# MATH 172 MIDTERM EXAM (29.03.2011)

| Q1 | | | Q2 | | | Q3 | Q4 | | Total |
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**Name:** **No:**

**ATTENTION: Duration is 75 minutes to solve 4 questions on 4 pages.. Simply giving a final result is not sufficient to answer any question, so show all the steps you pursued to get any final result. Otherwise your answer will not be evaluated as a correct answer.**

**1. Evaluate the following integrals. (10 points each)**

**(a)** $\int (e^x - e^{-x})^2 \, dx$

**Solution:**

$$\int (e^x - e^{-x})^2 dx = \int (e^{2x} + e^{-2x} - 2) dx = \frac{1}{2} e^{2x} - \frac{1}{2} e^{-2x} - 2x + C$$

**(b)** $\int \frac{x}{x^2 - 2} \ln(x^2 - 2) dx$

**Solution:**

$$\frac{1}{2} \int \frac{2x}{x^2 - 2} \ln(x^2 - 2) dx = \frac{1}{2} \int u du = \frac{1}{2} \frac{u^2}{2} + C = \frac{1}{4} \ln^2(x^2 - 2) + C \qquad \begin{cases} u = \ln(x^2 - 2) \\ du = \dfrac{2x}{x^2 - 2} dx \end{cases}$$

**(c)** $\int_1^e \left( \frac{x^2 + x}{x^2} \right) dx$

**Solution:**

$$\int_1^e \left( \frac{x^2 + x}{x^2} \right) dx = \int_1^e \left( 1 + \frac{1}{x} \right) dx = \left( x + \ln|x| \right) \Big|_1^e = e + \underbrace{\ln e}_{=1} - 1 - \underbrace{\ln 1}_{=0} = e$$

**2. Find the following integrals (10 points each)**

**(a)** $\int \frac{\ln x}{x^2} dx$

**Solution:**

$$\int \frac{\ln x}{x^2} dx = -\frac{1}{x}\ln x - \int -\frac{1}{x}\frac{1}{x} dx \begin{cases} u = \ln x & \Rightarrow & du = \frac{1}{x} dx \\ dv = \frac{dx}{x^2} & \Rightarrow & v = -\frac{1}{x} \end{cases}$$

$$= -\frac{1}{x}\ln x + \int x^{-2} dx$$

$$= -\frac{1}{x}\ln x - \frac{1}{x} + C = -\frac{1}{x}\left(\ln x + 1\right) + C$$

**(c)** $\int \frac{x+1}{x^3 - x} dx$

**Solution:**
First we have to express the rational function in terms of partial fraction:

$$\frac{x+1}{x^3 - x} = \frac{\cancel{x+1}}{x\cancel{(x+1)}(x-1)} = \frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} = \frac{A(x-1) + Bx}{x(x-1)}$$

$$1 = A(x-1) + Bx$$

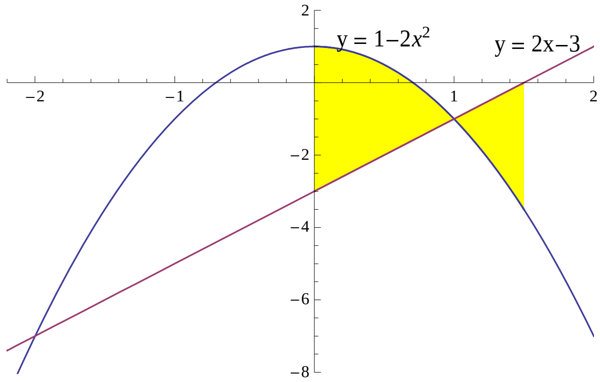
$$x = 0 \quad \Rightarrow \quad A = -1 \quad \therefore \quad x = 1 \quad \Rightarrow \quad B = 1$$

$$\int \frac{x+1}{x^3 - x} dx = \int \frac{-1}{x} dx + \int \frac{1}{x-1} dx$$

$$= -\ln|x| + \ln|x-1| + C$$

$$= \ln\left|\frac{x-1}{x}\right| + C$$

**3. Find the area of the shaded region between $y = 1 - 2x^2$ and $y = 2x - 3$ from *x* = 0 to *x* = 0 by using the integral calculus. First find the intersection points of two curves. (20 points)**




**Solution:**

The curves intersect each other at

$$y = 1 - 2x^2 = 2x - 3 \quad \Rightarrow \quad 2x^2 + 2x - 4 = 2(x+2)(x-1) = 0$$

$$\Rightarrow \quad (x = -2, y = -7) \ \& \ (x = 1, y = -1)$$

The area of the shaded region is from *x* = 0 to *x* = 3/2 shown as yellow. As seen from the plot the integral should be taken from *x* = 0 to *x* = 1 and from *x* = 1 to *x* = 3/2 because in each part, the upper and lower curves are not the same.

$$area = \int_0^1 \underbrace{\left(1 - 2x^2 - 2x + 3\right)}_{-2x^2 - 2x + 4} dx + \int_1^{3/2} \underbrace{\left(2x - 3 - 1 + 2x^2\right)}_{2x^2 + 2x - 4} dx =$$

$$= \left(-\frac{2x^3}{3} - x^2 + 4x\right)\Bigg|_0^1 + \left(\frac{2x^3}{3} + x^2 - 4x\right)\Bigg|_1^{3/2}$$

$$= \left\{-\frac{2}{3} - 1 + 4\right\} + \left\{\frac{2}{3}\frac{27}{8} + \frac{9}{4} - 4\frac{3}{2} - \frac{2}{3} - 1 + 4\right\} = \frac{7}{3} + \frac{9}{2} - \frac{2}{3} - 3 = \frac{19}{6}$$

**4. (a) The marginal-revenue function is $\frac{dr}{dq} = 500 - \frac{200}{q+1}$. Find the *p* demand function. (10 points)**

**Solution:**

The revenue function, *r*, can be obtained from the marginal-revenue function:

$$r = \int \frac{dr}{dq} dq = \int \left( 500 - \frac{200}{q+1} \right) dq = 500q - 200\ln(q+1) + C .$$

Here we have to find the *C* constant of integration by assuming that when no unit is sold, total revenue is zero as

$$r_{q=0} = \left(500q - 200\ln(q+1) + C\right)_{q=0} = C = 0 .$$

We know that the general relationship of revenue in term of *p* price per unit and *q* quantity *r* = *pq*. Hence the demand function (*p* price per unit) is

$$p = \frac{r}{q} = \frac{500q - 200\ln(q+1)}{q} = 500 - \frac{200}{q}\ln(q+1)$$

**b) The demand equation is $p = 100 - q^2$ and a supply equation of a product is $p = 2q + 20$. Determine the producers' surplus (*PS*) under the market equilibrium. (10 points)**

**Solution:**

First we have to find the market equilibrium where the demand and supply functions are equal to each other as follows:

$$f(q_0) = g(q_0)$$

$$100 - q_0^2 = 2q_0 + 20 \quad \Rightarrow \quad q_0^2 + 2q_0 - 80 = (q_0 + 10)(q_0 - 8) = 0 .$$

$$\Rightarrow \quad q_0 = 8 \qquad \Rightarrow \quad p_0 = 2q_0 + 20 = 36$$

Then we can find the producers' surplus (*PS*):

$$PS = \int_0^{q_0} \left(p_0 - f(q)\right) dq = \int_0^8 \underbrace{\left[36 - 2q - 20\right]}_{16-2q} dq = \left(16q - \frac{\not{2} q^2}{\not{2}}\right)_0^8 = 128 - 64 = 64$$