**MATH 172 FINAL EXAM (29.12.2009)**

**Name:**
**Instructor:**
**Section:**

| Q1 | | Q2 | Q3 | Q4 | | Q5 | Q6 | Q7 | Total |
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**ATTENTION: There are 7 questions on 6 pages. Solve all of them. Duration is 100 minutes. Simply giving a final result is not sufficient to answer any question, so show all the steps you pursued to get any final result. Otherwise your answer will not be evaluated as a correct answer.**

**1.** Suppose the demand equations for the related products *A* and *B* are

$$q_A = e^{-p_A - p_B} \text{ and } q_B = \frac{16}{p_A^2 p_B^2}$$

where $q_A$ and $q_B$ are the number of units of *A* and *B* demanded when the unit prices (in thousands of TL) are $p_A$ and $p_B$, respectively.

a) Classify *A* and *B* as competitive, complementary, or neither.
b) If the unit prices of *A* and *B* are 1000TL and 2000TL, respectively, estimate the change in the demand for *B* when the price of *A* is decreased by 250TL and the price of *B* is held constant.(10 Points)

**Solution:**

**a)** $$\frac{\partial q_A}{\partial p_B} = -e^{-p_A - p_B} < 0 \quad \text{and} \quad \frac{\partial q_B}{\partial p_A} = -\frac{32}{\underbrace{p_A^3 p_B^2}_{>0}} < 0$$

*A* and *B* are complementary products

**b)** $$\left.\frac{\partial q_B}{\partial p_A}\right|_{\substack{p_A=1\\ p_B=2}} = -\left.\frac{32}{p_A^3 p_B^2}\right|_{\substack{p_A=1\\ p_B=2}} = -\frac{32}{4} = -8$$

where it means that when the price of $p_A$ is increased by one unit (1000TL), then the demand for *B* is reduced by 8 units. So the change in the demand for B when the price of A is decreased 250 TL (1/4 reduction) is just increased 2 units.

**(b)** The profit (in *TL*) of a business is given by

$$P = P(q) = 369q - 2.1q^2 - 400$$

where *q* is the number of units of the product sold. Find the average profit on the interval from *q* = 0 to *q* = 100. (10 Points)

**Solution:**

$$\overline{P} = \frac{1}{b-a}\int_a^b P\,dq = \frac{1}{100-0}\int_0^{100}\left(400q - 2.1q^2 - 400\right)dq$$

$$= \frac{1}{100}\left(400\frac{q^2}{2} - 2.1\frac{q^3}{3} - 400q\right)_0^{100} = \frac{1}{100}\left(200q^2 - 0.7q^3 - 400q\right)_0^{100}$$

$$= \frac{1}{100}\left(2.000.000 - 700.000 - 40.000\right) = 12.600TL$$

**2.** Take the indefinite integral of $\int 2xe^{-3x}dx$ . (10 Points)

**Solution:**

$$\int 2xe^{-3x}dx = -\frac{2}{3}xe^{-3x} - \int \frac{e^{-3x}}{-3}2dx \qquad \Rightarrow \qquad \begin{cases} 2x = u & \Rightarrow \quad 2dx = du \\ e^{-3x}dx = dv & \Rightarrow \quad \dfrac{e^{-3x}}{-3} = v \end{cases}$$

$$= -\frac{2}{3}xe^{-3x} + \frac{2}{3}\frac{e^{-3x}}{-3} + C = -\frac{2}{3}xe^{-3x} - \frac{2}{9}e^{-3x} + C = -\frac{2}{3}e^{-3x}\left(x + \frac{1}{3}\right) + C$$

**3.** Compute $(BA^T)^T$ matrix if the $A$ and $B$ matrix are given below: (10 Points)

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

**Solution:**

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \qquad \Rightarrow \qquad A^T = \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$BA^T = \begin{pmatrix} 0 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}_{3x3} \begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{pmatrix}_{3x2} = \begin{pmatrix} 0 & -1 \\ 3 & -1 \\ 0 & 2 \end{pmatrix}_{3x2}$$

$$(BA^T)^T = \begin{pmatrix} 0 & 3 & 0 \\ -1 & -1 & 2 \end{pmatrix}_{2x3}$$

**4. (a)** If $2z^2 = 2x + yz$, find $\dfrac{\partial^2 z}{\partial x \partial y}$. (10 Points)

**Solution:**

$$\frac{\partial}{\partial x}\left(2z^2 = 2x + yz\right) \;\Rightarrow\; 4z\frac{\partial z}{\partial x} = 2 + y\frac{\partial z}{\partial x} \;\Rightarrow\; \frac{\partial z}{\partial x} = \frac{2}{4z - y}$$

