MATH 172 A FINAL EXAM (26.05.2009)

Name:
Instructor:
Section:

Q1		Q2	Q3	Q4		Q5	Q6	Q7	Total

ATTENTION: There are 7 questions on 6 pages. Solve all of them. Duration is 100 minutes. Simply giving a final result is not sufficient to answer any question, so show all the steps you pursued to get any final result. Otherwise your answer will not be evaluated as a correct answer.

1. (a) The demand function for margarine is $q_M = 1000 - 50p_M + 2p_B$ and the demand function for butter is $q_B = 500 + 4p_M - 20p_B$ where q_M and q_B are the quantities demanded for the margarine and the butter respectively, and p_M and p_B are their respective prices per unit. Determine (***i***) the marginal demand for margarine with respect to p_B, (***ii***) the marginal demand for butter with respect to p_M, (***iii***) whether the margarine and butter products are competitive, complimentary or neither. (10 Points)

<u>Solution:</u>

(***i***) $\dfrac{\partial q_M}{\partial p_B} = 2 > 0$ (***ii***) $\dfrac{\partial q_B}{\partial p_M} = 4 > 0$

(***iii***) From (*i*) and (*ii*), the margarine and butter are competitive products.

(b) A manufacturer's marginal-cost function is

$$\frac{dc}{dq} = \frac{1000}{\sqrt{3q+70}}$$

If *c* is in TL, determine the cost involved to increase production from 10 to 33 units. (10 Points)

<u>Solution:</u>

$$c = \int_{10}^{33} \frac{dc}{dq}dq = \int_{10}^{33} \frac{1000}{\sqrt{3q+70}}dq = 1000\int_{10}^{33} \frac{1}{\sqrt{u}}\frac{du}{3} \quad \Rightarrow \begin{cases} u = 3q+70 \\ du = 3dq \end{cases} \quad \Rightarrow \begin{cases} q = 33 \to u = 169 \\ q = 10 \to u = 100 \end{cases}$$

$$= \frac{1000}{3}\frac{u^{1/2}}{1/2}\bigg|_{100}^{169} = \frac{2000}{3}\underbrace{\left(\sqrt{169} - \sqrt{100}\right)}_{=3} = 2000TL$$

2. Take the indefinite integral of $\int 9x^2 \ln x dx$. (10 Points)

Solution:

$$\int 9x^2 \ln x dx = 3x^3 \ln x - \int 3x^3 \frac{1}{x} dx \quad \Rightarrow \quad \begin{cases} \ln x = u & \Rightarrow \quad \frac{1}{x} dx = du \\ 9x^2 dx = dv & \Rightarrow \quad 9\frac{x^3}{3} = 3x^3 = v \end{cases}$$

$$= 3x^3 \ln x - 3\frac{x^3}{3} + C = 3x^3 \ln x - x^3 + C$$

3. Solve for *x* and *y* of the matrix equation given below: (10 Points)

$$\begin{bmatrix} 1 & x \\ 2 & y \end{bmatrix} \begin{bmatrix} 2 & 1 \\ x & 3 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 3 & y \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 1 & x \\ 2 & y \end{bmatrix} \begin{bmatrix} 2 & 1 \\ x & 3 \end{bmatrix} = \begin{bmatrix} 2+x^2 & 1+3x \\ 4+xy & 2+3y \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 3 & y \end{bmatrix}$$

$$\begin{aligned} &2+x^2 = 3 \\ &1+3x = 4 \quad \Rightarrow \quad x = 1 \\ &4+xy = 3 \\ &2+3y = y \quad \Rightarrow \quad y = -1 \end{aligned}$$

4. (a) Let $x^2+2xy-2z^2+xz+2=0$. Use implicit differentiation to evaluate $\frac{\partial z}{\partial x}$ when x = 1, y = -1 and z = 3. (10 Points)

Solution:

$$\frac{\partial}{\partial x}\left(x^2+2xy-2z^2+xz+2\right)=2x+2y-4z\frac{\partial z}{\partial x}+z+x\frac{\partial z}{\partial x}=0$$

Collecting and arranging $\frac{\partial z}{\partial x}$

$$\frac{\partial z}{\partial x}\left(4z-x\right)=2x+2y+z \qquad \Rightarrow \qquad \frac{\partial z}{\partial x}=\frac{2x+2y+z}{4z-x}$$

$$\left.\frac{\partial z}{\partial x}\right|_{\substack{x=1\\y=-1\\z=3}}=\frac{2-2+3}{4\times3-1}=\frac{3}{11}$$

(b) If $z=\ln\frac{x}{y}+e^y-xy$ where $x=s^2$ and $y=r+s$, evaluate $\frac{\partial z}{\partial r}$ when r = 0 and s = -1. (10 Points)

Solution:

$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\underbrace{\frac{\partial x}{\partial r}}_{=0}+\underbrace{\frac{\partial z}{\partial y}}_{-\frac{1}{y}+e^y-x}\underbrace{\frac{\partial y}{\partial r}}_{=1}=-\frac{1}{y}+e^y-x$$

$$\left.\frac{\partial z}{\partial r}\right|_{\substack{r=0\\s=-1}}=-\frac{1}{-1}+e^{-1}-1=e^{-1} \qquad \begin{matrix} r=0 \\ s=-1 \end{matrix} \Rightarrow \begin{cases} x=1 \\ y=-1 \end{cases}$$

5. Examine the function $f(\omega, z) = 2\omega^3 + 2z^3 - 6\omega z + 7$ for relative extrema using the second derivative test. (15 Points)

