PROBLEM SET 3 – SOLUTIONS

1–a) $\int \frac{x+1}{e^x}\,dx \qquad u = x+1 \quad du = dx \quad dv = e^{-x}dx \quad v = -e^{-x}$

$$\int \frac{x+1}{e^x}\,dx = -\frac{x+1}{e^x} - \int -e^{-x}dx = -\frac{x+1}{e^x} + \int e^{-x}dx = -\frac{x+2}{e^x} + C$$

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b) $\int_e^3 \sqrt[3]{x}\,\ln(x^5)\,dx = 5\int_e^3 x^{1/3}\ln x\,dx$

$u = \ln x \quad du = dx/x \quad dv = x^{1/3}dx \quad v = \frac{3}{4}x^{4/3}$

$$5\left\{\left[\ln x\,\frac{3}{4}x^{4/3} - \int \frac{3}{4}x^{4/3}\,\frac{1}{x}\,dx\right]\Big|_e^3\right\} = 5\left\{\left[\frac{3\ln x}{4}x^{4/3} - \frac{4}{3}\int x^{1/3}dx\right]\Big|_e^3\right\}$$

$$= 5\left\{\left(\frac{3\ln x}{4}x^{4/3} - \frac{9}{16}x^{4/3}\right)\Big|_e^3\right\} = 5\left\{\frac{3}{4}3^{4/3}\left(\ln 3 - \frac{3}{4}\right) - \frac{3}{4}e^{4/3}\left(\frac{1}{4}\right)\right\}$$

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c) $\int (x - e^{-x})^2dx = \int (x^2 - 2xe^{-x} + e^{-2x})\,dx = \frac{x^3}{3} - \frac{e^{-2x}}{2} - 2\int xe^{-x}dx$

$\int xe^{-x}dx \Rightarrow u = x \quad du = dx \quad dv = e^{-x}dx \quad v = -e^{-x}$

$$\int xe^{-x}dx = -xe^{-x} - \int -e^{-x}dx = -xe^{-x} - \int e^{-x}(-dx)$$

$$= -xe^{-x} - e^{-x} + C = -e^{-x}(x+1) + C$$

$$\Rightarrow \int (x-e^{-x})^2dx = \frac{x^3}{3} - \frac{e^{-2x}}{2} + 2e^{-x}(x+1) + C$$

1–d) $\int 2(2x-1)\ln(x-1)\,dx$

$u = 2\ln(x-1) \quad du = \frac{2}{x-1}dx \quad dv = (2x-1)dx \quad v = x^2 - x = x(x-1)$

$$\int 2(2x-1)\ln(x-1)dx = 2x(x-1)\ln(x-1) - \int x(x-1)\frac{2}{x-1}dx$$

$$= 2x(x-1)\ln(x-1) - \int 2x\,dx = 2x(x-1)\ln(x-1) - x^2 + C$$

1-e) $\displaystyle\int \frac{\ln(x+1)}{\sqrt{x+1}}\,dx$

$u = \ln(x+1) \quad du = \dfrac{dx}{x+1} \qquad dv = (x+1)^{-1/2}\,dx \qquad v = 2(x+1)^{1/2}$

$$\int \frac{\ln(x+1)}{\sqrt{x+1}}\,dx = 2(x+1)^{1/2}\ln(x+1) - 2\int (x+1)^{-1/2}\,dx$$

$$= 2(x+1)^{1/2}\ln(x+1) - 4(x+1)^{1/2} + C = 2\sqrt{x+1}\left[\ln(x+1) - 2\right] + C$$

1-f) $\displaystyle\int (\ln x)^3\,dx = \int \ln^3 x\,dx$

$u = (\ln x)^3 \quad du = \dfrac{3(\ln x)^2}{x}\,dx \qquad dv = dx \qquad v = x$

$\displaystyle\int (\ln x)^3\,dx = x(\ln x)^3 - 3\int (\ln x)^2\,dx$ (use integration parts again)

$u = (\ln x)^2 \quad du = \dfrac{2\ln x}{x}\,dx \qquad dv = dx \qquad v = x$

$\displaystyle\int (\ln x)^3\,dx = x(\ln x)^3 - 3\left[x(\ln x)^2 - 2\int \ln x\,dx\right]$ (again)