$$\frac{\partial}{\partial y}\left(2z^2 = 2x + yz\right) \;\Rightarrow\; 4z\frac{\partial z}{\partial y} = z + y\frac{\partial z}{\partial y} \;\Rightarrow\; \frac{\partial z}{\partial y} = \frac{z}{4z - y}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{2}{4z - y}\right) = \frac{-2\left(4\dfrac{\partial z}{\partial y} - 1\right)}{\left(4z - y\right)^2} = \frac{-2\left(\dfrac{4z}{4z - y} - 1\right)}{\left(4z - y\right)^2} = \frac{-2y}{4z - y}$$

**(b)** If $u = e^{xyz}$ where $x = s^3$, $y = \ln\left(r - s\right)$, and $z = \sqrt{rs^2}$ evaluate $\dfrac{\partial u}{\partial r}$ when $r$ = 2 and $s$ = 1. (10 Points)

**Solution:**

$$\frac{\partial u}{\partial r} = \underbrace{\frac{\partial u}{\partial x}}_{yze^{xyz}}\underbrace{\frac{\partial x}{\partial r}}_{0} + \underbrace{\frac{\partial u}{\partial y}}_{xze^{xyz}}\underbrace{\frac{\partial y}{\partial r}}_{\frac{1}{r-s}} + \underbrace{\frac{\partial u}{\partial z}}_{xye^{xyz}}\underbrace{\frac{\partial z}{\partial r}}_{\frac{s}{2\sqrt{r}}} = \left(\frac{z}{r - s} + \frac{ys}{2\sqrt{r}}\right)xe^{xyz}$$

$$\left.\frac{\partial u}{\partial r}\right|_{\substack{r=2\\ s=1}} = \left(\frac{\sqrt{2}}{2 - 1} + \frac{0 \times 1}{2\sqrt{2}}\right)e^0 = \sqrt{2} \qquad \begin{matrix} r = 2 \\ s = 1 \end{matrix} \;\Rightarrow\; \begin{cases} x = 1 \\ y = 0 \\ z = \sqrt{2} \end{cases}$$

**5.** Examine the function $f(x,y)=\ln(xy)+2x^3-xy-6x$ for relative extrema using the second derivative test. (15 Points)

**Solution:**
The first derivatives of $f(x,y)$ function:

$$f_y=\frac{\partial f}{\partial y}=\frac{1}{y}-x=0 \qquad \Rightarrow \qquad y=\frac{1}{x}$$

$$f_x=\frac{\partial f}{\partial x}=\frac{1}{x}+6x^2-y-6=6x^2-6=0 \Rightarrow \qquad x=\pm1 \quad \& \quad y=\pm1$$

From the first partial derivatives, the critical points are (1, 1) and (-1, -1).

The second derivatives are as follows

$$f_{xx}=\frac{\partial^2 f}{\partial x^2}=-\frac{1}{x^2}+12x, \qquad f_{yy}=\frac{\partial^2 f}{\partial y^2}=-\frac{1}{y^2}, \qquad f_{xy}=\frac{\partial^2 f}{\partial x \partial y}=-1$$

The function $D(x,y)$ for the second-derivative test is given by

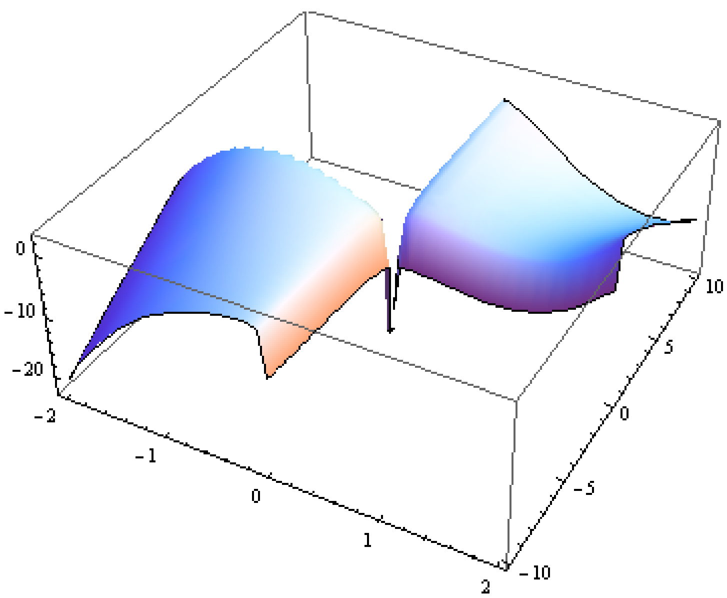
$$D(x,y)=f_{xx}f_{yy}-f_{xy}^2=\left(-\frac{1}{x^2}+12x\right)\left(-\frac{1}{y^2}\right)-(-1)^2=-\frac{1}{y^2}\left(-\frac{1}{x^2}+12x\right)-1$$