<u>Solution:</u>

The first derivatives of $f(\omega, z)$ function:

$$f_\omega = \frac{\partial f}{\partial \omega} = 6\omega^2 - 6z = 6\left(\omega^2 - z\right) = 0 \qquad \Rightarrow \quad \omega^2 = z$$

$$f_z = \frac{\partial f}{\partial z} = 6z^2 - 6\omega = 6\left(z^2 - \omega\right) = 0 \qquad \Rightarrow \quad \left(\omega^2\right)^2 - \omega = \underbrace{\omega\left(\omega^3 - 1\right) = 0}_{\omega=0 \text{ or } \omega=1}$$

From the first partial derivatives, the solutions for ω are $\omega = 0$ or $\omega = 1$, and the corresponding solutions for z are $z = 0$ or $z = 1$, respectively. The critical points are (0,0) and (1,1) points.

The second derivatives are as follows

$$f_{\omega\omega} = \frac{\partial^2 f}{\partial \omega^2} = 12\omega, \qquad f_{zz} = \frac{\partial^2 f}{\partial z^2} = 12z, \qquad f_{\omega z} = \frac{\partial^2 f}{\partial \omega \partial z} = -6$$

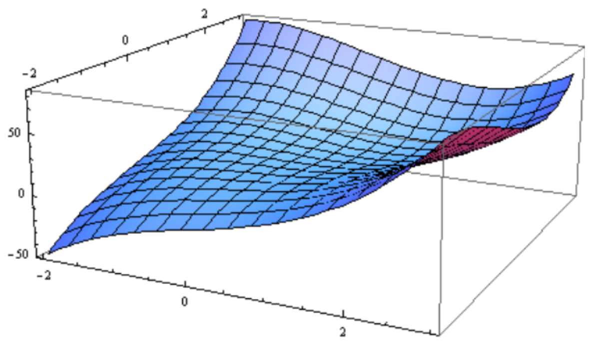
The function $D(\omega, z)$ for the second-derivative test is given by

$$D(\omega, z) = f_{\omega\omega} f_{zz} - f_{\omega z}^2 = 144\omega z - (-6)^2 = 144\omega z - 36$$

The extrema at (0, 0), (1,1) critical points are as follows:

<u>The (0, 0) point:</u> $D(0,0) = -36 < 0$, a saddle point

<u>The (1, 1) point:</u> $D(1,1) = 144 - 36 = 108 > 0$, $f_{\omega\omega}(1,1) = 12 > 0$ so a relative minimum



6. By using matrix reduction, solve the given system and determine whether it has unique solution or infinitely many solutions. (15 Points)

$$\begin{aligned} x+y+7z&=0\\ x-y-z&=0\\ 2x-3y-6z&=0\\ 3x+y+13z&=0 \end{aligned}$$

<u>Solution:</u>

$$\begin{bmatrix} 1 & 1 & 7\\ 1 & -1 & -1\\ 2 & -3 & -6\\ 3 & 1 & 1 \end{bmatrix} 3 \xrightarrow[\substack{-2R_1+R_3\\ -3R_1+R_4}]{-R_1+R_2} \begin{bmatrix} 1 & 1 & 7\\ 0 & -2 & -8\\ 0 & -5 & -20\\ 0 & -2 & -8 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_2} \begin{bmatrix} 1 & 1 & 7\\ 0 & 1 & 4\\ 0 & -5 & -20\\ 0 & -2 & -8 \end{bmatrix}$$

$$\xrightarrow[\substack{5R_2+R_3\\ 2R_2+R_4}]{-R_2+R_1} \begin{bmatrix} 1 & 0 & 3\\ 0 & 1 & 4\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix}$$

The number of equation is less than the unknowns (x, y, z); therefore it has infinitely many solutions as

$$\begin{aligned} x+3z=0 \quad &\Rightarrow \quad x=-3r\\ y+4z=0 \quad &\Rightarrow \quad y=-4r\\ & \quad\quad\;\; z=r \end{aligned}$$

where r is any real number.

7. Solve the given system by using the inverse of its coefficient matrix. (15 Points)

$$\begin{aligned} 3x + y + 4z &= 1 \\ x + z &= 0 \\ 2y + z &= 0 \end{aligned}$$

<u>Solution:</u>

The equation can be written in matrix form as:

$$\mathbf{AX = B} \quad \text{with} \quad \mathbf{A} = \begin{bmatrix} 3 & 1 & 4 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The inverse of the coefficient matrix **A** can be found as follows:

$$[\mathbf{A}|\mathbf{I}] = \left[\begin{array}{ccc|ccc} 3 & 1 & 4 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & 1 & 4 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-3R_1 + R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -3 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{-2R_2 + R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -3 & 0 \\ 0 & 0 & -1 & -2 & 6 & 1 \end{array}\right] \xrightarrow{-R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & -3 & 0 \\ 0 & 0 & 1 & 2 & -6 & -1 \end{array}\right]$$

$$\xrightarrow[-R_3 + R_2]{-R_3 + R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 7 & 1 \\ 0 & 1 & 0 & -1 & 3 & 1 \\ 0 & 0 & 1 & 2 & -6 & -1 \end{array}\right] = [\mathbf{I}|\mathbf{A}^{-1}]$$

Hence we may solve the system of equations using the inverse of **A** by

$$\mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{A}^{-1}\mathbf{B} = \begin{bmatrix} -2 & 7 & 1 \\ -1 & 3 & 1 \\ 2 & -6 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

The solution of the equation system is (-2, -1, 2).