$u = \ln x \quad du = \dfrac{dx}{x} \qquad dv = dx \qquad v = x \qquad \displaystyle\int \ln x\,dx = x[\ln x - 1]$

$\displaystyle\int (\ln x)^3\,dx = x(\ln x)^3 - 3\left[x(\ln x)^2 - 2x(\ln x - 1)\right] + C$ or

$$= x\left(\ln^3 x - 3\ln^2 x + 6\ln x - 6\right) + C$$

2-a) $\displaystyle\int \frac{14x^3 + 24x}{(x^2+1)(x^2+2)}\,dx \qquad \frac{14x^3+24x}{(x^2+1)(x^2+2)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+2}$

$$14x^3 + 24x = (x^2+2)(Ax+B) + (x^2+1)(Cx+D)$$

$$= (A+C)x^3 + (B+D)x^2 + (2A+C)x + (2B+D)$$

$$\left.\begin{array}{l} A+C=14 \\ B+D=0 \\ 2A+C=24 \\ 2B+D=0 \end{array}\right\} \left.\begin{array}{l} A=10 \\ B=0 \\ C=4 \\ D=0 \end{array}\right\} \quad \int \frac{14x^3+24x}{(x^2+1)(x^2+2)}\,dx = \int\left(\frac{10x}{x^2+1} + \frac{4x}{x^2+2}\right)dx$$

$$= 5\int \frac{2\,dx}{x^2+1} + 2\int \frac{2\,dx}{x^2+2} = 5\ln(x^2+1) + 2\ln(x^2+2) + C$$

or $\quad = \ln\left[(x^2+1)^5 (x^2+2)^2\right] + C$

2-b) $\int \frac{2x^2-5x-2}{(x-2)^2(x-1)}\,dx$ $\qquad \frac{2x^2-5x-2}{(x-2)^2(x-1)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$

$2x^2-5x-2 = A(x-2)^2 + B(x-1)(x-2) + C(x-1)$

if $x=1$ then $-5=A$
if $x=2$ then $-4=C$
if $x=0$ then $-2 = 4A+2B-C = -20+2B+4 \Rightarrow B=7$

$\int\left(\frac{-5}{x-1} + \frac{7}{x-2} + \frac{-4}{(x-2)^2}\right)dx = -5\ln|x-1| + 7\ln|x-2| + \frac{4}{x-2} + c$

2-c) $\int \frac{5x^4+9x^2+3}{x(x^2+1)}\,dx$

$\frac{5x^4+9x^2+3}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$

$5x^4+9x^2+3 = A(x^2+1)^2 + (Bx+C)\,x(x^2+1) + (Dx+E)\,x$

$= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A$

$A=3,\ B=2,\ C=0,\ D=1,\ E=0$

$\int \frac{5x^4+9x^2+3}{x(x^2+1)^2}\,dx = \int\left(\frac{3}{x} + \frac{2x}{x^2+1} + \frac{x}{(x^2+1)^2}\right)dx$

$= 3\ln|x| + \ln|x^2+1| - \frac{1}{2(x^2+1)} + c$

2-d) $\int \frac{5x^2+2}{x^3+x}\,dx$ $\qquad \frac{5x^2+2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

$5x^2+2 = A(x^2+1) + (Bx+C)x$

$5x^2+2 = (A+B)x^2 + Cx + A \Rightarrow A+B=5,\ C=0,\ A=2,\ B=3$

$\int \frac{5x^2+2}{x^3+x}\,dx = \int\left(\frac{2}{x} + \frac{3x}{x^2+1}\right)dx = 2\ln|x| + \frac{3}{2}\ln(x^2+1) + c$

3- $\text{Area} = \int_0^1 x^2 e^x \, dx \qquad u = x^2 \quad du = 2x\,dx$

$$dv = e^x dx \quad v = e^x$$

$$\int x^2 e^x dx = x^2 e^x - 2\int x e^x \, dx$$

Similarly $\int x e^x dx = x e^x - \int e^x dx = e^x(x-1)$

$$\Rightarrow \int_0^1 x^2 e^x dx = \left\{ x^2 e^x - 2\left[e^x(x-1)\right]\right\}\Big|_0^1 = \left[e^x(x^2-2x+2)\right]\Big|_0^1$$

$= e-2$ square units.

