad x=-1\text{ or }4$$



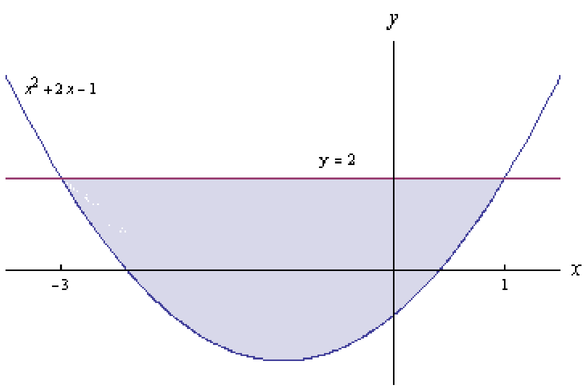
$$Area=\int_{-1}^{4}\left(x+5-(x-1)^2\right)dx=\int_{-1}^{4}\left(-x^2+3x+4\right)dx=\frac{43}{2}$$

b) $y=x^2+2x-1,\ y=2$

Solution:

The intercepts of the two plots are:

$$y=x^2+2x-1=2\quad\Rightarrow\quad(x+3)(x-1)=0\qquad\Rightarrow\quad x=-3\text{ or }1$$



$$Area=\int_{-3}^{1}\left(2-x^2-2x+1\right)dx=\int_{-3}^{1}\left(-x^2-2x+3\right)dx=-\frac{x^3}{3}-x^2+3x\Bigg|_{-3}^{1}=\frac{32}{3}$$

5) Profit of a company is a function of units sold (q) and is given by the following function: $f(q) = 4 - q - \frac{1}{q}$, q changes between 10 and 30. Evaluate the average profit per unit using the integral;

$$I = \frac{1}{20}\int_{10}^{30} f(q)dq \qquad \text{answer: } -16 - \frac{\ln 3}{20}$$

Solution:

$$I = \frac{1}{20}\int_{10}^{30} f(q)dq = \frac{1}{20}\int_{10}^{30}\left(4 - q - \frac{1}{q}\right)dq = \frac{1}{20}\left(4q - \frac{q^2}{2} - \ln q\right)\Bigg|_{10}^{30} = -16 - \frac{\ln 3}{20}$$

6) Demand and supply equations are given respectively. Determine consumer and producer surpluses under market equilibrium.

a) $p = 100 - q^2,\ p = 2q + 20$

Solution:

Equilibrium point is

$$p = 100 - q^2 = 2q + 20 \Rightarrow q^2 + 2q - 80 = 0 = (q+10)(q-8) \Rightarrow \cancel{q = -10} \quad or \quad q_0 = 8$$

$$p_0 = 2q_0 + 20\Big|_{q_0 = 8} = 36$$

$$CS = \int_0^{q_0}\left(f(q) - p_0\right)dq = \int_0^8\left(100 - q^2 - 36\right)dq = 64q - \frac{q^3}{3}\Bigg|_0^8 = \frac{1024}{3}$$

$$PS = \int_0^{q_0}\left(p_0 - g(q)\right)dq = \int_0^8\left(36 - 2q - 20\right)dq = 16q - q^2\Big|_0^8 = 64$$

b) $p = 1500 - q^2,\ p = 700 + q^2$

Solution:

Equilibrium point is

$$p = 1500 - q^2 = 700 + q^2 \Rightarrow 2q^2 = 800 \Rightarrow q_0 = \sqrt{400} = 20$$

$$p_0 = 700 + q_0^2\Big|_{q_0 = 20} = 1100$$

$$CS = \int_0^{q_0}\left(f(q) - p_0\right)dq = \int_0^{20}\left(1500 - q^2 - 1100\right)dq = 400q - \frac{q^3}{3}\Bigg|_0^{20} = \frac{16000}{3}$$

$$PS = \int_0^{q_0}\left(p_0 - g(q)\right)dq = \int_0^{20}\left(1100 - 700 - q^2\right)dq = 400q - \frac{q^3}{3}\Bigg|_0^{20} = \frac{16000}{3}$$

7) Marginal cost function of a product is given; a) Determine the marginal cost when 90 units are produced, b) If fixed cost is $500, find the total cost of producing 90 units.

$$\frac{dc}{dq} = 10 - \frac{100}{q+10}$$

Solution:

$$c = \int \frac{dc}{dq} dq = \int \left(10 - \frac{100}{q+10} \right) dq = 10q - 100\ln(q+10) + C$$

When q = 0, $c = 500 = 10q - 100\ln(q+10) + C\Big|_{q=0} = -100\ln 10 + C \quad \Rightarrow \quad C = 500 + 100\ln 10$

Hence the cost function is given by

$$c = 10q - 100\ln(q+10) + 500 + 100\ln 10 = 10q - 100\ln(\frac{q}{10}+1) + 500$$

To find the total cost of producing 90 units:

$$c = 10q - 100\ln(\frac{q}{10}+1) + 500\Big|_{q=90} = 900 - 100\ln 10 + 500 = 1400 - 100\ln 10 \approx \$1170$$

8) The demand equation for a product is $p = 0.01q^2 - 1.1q + 30$ and the supply equation is $p = 0.01q^2 + 8$. Determine consumers' surplus and producers' surplus when market equilibrium has been established.

Solution:

Equilibrium point is $p = 0.01q^2 - 1.1q + 30 = 0.01q^2 + 8 \quad \Rightarrow \quad q_0 = \frac{22}{1.1} = 20$

$$p_0 = 0.01q^2 + 8\Big|_{q_0=20} = 12$$

$$CS = \int_0^{q_0} \left(f(q) - p_0 \right) dq = \int_0^{20} \left(0.01q^2 - 1.1q + 30 - 12 \right) dq = 0.01\frac{q^3}{3} - 1.1\frac{q^2}{2} + 18q \Big|_0^{20} = \frac{500}{3}$$

$$PS = \int_0^{q_0} \left(p_0 - g(q) \right) dq = \int_0^{20} \left(12 - 0.01q^2 - 8 \right) dq = 4q - 0.01\frac{q^3}{3} \Big|_0^{20} = \frac{160}{3}$$