The extrama at (1, 1) and (-1, -1) critical points are as follows:

The (1, 1) point: $D(1,1)=-(-1+12)-1=-12<0$, a saddle point

The (-1, -1) point: $D(-1,-1)=-(-1+12(-1))-1=12>0$,

$f_{xx}(-1,-1)=-1+12(-1)=-13<0$ so a relative maximum



**6.** By using matrix reduction, solve the given system and determine whether it has unique solution or infinitely many solutions. (15 Points)

$$3x - 4y = 0$$
$$x + 5y = 0$$
$$4x - y = 0$$

**<u>Solution:</u>**

$$\begin{bmatrix} 3 & -4 \\ 1 & 5 \\ 4 & -1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & 5 \\ 3 & -4 \\ 4 & -1 \end{bmatrix} \xrightarrow[-4R_1+R_3]{-3R_1+R_2} \begin{bmatrix} 1 & 5 \\ 0 & -19 \\ 0 & -21 \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{19}R_2} \begin{bmatrix} 1 & 5 \\ 0 & 1 \\ 0 & -21 \end{bmatrix} \xrightarrow[21R_2+R_3]{-5R_2+R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

The number of equation is the same as the unknowns (*x*, *y*); therefore it has unique (trivial) solutions:

$x = 0 \quad \& \quad y = 0$.

**7. Solve the given system by using the inverse of its coefficient matrix. (15 Points)**

$$x+y+z=2$$
$$x-y+z=-2$$
$$x-y-z=0$$

**Solution:**

The equation can be written in matrix form as:

$$\mathbf{AX = B} \quad \text{with} \quad \mathbf{A}=\begin{bmatrix}1 & 1 & 1\\ 1 & -1 & 1\\ 1 & -1 & -1\end{bmatrix}, \quad \mathbf{X}=\begin{bmatrix}x\\ y\\ z\end{bmatrix}, \quad \mathbf{B}=\begin{bmatrix}2\\ -2\\ 0\end{bmatrix}$$

The inverse of the coefficient matrix **A** can be found as follows:

$$\left[\mathbf{A}|\mathbf{I}\right]=\left[\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 1 & -1 & 1 & 0 & 1 & 0\\ 1 & -1 & -1 & 0 & 0 & 1\end{array}\right]\xrightarrow[-R_1+R_3]{-R_1+R_2}\left[\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 0 & -2 & 0 & -1 & 1 & 0\\ 0 & -2 & -2 & -1 & 0 & 1\end{array}\right]\xrightarrow{-\frac{1}{2}R_2}\left[\begin{array}{ccc|ccc}1 & 1 & 1 & 1 & 0 & 0\\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0\\ 0 & -2 & -2 & -1 & 0 & 1\end{array}\right]$$

$$\xrightarrow[2R_2+R_3]{-R_2+R_1}\left[\begin{array}{ccc|ccc}1 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0\\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0\\ 0 & 0 & -2 & 0 & -1 & 1\end{array}\right]\xrightarrow{-\frac{1}{2}R_3}\left[\begin{array}{ccc|ccc}1 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & 0\\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0\\ 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2}\end{array}\right]$$

$$\xrightarrow{-R_3+R_1}\left[\begin{array}{ccc|ccc}1 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2}\\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & 0\\ 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2}\end{array}\right]=\left[\mathbf{I}|\mathbf{A}^{-1}\right]$$

Hence we may solve the system of equations using the inverse of **A** by

$$\mathbf{X}=\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\mathbf{A}^{-1}\mathbf{B}=\begin{bmatrix}\frac{1}{2} & 0 & \frac{1}{2}\\ \frac{1}{2} & -\frac{1}{2} & 0\\ 0 & \frac{1}{2} & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}2\\ -2\\ 0\end{bmatrix}=\begin{bmatrix}1\\ 2\\ -1\end{bmatrix}$$

The solution of the equation system is (1, 2, -1).