MATH 172 FINAL EXAM (04.08.2011) A

Name:

Q1	Q2	Q3	Q4	Total 1

ATTENTION: There are 4 questions on 4 pages. Solve all of them. Duration is 90 minutes. Simply giving a final result is not sufficient to answer any question, so show all the steps you pursued to get any final result. Otherwise your answer will not be evaluated as a correct answer.

1. a) **A company's production function is given by** $P = 2LK + \dfrac{2000}{\sqrt{LK}}$ **, where** ***P*** **is the total output generated by** ***L*** **units of labor and** ***K*** **units of capital. Determine the marginal production function with respect to** ***L*** **when** ***K*** **= 40 units and** ***L*** **= 10 units. (10 Points)**

Solution:

$$P = 2LK + 2000\left(LK\right)^{-1/2} \quad \Rightarrow \quad \frac{\partial P}{\partial L} = 2K - 1000K\left(LK\right)^{-3/2} = 2K - \frac{1000K}{\left(LK\right)^{3/2}} = 2K - \frac{1000}{L\sqrt{LK}}$$

$$\left.\frac{\partial P}{\partial L}\right|_{\substack{L=10\\ k=40}} = 2(40) - \frac{1000}{10\sqrt{(40)(10)}} = 80 - 5 = 75$$

b) Evaluate the integral $\int \frac{1-x}{e^x}\,dx$. **(10 Points)**

Solution:

$$\int \frac{1-x}{e^x}dx = -\left(1-x\right)e^{-x} - \int -e^{-x}(-dx) = -\left(1-x\right)e^{-x} + e^{-x} + C \qquad \begin{cases} u = 1-x & \Rightarrow \quad du = -dx \\ dv = e^{-x}dx & \Rightarrow \quad v = -e^{-x} \end{cases}$$

$$= xe^{-x} + C$$

2.a) Let $q_A = 100 - 3p_A + 4p_B^{1/2}$ and $q_B = 20\, p_A / p_B$ be demand functions, where p_A and p_B are prices for products A and B, respectively. Determine: (*i*) the marginal demand for A with respect to p_B , (*ii*) the marginal demand for B with respect to p_A , (*iii*) whether A and B are competitive, complementary, or neither. (10 Points)

<u>Solution:</u>

(***i***) $\dfrac{\partial q_A}{\partial p_B} = 2p_B^{-1/2} > 0$ (***ii***) $\dfrac{\partial q_B}{\partial p_A} = 20p_B^{-1} > 0$

(***iii***) From (*i*) and (*ii*), the products *A* and *B* are competitive products.

b) Examine the function $f(x, y) = x^2 + y^3 - xy + 7$ for relative extrema using the second derivative test. (15 Points)

<u>Solution:</u>

The first derivatives of *f*(*x*, *y*) function:

$$f_x = \frac{\partial f}{\partial x} = 2x - y = 0 \qquad \Rightarrow \qquad y = 2x$$

$$f_y = \frac{\partial f}{\partial y} = 3y^2 - x = 12x^2 - x = x(12x - 1) = 0 \quad \Rightarrow \quad x = 0 \quad \text{or} \quad x = \tfrac{1}{12}$$

The critical points are at (0,0) and (1/12, 1/6). The second derivatives are as follows

$$f_{xx} = 2, \qquad f_{xy} = -1, \qquad f_{yy} = 6y$$

The function *D*(*x*, *y*) for second-derivative test is given by

$$D(x, y) = f_{xx} f_{yy} - f_{xy}^2 = 12y - 1$$

<u>The (0, 0) point:</u> $D(0,0) = -1 < 0$, a saddle point

<u>The (1/12, 1/6) point:</u> $D(\tfrac{1}{12}, \tfrac{1}{6}) = 12y - 1 = 12\dfrac{1}{6} - 1 = 1 > 0$, $f_{xx} = 2 > 0$ a relative minimum

3.a) If $\ln\sqrt{x(z+1)}-3y=2x$ **, then evaluate** $\dfrac{\partial z}{\partial x}$ **using the implicit differentiation. (10 Points)**