4- $q_1 = 100 \qquad q_2 = 500 \qquad C = 4000 + 10q + 0.1q^2$

$$\bar{C} = \frac{1}{500-100}\int_{100}^{500} (4000 + 10q + 0.1q^2)\, dq$$

$$= \frac{1}{400}\left(4000q + 5q^2 + \frac{0.1q^3}{3}\right)\Bigg|_{100}^{500} \approx \$\, 17.333,33$$

5- $\bar{f} = \frac{1}{1-0}\int_0^1 t^2 e^{-t} dt = \int_0^1 t^2 e^{-t} dt$

$$\left.\begin{matrix} u = t^2 & du = 2t\,dt \\ dv = e^{-t}dt & v = -e^{-t} \end{matrix}\right\} \int t^2 e^{-t} dt = t^2(-e^{-t}) - \int -e^{-t}(2t\,dt)$$

$$= -e^{-t}t^2 + 2\int t e^{-t} dt \qquad \left(\int t e^{-t} dt = -e^{-t}(t+1)\right)$$

$$\int_0^1 t^2 e^{-t} dt = -e^{-t}(t^2 + 2t + 2)\Big|_0^1 = 2 - \frac{5}{e}$$

6-) $p = \dfrac{200(q+3)}{q^2+7q+6}$ (demand) $CS = ?$ for $\left(10, \dfrac{325}{22}\right)$
(q_0, P_0)

$$CS = \int_0^{10} \left[\frac{200(q+3)}{q^2+7q+6} - \frac{325}{22}\right] dq$$

$$\frac{200(q+3)}{q^2+7q+6} = \frac{200(q+3)}{(q+6)(q+1)} = \frac{A}{q+6} + \frac{B}{q+1}$$

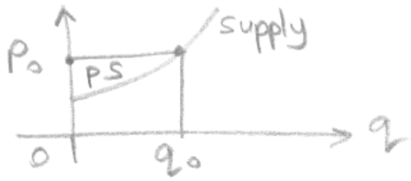
$$200(q+3) = A(q+1) + B(q+6)$$

if $q=-1$ and $q=-6$ $\Rightarrow$ $A=120$, $B=80$

$$CS = \int_0^{10} \left(\frac{120}{q+6} + \frac{80}{q+1} - \frac{325}{22}\right) dq$$

$$= 120\ln(16) + 80\ln(11) - \frac{3250}{22} - 120\ln(6) \approx \$161.80$$

7-) $P = 10(q+10)e^{(0.1q+1)}$



$q = 20$ $\quad P = 10(30)e^3 = 300e^3$

$(q_0, P_0) = (20, 300e^3)$

$$PS = \int_0^{q_0} (P_0 - P)\, dq = \int_0^{20} \left[300e^3 - 10(q+10)e^{(0.1q+1)}\right] dq$$

$$PS = 300e^3 \int_0^{20} dq - 10 \underbrace{\int_0^{20} (q+10)e^{0.1q+1}}_{I} dq$$

$\int_0^{20} dq \rightarrow q\Big|_0^{20} = 20$

For I, $u = q+10$ $\quad du = dq$, $\quad dv = e^{0.1q+1} dq$, $\quad v = 10e^{0.1q+1}$

$$I = \left[(q+10)\,10e^{0.1q+1}\right]_0^{20} - \int_0^{20} 10e^{0.1q+1} dq = 300e^3 - 100e - 10\left(10e^{0.1q+1}\right)\Big|_0^{20}$$

$$= 300e^3 - 100e - 100e^3 + 100e = 200e^3$$

$$PS = 300e^3(20) - 10(200e^3) = 4.000e^3$$

8-) a) $\int 2^{3q+4}\,dq = \int e^{\ln 2\,(3q+4)}\,dq$

$u = \ln 2\,(3q+4) \qquad du = \ln 2(3)\,dq$

$\int 2^{3q+4}\,dq = \int e^{u} \frac{1}{3\ln 2}\,du = \frac{e^{\ln 2(3q+4)}}{3\ln 2} + c = \frac{2^{3q+4}}{3\ln 2} + c$

b) $\int \frac{\ln(xe^{2x})}{x}\,dx = \int \frac{\ln x + \ln e^{2x}}{x}\,dx = \int \frac{\ln x + 2x}{x}\,dx$