Solution:

$$\ln\sqrt{x(z+1)}-3y=2x$$

$$\frac{1}{2}\ln x+\frac{1}{2}\ln(z+1)-3y=2x \qquad \Rightarrow \qquad \ln x+\ln(z+1)-6y=4x$$

$$\frac{\partial}{\partial x}\left(\ln x+\ln(z+1)-6y=4x\right)$$

$$\frac{\partial}{\partial x}\ln x+\frac{\partial}{\partial x}\ln(z+1)-6\frac{\partial y}{\partial x}=4\frac{\partial x}{\partial x}$$

$$\frac{1}{x}+\frac{1}{z+1}\frac{\partial z}{\partial x}-0=4 \qquad \Rightarrow \qquad \frac{\partial z}{\partial x}=\left(z+1\right)\left(4-\frac{1}{x}\right)$$

b) If $w=ye^{x-y}$ **where** $x=rs,\ \ y=s^2+r^2$ **, evaluate** $\dfrac{\partial w}{\partial r}$ **when** $r=1$ **and** $s=-1$**.** **(10 Points)**

Solution:

$$\frac{\partial w}{\partial r}=\underbrace{\frac{\partial w}{\partial x}}_{ye^{x-y}}\underbrace{\frac{\partial x}{\partial r}}_{s}+\underbrace{\frac{\partial w}{\partial y}}_{(1-y)e^{x-y}}\underbrace{\frac{\partial y}{\partial r}}_{2r}$$

$$=sye^{x-y}+2r\left(1-y\right)e^{x-y}=\left(sy+2r\left(1-y\right)\right)e^{x-y}$$

when $r=1$ **and** $s=-1$**,** $x=-1$ **and** $y=2$**.**

$$\left.\frac{\partial w}{\partial r}\right|_{\substack{r=1\\ s=-1\\ x=-1\\ y=2}}=\left(\left(sy+2r\left(1-y\right)\right)e^{x-y}\right)_{\substack{r=1\\ s=-1\\ x=-1\\ y=2}}$$

$$=\left(-2+2\left(1-2\right)\right)e^{-1-2}=-4e^{-3}$$

4.a) By using matrix reduction, solve the given system and determine whether it has unique solution or infinitely many solutions. (15 Points)

$$2y + x = 0$$
$$3x - 6z = 0$$
$$-y + x = 0$$

Solution:

$$\begin{bmatrix} 1 & 2 & 0 \\ 3 & 0 & -6 \\ 1 & -1 & 0 \end{bmatrix} \xrightarrow[-R_1+R_2]{-3R_1+R_2} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -6 & -6 \\ 0 & -3 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{6}R_3} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & -3 & 0 \end{bmatrix}$$

$$\xrightarrow[3R_2+R_3]{-2R_2+R_1} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{\frac{1}{3}R_3} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[-R_3+R_2]{2R_3+R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The number of equation is the same as the unknowns; therefore it has unique solutions as
$x = 0, \quad y = 0, \quad z = 0$.

4.b) Solve the given system by using the inverse of its coefficient matrix. (20 Points)

$$x - 2y = -3$$
$$2x - y = 6$$

Solution:

The equation can be written in matrix form as:

$$\mathbf{AX = B} \quad \text{with} \quad \mathbf{A} = \begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}, \qquad \mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -3 \\ 6 \end{bmatrix}$$

The inverse of the coefficient matrix **A** can be found as follows:

$$[\mathbf{A}|\mathbf{I}] = \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{array}\right] \xrightarrow{-2R_1+R_2} \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 3 & -2 & 1 \end{array}\right] \xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{cc|cc} 1 & -2 & 1 & 0 \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} \end{array}\right]$$

$$\xrightarrow{2R_2+R_1} \left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} \end{array}\right] = [\mathbf{I}|\mathbf{A}^{-1}]$$

Hence we may solve the system of equations using the inverse of **A** matrix by

$$\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} -3 \\ 6 \end{bmatrix} = \begin{bmatrix} 1+4 \\ 2+2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$$

The solution set is {5,4}.