$u = \ln x \qquad du = \frac{1}{x}\,dx \qquad \int u\,du + 2\int dx = \frac{u^2}{2} + 2x + c$

$= \frac{\ln^2 x}{2} + 2x + c$

c) $\int_5^{11} \frac{e^{\ln x}}{x}\,dx = \int \frac{x}{x}\,dx = \int dx = x \Big|_5^{11} = 11 - 5 = 6$

d) $\int \ln(4x)\,dx \qquad u = \ln(4x) \quad du = \frac{4}{4x}\,dx = \frac{dx}{x}$

$dv = dx \qquad v = x$

$\int \ln(4x)\,dx = x\ln(4x) - \int x \frac{1}{x}\,dx = x\ln(4x) - \int dx$

$= x\left[\ln(4x) - 1\right] + c$

8-e) $\int \frac{xe^x}{(x+1)^2}\,dx \qquad u = xe^x \quad du = e^x(x+1)\,dx$

$dv = (x+1)^{-2}\,dx \qquad v = -(x+1)^{-1}$

$v = -\frac{1}{x+1}$

$\int \frac{xe^x}{(x+1)^2}\,dx = -\frac{xe^x}{x+1} - \int -\frac{e^x(x+1)}{x+1}\,dx$

$= -\frac{xe^x}{x+1} + \int e^x\,dx = -\frac{xe^x}{x+1} + e^x + c = \frac{e^x}{x+1} + c$

8-f) $\int x^3 e^{x^2} dx$

$u = x^2 \qquad du = 2x\,dx \qquad dv = xe^{x^2} dx \qquad v = \int xe^{x^2} dx$

$t = x^2 \qquad dt = 2x\,dx \qquad v = \int e^t \frac{dt}{2} = \frac{1}{2} e^t = \frac{1}{2} e^{x^2}$

$\int x^3 e^{x^2} dx = x^2\left(\frac{1}{2} e^{x^2}\right) - \int \frac{1}{2} e^{x^2} (2x\,dx) = \frac{1}{2} x^2 e^{x^2} - \int x e^{x^2} dx$

$= \frac{1}{2} x^2 e^{x^2} - \frac{1}{2} e^{x^2} + C = \frac{1}{2} e^{x^2}(x^2 - 1) + C$

9-a) $\int_{-1}^{3} \left[(2x+3) - (x^2)\right] dx = \int_{-1}^{3} (-x^2 + 2x + 3)\,dx$

$= \left(-\frac{x^3}{3} + x^2 + 3x\right)\Big|_{-1}^{3} = \left[-\frac{3^3}{3} + 3^2 + 3(3)\right] - \left[-\frac{-1}{3} + (-1)^2 + 3(-1)\right]$

$= (-9 + 9 + 9) - \left(\frac{1}{3} + 1 - 3\right) = 9 + \frac{5}{3} = \frac{32}{3}$

9-b) Intersections

$x^3 - 2x^2 + 3 = x + 1$

$x^3 - 2x^2 + 3 - x - 1 = 0$

$x^2(x-2) - (x-2) = 0 \Rightarrow (x-2)(x^2-1) = 0 \qquad x = 2, \pm 1$

From the figure $x = 1, 2$ will be used.

$\int_0^1 (x^3 - 2x^2 - x + 2)\,dx + \int_1^2 (-x^3 + 2x^2 + x - 2)\,dx$

$\left(\frac{x^4}{4} - \frac{2x^3}{3} - \frac{x^2}{2} + 2x\right)\Big|_0^1 + \left(-\frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} - 2x\right)\Big|_1^2$

$= \left(\frac{1}{4} - \frac{2}{3} - \frac{1}{2} + 2\right) + \left(-\frac{2^4}{4} + \frac{2(2)^3}{3} + \frac{2^2}{2} - 2(2)\right) -$

$\left(-\frac{1}{4} + \frac{2}{3} + \frac{1}{2} - 2\right) = \frac{3}{2}$